The g-Good-Neighbor Conditional Diagnosability of k-Ary n-Cubes under the PMC Model and MM* Model

Jun Yuan, Aixia Liu, Xue Ma, Xiuli Liu, Xiao Qin, *Senior Member, IEEE*, and Jifu Zhang

Abstract—The diagnosability of a system is defined as the maximum number of faulty processors that the system can guarantee to identify, which plays an important role in measuring of the reliability of multiprocessor systems. In the work of Peng et al. in 2012, they proposed a new measure for fault diagnosis of systems, namely, g-good-neighbor conditional diagnosability. It is defined as the diagnosability of a multiprocessor system under the assumption that every fault-free node contains at least g fault-free neighbors, which can measure the reliability of interconnection networks in heterogeneous environments more accurately than traditional diagnosability. The k-ary n-cube is a family of popular networks. In this study, we first investigate and determine the R_g-connectivity of k-ary n-cube for $0 \leq g \leq n$. Based on this, we determine the g-good-neighbor conditional diagnosability of k-ary n-cube under the PMC model and MM* model for $k \geq 4, n \geq 3$ and $0 \leq g \leq n$. Our study shows the g-good-neighbor conditional diagnosability of k-ary n-cube is several times larger than the classical diagnosability of k-ary n-cube.

Index Terms—PMC diagnosis model, MM* diagnosis model, k-ary n-cube, conditional connectivity, fault diagnosability

1 Introduction

Advances in the semiconductor technology have made it possible to develop very high-performance large multiprocessor systems comprising hundreds of thousands of processors (nodes). Yet, it is almost impossible to build such a multiprocessor system without defects. Since all the processors run in parallel, the reliability of each processor in multiprocessor systems becomes of central importance for parallel computing. Therefore, in order to maintain the reliability of such multiprocessor systems, the fault processors should be found and replaced in time.

The process of identifying faulty nodes is called the diagnosis of the system. In 1967, Preparata et al. [37] proposed a model and a framework, called system-level diagnosis, which could test the processors automatically by the system itself. It is well known that system-level diagnosis appears to be an alternative to traditional circuit-level testing in a large multiprocessor system. In the more than four decades following this pioneering work, many terms for system-level diagnosis have been defined and various models (e.g., PMC, BGM, and comparison models) have been considered in the literature [5], [16], [32], [33], [37]. Among the proposed models, two well-known diagnosis models, i.e., the Preparata, Metze, and Chien (PMC) model [37] and the Maeng and Malek (MM) model [32], have been widely adopted.

In the PMC model, every node u is able to test another node v if there is a link that connects them, where u is called the tester and v is called the tested node. The outcome of a test performed by a fault-free tester is 1 (respectively, 0) if the tested node is faulty (respectively, fault-free), whereas the outcome of a test performed by a faulty tester is unreliable. In [18], Hakimi and Amin prove that a multiprocessor system is t-diagnosable if it is t-connected with at least $2t+1$ nodes. They also give a necessary and sufficient condition for verifying if a system is t-diagnosable under the PMC model. Recently, Mánik and Gramatová [30], [31] propose a diagnosis algorithm under the PMC model which use Boolean formalization. Fan et al. show the disjoint consecutive cycle (DCC, for short) linear congruential graphs, $G(F, 2p)$, is $2t$-diagnosable under the PMC model where $p \geq 3$ and $2 \leq t \leq p-1$ [17]. Ahlswede and Aydinian study the diagnosability of large multiprocessor networks [1]. The diagnosability of the well-known interconnection network hypercube and its several variations [24], [40] for example, the crossed cube [13], the Möbius cube [14], and the twisted cube [20] are shown to be n under the PMC model. A modification of the PMC model, the BGM model, proposed by Barsi et al. [5], use the same testing strategy as PMC model, but it assumes that a fault node is always tested as faulty regardless of the state of the tester. The rational is that tests consist of long sequences of stimuli and testing a faulty node is very likely to result in at least one mismatch, see [2], [5].

In the MM model, a node (called a comparator) sends the same task to its two neighbors and compares their responses. A comparison of nodes u and v performed by

- *J. Yuan, A. Liu, X. Ma, and X. Liu are with the School of Applied Science, Taiyuan University of Science and Technology, Taiyuan 030024, Shanxi, P.R. China. E-mail: fromyuanjun@tom.com, aixiayuanjun@sina.com, {893371074, 1002429753}@qq.com.*
- *X. Qin is with the Department of Computer Science and Software Engineering, Shelby Center for Engineering Technology, Auburn University, Auburn, AL 36849.E-mail: qinxiao@gmail.com.*
- *J. Zhang is with the School of Computer Science and Technology, Taiyuan University of Science and Technology, Taiyuan 030024, Shanxi, P.R. China. E-mail: zjf@tyust.edu.cn.*

Manuscript received 26 Dec. 2013; revised 11 Mar. 2014; accepted 8 Apr. 2014. Date of publication 16 Apr. 2014; date of current version 6 Mar. 2015.
Recommended for acceptance by P. Santi.
For information on obtaining reprints of this article, please send e-mail to: reprints@ieee.org, and reference the Digital Object Identifier below.
Digital Object Identifier no. 10.1109/TPDS.2014.2318305

a node w is denoted by $(u,v)_w$. A disagreement over a comparison performed by a fault-free comparator indicates the existence of a faulty node, whereas the outcome of a comparison performed by a faulty comparator is unreliable. The main advantage of this model is its simplicity since it is easy to compare a pair of nodes in multiprocessor systems. This approach seems attractive because no additional hardware is required and transient and permanent faults may be detected before the comparison program has completed. A paper by Sengupta and Dahbura [39] revealed important properties of a diagnosable system under this model. It suggested a special case of the MM model, called the MM* model. In this model, a comparison $(u,v)_w$ must be performed by w if u and v are neighbors of w in the system. It also presented a polynomial algorithm to identify faulty nodes in a general system under the MM* model if the system is diagnosable. The MM* model was adopted in [6], [8], [15], [19], [21], [22], [27], [29].

In classical measures of system-level diagnosability for multiprocessor systems, it has generally been assumed that any subset of processors can potentially fail at the same time. If there is a node v whose neighbors are faulty simultaneously, there is no way of knowing the faulty or fault-free status of v. As a consequence, the diagnosability of a system is upper bounded by its minimum degree. However, it always underestimates the resilience of large networks because the failure probability that all the neighbours of the same node is very small in many large scale parallel/distribute system. To overcome the shortcoming, Lai et al. [28] proposed a novel measure of diagnosability, called the conditional diagnosability, for measuring the diagnosability of a system under the assumption that for each node u all the processors directly connected to u cannot fail at the same time, i.e., at least one of neighbors of u is fault free. They also obtained conditional diagnosability results for hypercubes under the PMC model in [28]. In [19], Xu et al. showed the conditional diagnosability of the n-dimensional hypercube under the MM* model is $3n-5$ for $n \geq 5$. Under the PMC model, Zhu [47] studied the conditional diagnosability of bijective connection (BC) networks, which include hypercubes and a variety of hypercube variants, such as crossed cubes, twisted cubes, Möbius cubes, locally twisted cubes, and generalized twisted cubes. In [43], Xu et al. further generalized the previous result by studying this problem in a family of popular networks, i.e., the Matching Composition Networks (MCNs). The conditional diagnosability of Cayley graphs generated by transposition trees was studied first by Lin et al. [29]. Results concerning the conditional diagnosability of variants of the hypercube network under this model were also obtained [22], [44], [46], [47].

Motivated by the concepts of forbidden faulty sets, Peng et al. [36] then proposed the g-good-neighbor conditional diagnosability: which is defined as the maximum value t such that a graph G remains t-diagnosable under the condition that every healthy vertex v has at least g fault-free neighboring vertices. Besides, they showed that the g-good-neighbor conditional diagnosability of the n-dimensional hypercube Q_n.

The interconnection network considered in this study is the k-ary n-cube Q_n^k, proposed in [3], which is one of the most common multiprocessor systems for parallel computer/communication system. The k-ary n-cube is $2n$-regular with k^n vertices, edge symmetric, and vertex symmetric. The three most popular instances of k-ary n-cube are the ring $k=1$, the hypercube $k=2$, and the torus $n=2$. A number of distributed memory multiprocessors are based on a k-ary n-cube as the underlying topology, such as the iWarp [23], the J-machine [34], the Cray T3D [38] and the Blue Gene [7]. Recently, Chang et al. showed the conditional diagnosability of k-ary n-cube under the PMC Model is $8n-7$ for $k \geq 4$ and $n \geq 4$. Hsieh and Lee [22] showed the conditional diagnosability of k-ary n-cube under the MM* Model is $6n-5$ for $k \geq 4$ and $n \geq 4$.

In this paper, we study the g-good-neighbor conditional diagnosability under the PMC model and MM* model, and show that the g-good-neighbor conditional diagnosability of Q_n^k is $2^n(2n-g+1)-1$ for $k \geq 4, n \geq 3, 0 \leq g \leq n$ under the two models, which shows that the corresponding results based on the traditional fault model (where g is zero) tend to substantially underestimate network reliability of k-ary n-cube. The remainder of this paper is organized as follows: Section 2 provides terminology and preliminaries for diagnosing a system. In Section 3, we discuss the R_g-connectivity of Q_n^k. Sections 4 and 5 show the g-good-neighbor conditional diagnosability of Q_n^k under the PMC model and the MM* model, respectively. Finally, our conclusions are given in Section 6.

2 Preliminaries

2.1 Notations

Throughout this paper, an interconnection network is represented by an undirected simple graph G with the vertex set $V(G)$ and the edge set $E(G)$. A subgraph H of G (written $H \subseteq G$) is a graph with $V(H) \subseteq V(G), E(H) \subseteq E(G)$ and the endpoints of every edge in $E(H)$ belong to $V(H)$. Given a nonempty vertex subset V' of $V(G)$, the induced subgraph by V' in G, denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of G with both endpoints in V'. The degree $d_G(v)$ of a vertex v is the number of edges incident with v. A graph G is said to be k-regular if for any vertex $v, d_G(v)=k$. For any vertex v, we define the neighborhood $N_G(v)$ of v in G to be the set of vertices adjacent to v. Let $A \subseteq G$. We use $N_G(A)$ to denote the set $(\bigcup_{v \in V(A)} N_G(v)) \setminus V(A)$, $C_G(A)$ to denote the set $N_G(A) \cup V(A)$. For neighborhoods and degrees, we will usually omit the subscript for the graph when no confusion arises. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left. For graph-theoretical terminology and notation not defined here we follow [4].

2.2 The PMC model

A multiprocessor system is modeled as an undirected graph $G=(V(G),E)$, whose vertices represent processors and edges represent communication links. In the PMC model [37], two adjacent processors can perform tests on each other. For two adjacent vertices u and v in $V(G)$, the ordered pair (u,v) represents the test performed by u on v. The outcome of a test (u,v) is either 1 or 0 with the assumption that the testing result is regarded as reliable if the vertex u is fault-free. However, the outcome of a test (u, v) is unreliable, provided that the tester u itself is originally a faulty processor. Suppose that the vertex u of (u, v) is fault-free, then the result would be 0 (respectively, 1) if v is fault-free (respectively, faulty).

A test assignment T for a system G is a collection of tests for every adjacent pair of vertices. It can be modeled as a directed testing graph $T = (V(G), L)$, where $(u, v) \in L$ implies that u and v are adjacent in G. The collection of all test results for a test assignment T is called a syndrome. Formally, a syndrome is a function $\sigma : L \to (0, 1)$. The set of all faulty processors in the system is called a faulty set. This can be any subset of $V(G)$. The process of identifying all the faulty vertices is called the diagnosis of the system.

For a given syndrome σ, a subset of vertices $F \subseteq V(G)$ is said to be consistent with σ if syndrome σ can be produced from the situation that, for any $(u, v) \in L$ such that $u \in V - F, \sigma(u, v) = 1$ if and only if $v \in F$. This means that F is a possible set of faulty processors. Since a test outcome produced by a faulty processor is unreliable, a given set F of faulty vertices may produce a lot of different syndromes. On the other hand, different fault sets may produce the same syndrome. We use notation $\sigma(F)$ to represent the set of all syndromes which could be produced if F is the set of faulty vertices.

Two distinct sets F_1 and F_2 in $V(G)$ are said to be indistinguishable if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$, otherwise, F_1 and F_2 are said to be distinguishable. Besides, we say (F_1, F_2) is an indistinguishable pair if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$; else, (F_1, F_2) is a distinguishable pair.

2.3 The MM* model

In the MM model, to diagnose a system, a vertex sends the same task to two of its neighbors, and then compares their responses. To be consistent with the MM model, we have the following assumptions:

1) All faults are permanent.
2) A faulty processor produces incorrect outputs for each of its given tasks.
3) The output of a comparison performed by a faulty processor is unreliable.
4) Two faulty processors given the same input and task do not produce the same output.

The comparison scheme of a system G is modeled as a multigraph, denoted by $M(V(G), L)$, where L is the labeled-edge set. A labeled edge $(u, v)_w \in L$ represents a comparison in which two vertices u and v are compared by a vertex w, which implies $(u, w), (v, w) \in E(G)$. The collection of all comparison results in $M(V(G), L)$ is called the syndrome, denoted by σ^*, of the diagnosis. The result of the comparison $(u, v)_w$ in r is denoted by $r((u, v)_w)$. If the comparison $(u, v)_w$ disagrees, then $\sigma^*((u, v)_w) = 1$; otherwise, $\sigma^*((u, v)_w) = 0$. Hence, a syndrome is a function from L to $0, 1$. The MM* model is a special case of the MM model. In the MM* model, all comparisons of G are in the comparison scheme of G, i.e., if $(u, w), (v, w) \in E(G)$, then $(u, v)_w \in L$.

For a given syndrome σ^*, a set of faulty vertices $F \subset V(G)$ is said to be consistent with σ^* if σ^* can be produced from F, i.e., if the following conditions are satisfied, according to the assumptions of the MM model:

1) if $u, v \in F$ and $w \in V(G) - F$, then $\sigma^*((u, v)_w) = 1$,
2) if $u \in F$ and $v, w \in V(G) - F$, then $\sigma^*((u, v)_w) = 1$, and
3) if $u, v, w \in V(G) - F$, then $\sigma^*((u, v)_w) = 0$.

Since a faulty comparator can lead to an unreliable result, one set of faulty vertices may produce different syndromes. Let $\sigma^*(F)$ denote the set of all syndromes which F is consistent with. Two distinct sets $F_1, F_2 \subset V(G)$ are said to be distinguishable if $\sigma^*(F_1) \cap \sigma^*(F_2) = \emptyset$. Otherwise, they are said to be indistinguishable. (F_1, F_2) is a distinguishable pair (respectively, an indistinguishable pair) if F_1 and F_2 are distinguishable (respectively, indistinguishable).

2.4 Diagnosability

In this section, some known conceptions and results about the diagnosability of system are listed as follows.

Definition 2.4.1 [11]. *A system of n processors is t-diagnosable if all faulty processors can be identified without replacement, provided that the number of faults presented does not exceed t. The diagnosability of a system G denoted as $t(G)$, is the maximum value of t such that G is t-diagnosable.*

In [28], Lai et al. present a sufficient and necessary condition for a system to be t-diagnosable as follows.

Theorem 2.4.2 [28]. *A system $G = (V, E)$ is t-diagnosable if and only if F_1 and F_2 are distinguishable, for any two distinct subsets F_1 and F_2 of V with $|F_1| \leq t, |F_2| \leq t$.*

Note that the diagnosability of a system always underestimates the resilience of large networks because the failure probability that all the neighbours of the same node is very small in many large scale parallel/distribute system. For this reason, Lai et al. [28] introduce the conditional diagnosability. They consider the situation that any faulty set cannot contain all the neighbors of any vertex in a system.

Definition 2.4.3 [28]. *A system $G = (V, E)$ is conditionally t-diagnosable if F_1 and F_2 are distinguishable, for each pair of distinct faulty sets $F_1, F_2 \subseteq V$ with $|F_1| \leq t, |F_2| \leq t$ and $F_1 \not\supseteq N(v), F_2 \not\supseteq N(v)$ for any vertex $v \in V$. The conditional diagnosability $t_c(G)$ of a graph G is the maximum value of t such that G is conditionally t-diagnosable.*

Motivated by these concepts of conditionally t-diagnosability and forbidden faulty sets [28], [42], Peng et al. [36] then propose the g-good-neighbor conditional diagnosability by claiming that for every fault-free vertex in a system, it has at least g fault-free neighbors.

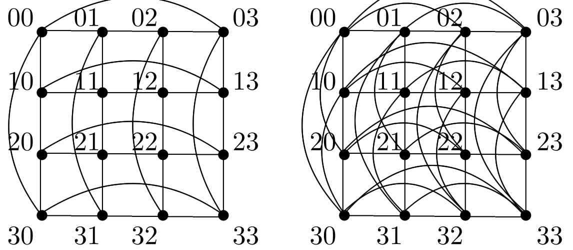
Definition 2.4.4 [36]. *A faulty set $F \subseteq V$ is called a g-good-neighbor conditional faulty set if $|N(v) \cap (V - F)| \geq g$ for every vertex v in $V - F$.*

Definiton 2.4.5 [36]. *A system $G = (V(G), E)$ is g-good-neighbor conditional t-diagnosable if each distinct pair of g-good-neighbor conditional faulty sets F_1 and F_2 of V with $|F_1| \leq t, |F_2| \leq t$ are distinguishable.*

Definiton 2.4.6 [36]. *The g-good-neighbor conditional diagnosability $t_g(G)$ of a graph G is the maximum value of t such that G is g-good-neighbor conditionally t-diagnosable.*

2.5 k-ary n-cube

The k-ary n-cube Q_n^k ($k \geq 2$ and $n \geq 1$) is a graph consisting of k^n vertices, each of which has the form $u_1 u_2 \cdots u_n$, where $0 \leq u_i \leq k - 1$ for $1 \leq i \leq n$. Two vertices $u = u_1 u_2 \cdots u_n$ and



(a). 4-ary 2-cube Q_2^4 (b). 4-ary 2-dimensional hypercube $Q(2,4)$

Fig. 1. 4-ary 2-cube Q_2^4 and 4-ary 2-dimensional hypercube Q(2,4).

$v = v_1v_2\cdots v_n$ are adjacent if and only if there exists an integer $j, 1 \le j \le n$, such that $u_j = v_j \pm 1 \pmod k$ and $u_l = v_l$ for every $l \ne j, 1 \le l \le n$. Such an edge (u,v) is called a j-dimensional edge. For brevity, we omit writing "(mod k)" in similar expressions for the remainder of the paper. Note that each vertex of Q_n^k has degree $2n$ when $k \ge 3$, and n when $k = 2$. Obviously, Q_1^k is a cycle of length k, Q_n^2 is an n-dimensional hypercube, and Q_2^k is a $k \times k$ wrap-around mesh. Fig. 1a shows the four-ary two-cube Q_2^4.

We can partition Q_n^k along the dimension j, by deleting all the j-dimensional edges, into k disjoint subcubes, $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$ (abbreviated as $Q[0], Q[1], \ldots, Q[k-1]$, if there is no ambiguity). It is clear that $Q_j[i]$ is a subgraph of Q_n^k induced by $\{u : u = u_1u_2\cdots u_j\cdots u_n \in V(Q_n^k)$ and $u_j = i\}$ and each $Q_j[i]$ is isomorphic to Q_{n-1}^k for every $0 \le i \le k-1$. Moreover, there is a perfect matching between $Q_j[i]$ and $Q_j[i+1]$ for $0 \le i \le k-1$.

Let X be a don't care symbol and let

$$X^t = \underbrace{XX\cdots X}_{t}.$$

For convenience of representation, we denote by an n-length string of symbols $X^m\alpha^l X^{n-m-l}$ the subgraph in Q_n^k induced by the vertex set $\{v = u_1u_2\cdots u_n \in V(Q_n^k) | u_{m+1}, u_{m+2}, \ldots, u_{m+l} = \alpha\}$. Let $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$ be a partition of Q_n^k along some dimension j. Clearly, $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$ can be denoted by $X^{j-1}0X^{n-j}, X^{j-1}1X^{n-j}, \ldots, X^{j-1}(k-1)X^{n-j}$, respectively.

The k-ary n-dimensional hypercube $Q(n,k)$ is a graph consisting of k^n vertices, each of which has the form $u_1u_2\cdots u_n$, where $0 \le u_i \le k-1$ for $1 \le i \le n$. Two vertices $u = u_1u_2\cdots u_n$ and $v = v_1v_2\cdots v_n$ are adjacent if and only if there exists exactly a dimension $j, 1 \le j \le n$, such that $u_j \ne v_j$ and $u_l = v_l$ for every $l \ne j$. Note that each vertex of $Q(n,k)$ has degree $n(k-1)$. Obviously, $Q(n,2)$ is an n-dimensional hypercube Q_n, and $Q(n,3)$ is a three-ary n-cube Q_n^3. Fig. 1b shows the four-ary two-dimensional hypercube $Q(2,4)$.

3 The R_g-Connectivity of k-Ary n-Cube Q_n^k

In order to get the g-good-neighbor conditional diagnosability of Q_n^k, we first need to discuss the R_g-connectivity of Q_n^k, which is closely related to g-good-neighbor conditional diagnosability, proposed by Latifi et al. [25]. A g-good-neighbor conditional cut of a graph G is a g-good-neighbor conditional faulty set F such that $G - F$ is disconnected. The cardinality of the minimum g-good-neighbor conditional cut is said to be the R_g-connectivity of G, denoted by $\kappa_g(G)$. As a more refined index than the traditional connectivity, the R_g-connectivity can be used to measure of conditional fault tolerance of networks. There are many results concerning the R_g-connectivity for particular classes of interconnection networks and small g's, such as [9], [12], [25], [26], [35], [41], [42], [45]. But, with regard to general integer g, little information has been found. In this section, we determine the R_g-connectivity of k-ary n-cube Q_n^k for $k \ge 4, n \ge 3$ and $0 \le g \le n$.

The following lemma discusses some properties of the subgraph of Q_n^k, which is isomorphic to the g-dimensional hypercube $Q_g, 0 \le g \le n$. By this Lemma, an upper boundary of the R_g-connectivity of Q_n^k is obtained.

Lemma 3.1. *Let H be an induced subgraph of k-ary n-cube Q_n^k such that H is isomorphic to the g-dimensional hypercube Q_g, and let $C_{Q_n^k}(H) = N_{Q_n^k}(H) \cup V(H)$, where $0 \le g \le n, k \ge 5$ and $n \ge 3$. Then $|N_{Q_n^k}(H)| = (2n-g)2^g$ and the minimum degree of $Q_n^k - C_{Q_n^k}(H)$ is not less than $2n-2$.*

Proof. By the symmetry of Q_n^k, without loss of generality, let $H = X^g0^{n-g}$, where $X \in \{0,1\}$. We first prove any pair of vertices in H have no common neighbors in $N_{Q_n^k}(H)$. Suppose, on the contrary, there are two vertices $u, v \in V(H)$ such that they have common neighbors in $N_{Q_n^k}(H)$. Say $u = u_1u_2\cdots u_g0^{n-g}, v = v_1v_2\cdots v_g0^{n-g}$. Then, by the definition of k-ary n-cube, we have either there is precise one dimension $i \in \{1,2,\ldots,g\}$ such that $u_i \ne v_i$ or there are precise two dimensions $i,j \in \{1,2,\ldots,g\}$ such that $u_i \ne v_i, u_j \ne v_j$. Consider the following two cases.

Case 1. There is precise one dimension $i \in \{1,2,\ldots,g\}$ such that $u_i \ne v_i$.

In this case, we have $u_i = v_i \pm 2 (\mathrm{mod} k)$. Since $u, v \in V(H)$, it follows that $u_i, v_i \in \{0,1\}$ and hence $u_i = v_i \pm 1$, contradicting $u_i = v_i \pm 2 (\mathrm{mod} k)$.

Case 2. There are precise two dimensions $i,j \in \{1,2,\ldots,g\}$ such that $u_i \ne v_i$ and $u_j \ne v_j$.

Without loss of generality, let $i = 1, j = 2$. Then, u and v precisely have two common neighbors $w_1 = u_1v_2u_3\ldots u_g0^{n-g}$ and $w_2 = v_1u_2u_3\cdots u_g0^{n-g}$ in Q_n^k. Since $u, v \in V(H)$, it follows that $u_l, v_l \in \{0,1\}$ for any $1 \le l \le g$. Therefore, both w_1 and w_2 belong to $V(H)$, that is u and v have no common neighbors in $N_{Q_n^k}(H)$.

Therefore, any pair of vertices in H have no common neighbors in $N_{Q_n^k}(H)$. Combining this with $H = X^g0^{n-g}$, $X \in \{0,1\}$, it is not difficult to see that (the subgraph induced by the neighbors of H is showed in Fig. 2):

□

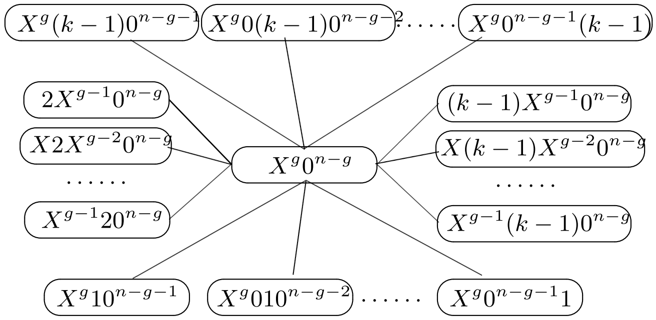


Fig. 2. The subgraph induced by the neighbors $N_{Q_n^k}(H)$ of H.

$$
\begin{aligned}
&\left|N_{Q_n^k}(H)\right| \\
&= |V(X^g 1 0^{n-g-1}) \cup V(X^g 0 1 0^{n-g-2}) \\
&\quad \cup \cdots \cup V(X^g 0^{n-g-1} 1)| + |V(X^g (k-1) 0^{n-g-1}) \\
&\quad \cup V(X^g 0 (k-1) 0^{n-g-2}) \cup \cdots \cup V(X^g 0^{n-g-1} (k-1))| \\
&\quad + |V(2 X^{g-1} 0^{n-g}) \cup V(X 2 X^{g-2} 0^{n-g}) \\
&\quad \cup \cdots \cup V(X^{g-1} 2 0^{n-g})| + |V((k-1) X^{g-1} 0^{n-g}) \\
&\quad \cup V(X (k-1) X^{g-2} 0^{n-g}) \cup \cdots \cup V(X^{g-1} (k-1) 0^{n-g})| \\
&= |V(X^g 1 0^{n-g-1})| + |V(X^g 0 1 0^{n-g-2})| + \cdots \\
&\quad + |V(X^g 0^{n-g-1} 1)| + |V(X^g (k-1) 0^{n-g-1})| \\
&\quad + |V(X^g 0 (k-1) 0^{n-g-2})| + \cdots \\
&\quad + |V(X^g 0^{n-g-1} (k-1))| + |V(2 X^{g-1} 0^{n-g})| \\
&\quad + |V(X 2 X^{g-2} 0^{n-g}) + \cdots \\
&\quad + |V(X^{g-1} 2 0^{n-g})| + |V((k-1) X^{g-1} 0^{n-g})| \\
&\quad + |V(X (k-1) X^{g-2} 0^{n-g})| + \cdots \\
&\quad + |V(X^{g-1} (k-1) 0^{n-g})| \\
&= 2^g \times (n-g) + 2^g \times (n-g) + 2^{g-1} \times g + 2^{g-1} \times g \\
&= 2^g \times (2n-g).
\end{aligned}
$$

We now shall show the minimum degree of $Q_n^k - C_{Q_n^k}(H)$ is not less than $2n-2$. For any vertex $x \in V(Q_n^k - C_{Q_n^k}(H))$, assume that $x = (u_1 u_2 \cdots u_g u_{g+1} \cdots u_n)$. Consider the following six cases:

Case 1. There is precise one element, which is 2 or $k-1$, in $\{u_1, u_2, \ldots, u_g\}$, and the others are 0 or 1; there is precise one element, which is 1 or $k-1$ in $u_{g+1}, u_{g+2}, u_{g+3}, \ldots, u_n$, and the others are all 0.

Without loss of generality, let $x = (2u_2 \cdots u_g 1 0^{n-g-1})$. Note $k \geq 5$. Then the neighbors of x in $C_{Q_n^k}(H)$ are $(1u_2 \cdots u_g 1 0^{n-g-1})$ and $(2u_2 \cdots u_g 0^{n-g})$. So $d_{Q_n^k - C_{Q_n^k}(H)}(x) = 2n-2$.

Case 2. Each of $u_1, u_2, \ldots, u_g$ is 0 or 1; and precise one of $u_{g+1}, u_{g+2}, \ldots, u_n$ is 2 or $k-2$, and the others are all 0.

Without loss of generality, let $x = (u_1 u_2 \cdots u_g 2 0^{n-g-1})$. Then the neighbor of x in $C_{Q_n^k}(H)$ is $(u_1 u_2 \cdots u_g 1 0^{n-g-1})$. So $d_{Q_n^k - C_{Q_n^k}(H)}(x) = 2n-1$.

Case 3. All of $u_1, u_2, \ldots, u_g$ are 0 or 1; and there are precise two elements, which are 1 or $k-1$ in $\{u_{g+1}, u_{g+2}, \ldots, u_n\}$, and the others are all 0.

Without loss of generality, let $x = (u_1 u_2 \cdots u_g 1 1 0^{n-g-2})$. Then the neighbors of x in $C_{Q_n^k}(H)$ are $(u_1 u_2 \cdots u_g 1 0^{n-g-1})$ and $(u_1 u_2 \cdots u_g 0 1 0^{n-g-2})$. So $d_{Q_n^k - C_{Q_n^k}(H)}(x) = 2n-2$.

Case 4. Precise two elements of $\{u_1, u_2, \ldots, u_g\}$ are 2 or $k-1$, and the others are 0 or 1; and all of $u_{g+1}, u_{g+2}, \ldots, u_n$ are 0.

Without loss of generality, let $x = (2 2 u_3 \cdots u_g 0^{n-g})$. Then the neighbors of x in $C_{Q_n^k}(H)$ are $(1 2 u_3 \cdots u_g 0^{n-g})$ and $(2 1 u_3 \cdots u_g 0^{n-g})$. So $d_{Q_n^k - C_{Q_n^k}(H)}(x) = 2n-2$.

Case 5. Precise one element of $u_1, u_2, \ldots, u_g$ is 3 or $k-2$, and the others are 0 or 1; all of $u_{g+1}, u_{g+2}, \ldots, u_n$ are 0.

Without loss of generality, let $x = (3u_2 \cdots u_g 0^{n-g})$. If $k = 5$, then the neighbors of x in $C_{Q_n^k}(H)$ are $(2u_2 \cdots u_g 0^{n-g})$ and $(4u_2 \cdots u_g 0^{n-g})$. Otherwise, the neighbor of x in $C_{Q_n^k}(H)$ is $(2u_2 \cdots u_g 0^{n-g})$. So $d_{Q_n^k - C_{Q_n^k}(H)}(x) \geq 2n-2$.

Case 6. x does not satisfy any one of the above five cases.

Then x is not adjacent to any vertex of $C_{Q_n^k}(H)$. So $d_{Q_n^k - C_{Q_n^k}(H)}(x) = 2n$.

The proof is complete. □

Corollary 3.2. *The R_g-connectivity $\kappa_g(Q_n^k) \leq (2n-g)2^g$ for $0 \leq g \leq n, k \geq 5$ and $n \geq 3$.*

Proof. Let H be an induced subgraph of k-ary n-cube such that $H \cong Q_g$. By Lemma 3.1, $|N_{Q_n^k}(H)| = (2n-g)2^g$ and the minimum degree $\delta(Q_n^k - C_{Q_n^k}(H)) \geq 2n-2$. Since $n \geq 3$, it follows that $2n-2 \geq n$. Thus $\delta(Q_n^k - N_{Q_n^k}(H)) = \delta(H) = n \geq g$. By the definitions of g-good-neighbor conditional cut and R_g-connectivity, we have $N_{Q_n^k}(H)$ is a g-good-neighbor conditional cut of Q_n^k and hence $\kappa_g(Q_n^k) \leq |N_{Q_n^k}(H)| = (2n-g)2^g$. The proof is complete. □

Next, we shall show $(2n-g)2^g$ is also the lower boundary of $\kappa_g(Q_n^k)$ for $0 \leq g \leq n, k \geq 5$ and $n \geq 3$. Before doing this, we need to have some useful topological properties of Q_n^k.

Lemma 3.3. *Let H be a connected subgraph of k-ary n-cube such that the minimum degree $\delta(H)$ of H is not less than g, where $0 \leq g \leq n, n \geq 3, k \geq 4$. Then $|V(H)| \geq 2^g$.*

Proof. The proof is by induction on g. Clearly, the result is true for the base cases $g = 0$ and 1. Assume the result is true for $g-1$. We now consider $g \geq 2$. Since $\delta(H) \geq g \geq 2, Q_n^k$ can be partitioned into $Q[0], Q[1], \ldots, Q[k-1]$ along some dimension l such that H is not a subgraph of any $Q[i]$ for $i = 0, 1, \ldots, k-1$. By symmetry, without loss of generality, we may assume $V(H) \cap V(Q[i]) \neq \emptyset$ for $i = 0, 1, \ldots, p-1$, where $2 \leq p \leq k$. For $0 \leq i \leq p-1$, let H_i be the induced subgraph $Q[i][V(H) \cap V(Q[i])]$ and u be an arbitrary vertex in H_i. It is sufficient to discuss the following two cases.

Case 1. $2 \leq p \leq k-1$.

Clearly, for $i = 0$ or $p-1, d_{H_i}(u) \geq d_H(u) - 1 \geq g-1$, and for $i = 1, 2, \ldots, p-2$, $d_{H_i}(u) \geq d_H(u) - 2 \geq g-2$. It follows that $\delta(H_0) \geq g-1, \delta(H_{p-1}) \geq g-1$, and $\delta(H_i) \geq g-2$ for $i = 1, 2, \ldots, p-2$. By induction, we have $|V(H_0)| \geq 2^{g-1}, |V(H_{p-1})| \geq 2^{g-1}$, and $|V(H_i)| \geq 2^{g-2}$ for $i = 1, 2, \ldots, p-2$. So,

$$
\begin{aligned}
|V(H)| &= |V(H_0)| + |V(H_1)| + \cdots + |V(H_{p-1})| \\
&\geq 2^{g-1} \times 2 = 2^g.
\end{aligned}
$$

Case 2. $p = k$.

Clearly, $d_{H_i}(u) \geq d_H(u) - 2 \geq g-2$. It follows that $\delta(H_i) \geq g-2$ for $i = 0, 1, \ldots, k-1$. By induction, $|V(H_i)| \geq 2^{g-2}$, and hence

$$
\begin{aligned}
|V(H)| &= |V(H_0)| + |V(H_1)| + \cdots + |V(H_{k-1})| \\
&\geq 2^{g-2} \times k \geq 2^g.
\end{aligned}
$$

The proof is complete. □

Lemma 3.4. *Let $Q_i[0], Q_i[1], \ldots, Q_i[k-1]$ be the decomposition of $Q_n^k (k \geq 5, n \geq 3)$ along some dimension i and let A be a* *connected subgraph of* Q_n^k *and* $A \subseteq Q_n^k[\bigcap_{i=1}^n((V(Q_i[0]) \cup V(Q_i[1]) \cup V(Q_i[2]))]$. *If* $0 \le g \le \delta(A) \le n$, *then* $|N_{Q_n^k}(A)| \ge 2^g(2n-g)$.

Proof. The proof proceeds by induction on g. Since $|N_{Q_n^k}(A)| \ge \kappa(Q_n^k) = 2n$, the statement is true for the base case $g=0$. Consider the case $g \ge 1$. Let $R = Q_n^k[\bigcap_{i=1}^n((V(Q_i[0]) \cup V(Q_i[1]) \cup V(Q_i[2]))]$. Since $A \subseteq R$ and $\delta(A) \ge g \ge 1$, there exists a dimension $i \in \{1, 2, \ldots, n\}$ such that at least two of $A \cap Q_i[0], A \cap Q_i[1], A \cap Q_i[2]$ are not empty.

Case 1. Exactly two of $A \cap Q_i[0], A \cap Q_i[1], A \cap Q_i[2]$ are not empty.

Without loss of generality, say $A_0 = A \cap Q_i[0] \ne \emptyset$, $A_1 = A \cap Q_i[1] \ne \emptyset$. By the construction of Q_n^k and $\delta(A) \ge g$, we can deduce that $\delta(A_0) \ge g-1$ and $\delta(A_1) \ge g-1$. Furthermore, by induction, we have $|N_{Q_i[0]}(A_0)| \ge 2^{g-1}(2n-g-1)$ and $|N_{Q_i[1]}(A_1)| \ge 2^{g-1}(2n-g-1)$. Since $\delta(A_0) \ge g-1$ and $\delta(A_1) \ge g-1$, by Lemma 3.3, we can deduce that $|V(A_0)| \ge 2^{g-1}$, $|V(A_1)| \ge 2^{g-1}$. Combining this with the construction of Q_n^k, we can deduce $|N_{Q_i[k-1]}(V(A_0))| = |V(A_0)| \ge 2^{g-1}$, $|N_{Q_i[2]}(V(A_1))| = |V(A_1)| \ge 2^{g-1}$. Therefore,

$$\begin{aligned}\left|N_{Q_n^k}(A)\right| &\ge |N_{Q_i[k-1]}(V(A_0))| + |N_{Q_i[0]}(A_0)| \\ &\quad + |N_{Q_i[1]}(A_1)| + |N_{Q_i[2]}(V(A_1))| \\ &\ge 2^g(2n-g).\end{aligned}$$

Case 2. All of $A \cap Q_i[0], A \cap Q_i[1], A \cap Q_i[2]$ are not empty.Denote $A_0 = A \cap Q_i[0], A_1 = A \cap Q_i[1], A_2 = A \cap Q_i[2]$. By the construction of Q_n^k and $\delta(A) \ge g$, we can deduce that $\delta(A_0) \ge g-1, \delta(A_1) \ge g-2$ and $\delta(A_2) \ge g-1$. By induction, we have $|N_{Q_i[0]}(A_0)| \ge 2^{g-1}(2n-g-1), |N_{Q_i[1]}(A_1)| \ge 2^{g-2}(2n-g)$ and $|N_{Q_i[2]}(A_2)| \ge 2^{g-1}(2n-g-1)$. Since $\delta(A_0) \ge g-1$ and $\delta(A_2) \ge g-1$, by Lemma 3.3, we can deduce that $|V(A_0)| \ge 2^{g-1}$, $|V(A_2)| \ge 2^{g-1}$. Clearly, $|N_{Q_i[k-1]}(V(A_0))| = |V(A_0)| \ge 2^{g-1}, |N_{Q_i[3]}(V(A_2))| = |V(A_2)| \ge 2^{g-1}$. Therefore,

$$\begin{aligned}\left|N_{Q_n^3}(A)\right| &\ge |N_{Q_i[k-1]}(V(A_0))| + |N_{Q_i[0]}(A_0)| \\ &\quad + |N_{Q_i[1]}(A_1)| + |N_{Q_i[2]}(A_2)| \\ &\quad + |N_{Q_i[3]}(V(A_2))| \\ &> 2^g(2n-g).\end{aligned}$$

The proof is complete. □

The g-restricted connectivity of G is closed related to the κ_g-connectivity. A vertex cut of G is called a g-restricted cut if $G-S$ is disconnected and every component of $G-S$ has more than g vertices. The g-restricted connectivity of G, denoted by $\widetilde{\kappa}_g(G)$, is defined as the cardinality of a minimum g-restricted cut. Clearly, $\kappa_0(G) = \widetilde{\kappa}_0(G) = \kappa(G)$ and $\kappa_1(G) = \widetilde{\kappa}_1(G)$. In 2004, Day determined the 1-restricted connectivity of $Q_n^k (k \ge 4)$ [10].

Lemma 3.5 [3], [10]. *The g-restricted connectivity of* $Q_n^k (k \ge 4)$, $\widetilde{\kappa}_g(Q_n^k) = 2(g+1)n - 2g$ *for* $n \ge 2$ *and* $g = 0, 1$.

The following Lemma shows $(2n-g)2^g$ is the lower boundary of $\kappa_g(Q_n^k)$ for $0 \le g \le n, k \ge 5$ and $n \ge 3$.

Lemma 3.6. *The* R_g*-connectivity* $\kappa_g(Q_n^k) \ge (2n-g)2^g$ *for* $0 \le g \le n, k \ge 5$ *and* $n \ge 3$.

Proof. Let F be an arbitrary R_g-cut of k-ary n-cube Q_n^k, where $0 \le g \le n, k \ge 5$ and $n \ge 3$. It is sufficient to show that $|F| \ge 2^g(2n-g)$. We shall prove $|F| \ge 2^g(2n-g)$ by induction on g. By Lemma 3.5, the statement is true for the base cases $g = 0, 1$. We now consider $g \ge 2$. Let $Q[0], Q[1], \ldots, Q[k-1]$ be a decomposition of Q_n^k along dimension $l (1 \le l \le n)$ and let $F_i = F \cap V(Q[i])$ for any $i = 0, 1, \ldots, k-1$ (F_i is allowed to be empty.). We discuss the following two cases.

Case 1. $Q[i] - F_i$ is disconnected for any $i = 0, 1, \ldots, k-1$.

Since F is an R_g-cut of Q_n^k, we have $\delta(Q_n^k - F) \ge g$. Note that the l-dimensional edge set between $Q[i]$ and $Q[i+1]$ is a perfect matching of $V(Q[i])$ and $V(Q[i+1])$. Thus, for any $i = 0, 1, \ldots, k-1$, $\delta(Q[i] - F_i) \ge g-2$. It follows that F_i is an R_{g-2}-cut of $Q[i]$. By induction, $|F_i| \ge 2^{g-2}(2n-g)$ for $i = 0, 1, \ldots, k-1$. Therefore, $|F| = |F_0| + |F_1| + \cdots + |F_{k-1}| \ge k \times 2^{g-2}(2n-g) \cdots 2^g(2n-g)$.

Case 2. There exists a $Q[i]$ such that $Q[i] - F_i$ is connected.

Assume that $G_{i_0} = Q[i_0] - F_{i_0}$ is connected and G is the component of $Q_n^k - F$ such that $G_{i_0} \subseteq G$. Without loss of generality, assume that $V(G) \cap V(Q[j]) \ne \emptyset$ for $j = 0, 1, \ldots, p-1 (2 \le p \le k)$. Clearly, $0 \le i_0 \le p-1$.

Case 2.1. $p \le k-1$.

Let x be an arbitrary vertex in G_{i_0} and let $x_i \in V(Q[i]) (i \ne i_0)$ be the vertex such that just the ith coordinate of x_i is different from that of x. Since $p \le k-1$, it follows that there exists an integer i such that $x_i \in F - F_{i_0}$. Thus $|F| = |F - F_{i_0}| + |F_{i_0}| \ge |V(G_{i_0})| + |F_{i_0}| = |V(Q[i_0])| = k^{n-1} \ge 2^g(2n-g)$.

Case 2.2. $p = k$.

For $i = 0, 1, \ldots, k-1$, denote $G \cap Q[i]$ by G_i.

Case 2.2.1. There are at least four of $Q[i] - F_i$ for $i = 0, 1, \ldots, k-1$, such that each of them is disconnected.

Say $Q[i_1] - F_{i_1}, Q[i_2] - F_{i_2}, Q[i_3] - F_{i_3}, Q[i_4] - F_{i_4}$ are disconnected. Clearly, $\delta(Q_n^k - F) \ge g$. Note that the l-dimensional edge set between $Q[i]$ and $Q[i+1]$ is a perfect matching of $V(Q[i])$ and $V(Q[i+1])$. We can deduce $F_{i_1} = F \cap Q[i_1], F_{i_2} = F \cap Q[i_2], F_{i_3} = F \cap Q[i_3], F_{i_4} = F \cap Q[i_4]$ are $R_{g-2}-$cuts of $Q[i_1], Q[i_2], Q[i_3], Q[i_4]$, respectively. By induction, $|F_{i_1}| \ge 2^{g-2}(2n-g), |F_{i_2}| \ge 2^{g-2}(2n-g), |F_{i_3}| \ge 2^{g-2}(2n-g), |F_{i_4}| \ge 2^{g-2}(2n-g)$. Thus $|F| \ge |F_{i_1}| + |F_{i_2}| + |F_{i_3}| + |F_{i_4}| \ge 2^g(2n-g)$.

Case 2.2.2. There are at most three of $Q[i] - F_i, i = 0, 1, \ldots, k-1$ such that each of them is disconnected.

By contradiction, suppose $|F| < 2^g(2n-g)$. We now shall show two important claims.

Claim 1. $|V(G)| > k^n - \frac{13}{2} \times 2^{g-2}(2n-g)$.

Suppose there are exactly three of $Q[i] - F_i, i = 0, 1, \ldots, k-1$ such that each of them is disconnected. Say $Q[i_1] - F_{i_1}, Q[i_2] - F_{i_2}, Q[i_3] - F_{i_3}$ are disconnected.

If i_1, i_2, i_3 are three continuous integers(mod k), then without loss of generality, assume $i_1 = 0, i_2 = 1, i_3 = 2$. Clearly, F_0, F_1, F_2 are R_{g-2}-cuts of $Q[0], Q[1]$ and $Q[2]$, respectively. By induction, $|F_i| \ge 2^{g-2}(2n-g)$ for $i = 0, 1, 2$. Let $H_i = Q[i] - F_i - G_i$ for $i = 0, 1, 2$. Then, by the construction of Q_n^k, we have $|V(H_0)| \leq |F_{k-1}|$, $|V(H_2)| \leq |F_3|$ and

$$\begin{aligned}|V(H_1)| &\leq \min\{|V(H_0)| + |F_0|, |V(H_2)| + |F_2|\} \\ &\leq \frac{|V(H_0)| + |F_0| + |V(H_2)| + |F_2|}{2} \\ &\leq \frac{|F_{k-1}| + |F_0| + |F_3| + |F_2|}{2} \\ &\leq \frac{|F| - |F_1|}{2}.\end{aligned}$$

Since $|F| < 2^g(2n-g), |F_i| \geq 2^{g-2}(2n-g)$ for $i = 0,1,2,$ and $|F| \geq |F_0| + |F_1| + |F_2| + |F_3| + |F_{k-1}|$, we have that $|F_3| + |F_{k-1}| < 2^{g-2}(2n-g)$. Therefore,

$$\begin{aligned}|V(G)| &= |V(Q_n^k)| - |V(H_0)| - |V(H_1)| - |V(H_2)| - |F| \\ &\geq k^n - |F_{k-1}| - \frac{|F| - |F_1|}{2} - |F_3| - |F| \\ &> k^n - 2^{g-2}(2n-g) - \frac{2^g(2n-g) - 2^{g-2}(2n-g)}{2} \\ &\quad - 2^g(2n-g) \\ &= k^n - \frac{13}{2} \times 2^{g-2}(2n-g).\end{aligned}$$

If i_1, i_2 are continuous integers(mod k), but i_1, i_2, i_3 are not continuous, then without loss of generality, assume $i_1 = 0, i_2 = 1$. Clearly, we can show that F_0, F_1, F_{i_3} are R_{g-2}-cuts of $Q[0], Q[1]$ and $Q[i_3]$, respectively. By induction, $|F_i| \geq 2^{g-2}(2n-g)$ for $i = 0, 1, i_3$. Let $H_i = Q[i] - F_i - G_i$ for $i = 0, 1, i_3$. Then $|V(H_0)| \leq |F_{k-1}|, |V(H_1)| \leq |F_2|$, $|V(H_{i_3})| \leq \min\{|F_{i_3-1}|, |F_{i_3+1}|\} \leq \frac{|F_{i_3-1}| + |F_{i_3+1}|}{2} \leq \frac{|F| - |F_0| - |F_1| - |F_{i_3}|}{2}$. Therefore,

$$\begin{aligned}&|V(G)| \\ &= |V(Q_n^k)| - |V(H_0)| - |V(H_1)| - |V(H_{i_3})| - |F| \\ &\geq k^n - |F_{k-1}| - |F_2| - \frac{|F| - |F_0| - |F_1| - |F_{i_3}|}{2} - |F| \\ &> k^n - 2 \times 2^{g-2}(2n-g) \\ &\quad - \frac{2^g(2n-g) - 3 \times 2^{g-2}(2n-g)}{2} - 2^g(2n-g) \\ &= k^n - \frac{13}{2} \times 2^{g-2}(2n-g).\end{aligned}$$

Suppose that i_1, i_2 and i_3 are not continuous integers pairwise. Clearly, we can show that $F_{i_1}, F_{i_2}, F_{i_3}$ are R_{g-2}-cuts of $Q[i_1], Q[i_2]$ and $Q[i_3]$, respectively. By induction, $|F_{i_j}| \geq 2^{g-2}(2n-g)$ for $j = 1, 2, 3$. Let $H_{i_j} = Q[i_j] - F_{i_j} - G_{i_j}$ for $j = 1, 2, 3$. Then

$$|V(H_{i_1})| \leq \min\{|F_{i_1-1}|, |F_{i_1-1}|\} \leq \frac{|F_{i_1-1}| + |F_{i_1-1}|}{2},$$

$$|V(H_{i_2})| \leq \min\{|F_{i_2-1}|, |F_{i_2-1}|\} \leq \frac{|F_{i_2-1}| + |F_{i_2-1}|}{2}$$

and

$$|V(H_{i_3})| \leq \min\{|F_{i_3-1}|, |F_{i_3+1}|\} \leq \frac{|F_{i_3-1}| + |F_{i_3+1}|}{2}.$$

Since $|F| < 2^g(2n-g)$ and $|F_{i_j}| \geq 2^{g-2}(2n-g)$ for $j = 1, 2, 3$, we have that

$$\begin{aligned}\sum_{j=1}^{3} |V(H_{i_j})| &\leq \frac{1}{2} \sum_{j=1}^{3} \left(|F_{i_j-1}| + |F_{i_j+1}|\right) \\ &\leq \frac{1}{2}\left(|F| - \sum_{j=1}^{3} |F_{i_j}|\right) \\ &< 72^{g-3}(2n-g).\end{aligned}$$

Therefore,

$$\begin{aligned}|V(G)| &= |V(Q_n^k)| - \sum_{j=1}^{3} |V(H_{i_j})| - |F| \\ &> k^n - 2^{g-3}(2n-g) - 2^g(2n-g) \\ &> k^n - \frac{13}{2} \times 2^{g-2}(2n-g).\end{aligned}$$

Using a similar discussion, we can deduce that the Claim is true when there are at most two of $Q[i] - F_i, i = 0, 1, \ldots, k-1$ such that each of them is disconnected. The proof of Claim 1 is complete.

Claim 2. For any decomposition $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$ of Q_n^k along some dimension $1 \leq j \leq n$, G intersects with every one of $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$.

Suppose on the contrary that there exists a decomposition $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$ of Q_n^k along some dimension $1 \leq j \leq n$ such that G does not intersect with every one of $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$. Without loss of generality, assume that $G \cap Q_j[0], G \cap Q_j[1], \ldots, G \cap Q_j[l-1]$ are not empty. Clearly, $l \leq k-1$. Denote $G \cap Q_j[i]$ by G_i for $0 \leq i \leq l-1$. Let $F_j[i] = F \cap V(Q_j[i])$ for $i = 0, 1, \ldots, k-1$.

If $Q_j[i] - F_j[i]$ is disconnected for any $i = 0, 1, \ldots, k-1$, then by Case 1, we have $|F| \geq 2^g(2n-g)$, a contradiction. So suppose that there exists a $Q_j[i]$ such that $Q_j[i] - F_j[i]$ is connected. Let $Q_j[i_0] - F_j[i_0]$ be connected, and H be the component of $Q_n^k - F$ such that $Q_j[i_0] - F_j[i_0]$ is a subgraph of H. If H does not intersect with every one of $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$, then similar to Case 2.1, we can conclude $|F| \geq 2^g(2n-g)$, a contradiction. So suppose H intersects with every one of $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$. By the assumption that G does not intersect with every one of $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$, we can conclude $H \neq G$. If there are at least four of $Q_j[i] - F_j[i], i = 0, 1, \ldots, k-1$, such that each of them is disconnected, then similar to Case 2.2.1, we have $|F| \geq 2^g(2n-g)$, a contradiction. Therefore, there are at most three of $Q_j[i] - F_j[i], i = 0, 1, \ldots, k-1$, such that each of them is disconnected. Similar to the Claim 1 of Case 2.2.2, we have $|V(H)| \geq k^n - \frac{13}{2} \times 2^{g-2}(2n-g)$. Since F is an R_{g-1}-cut of Q_n^k, by induction, we have $|F| \geq 2^{g-1}[2n-g+1]$. So

$$\begin{aligned}k^n &= |V(Q_n^k)| \geq |V(G)| + |V(H)| + |F| \\ &> 2\left[k^n - \frac{13}{2} \times 2^{g-2}(2n-g)\right] + 2^{g-1}[2n-g+1].\end{aligned}$$

It is easy to show

$$k^n < 2\left[k^n - \frac{13}{2} \times 2^{g-2}(2n-g)\right] + 2^{g-1}[2n-g+1],$$

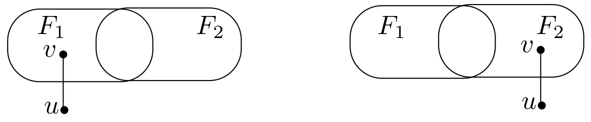


Fig. 3. Illustration of a distinguishable pair (F_1, F_2) under the PMC model.

a contradiction. The proof of Claim 2 is complete.

Let A be an another component of $Q_n^k - F$ except G. Then by Claim 2 and the above discussion, we can deduce that for any decomposition $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$ of Q_n^k along some dimension $1 \leq j \leq n$, A intersects at most three of $Q_j[0], Q_j[1], \ldots, Q_j[k-1]$. Without loss of generality, let $A \subseteq Q_n^k[\bigcap_{j=1}^n (V(Q_j[0]) \cup V(Q_j[1]) (Q_j[2]))]$. Since F is an R_g-cut of Q_n^k, it follows that $\delta(A) \geq g$. Then by Lemma 3.4, $|F| \geq |N_{Q_n^3}(A)| \geq 2^g(2n-g)$, a contradiction. □

Latifi et al. [25], and Wu and Guo [42] have studied the R_g-connectivity of the hypercube and got the following result.

Theorem 3.7 [25], [42]. *Assume that $n \geq 3$ and $0 \leq g \leq n-2$. Then the R_g-connectivity of n-dimensional hypercube Q_n, $\kappa_g(Q_n) = (n-g)2^g$.*

By the definition of k-ary n-cube, we have $Q_n^4 = Q_{2n}$. So, by Theorem 3.7, we can deduce the following corollary.

Corollary 3.8. *Assume that $n \geq 2$ and $0 \leq g \leq 2n-2$. Then the R_g-connectivity of four-ary n-cube Q_n^4, $\kappa_g(Q_n^4) = (2n-g)2^g$.*

Combining Corollaries 3.2 and 3.8, and Lemma 3.6, we can obtain the R_g-connectivity of Q_n^k for $0 \leq g \leq n, n \geq 3$ and $k \geq 4$.

Theorem 3.9. *Assume that $k \geq 4, n \geq 3$ and $0 \leq g \leq n$. Then the R_g-connectivity of k-ary n-cube Q_n^k, $\kappa_g(Q_n^k) = (2n-g)2^g$.*

4 THE g-GOOD-NEIGHBOR CONDITIONAL DIAGNOSABILITY OF k-ARY n-CUBE Q_n^k UNDER THE PMC MODEL

In this section, we shall show the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the PMC model.

Let $G = (V, E)$ be an undirected graph of a system G. Let F_1 and F_2 be two distinct subsets of V, and let the symmetric difference $F_1 \triangle F_2 = (F_1 \cup F_2) - (F_1 - F_2)$. In 1984, Dahbura and Masson [11] proposed a sufficient and necessary condition for two distinct subsets F_1 and F_2 to be a distinguishable-pair under the PMC model.

Theorem 4.1 [11]. *For any two distinct subsets F_1 and F_2 of V, (F_1, F_2) is a distinguishable pair under the PMC model if and only if there is a vertex $u \in V - (F_1 \cup F_2)$ and there is another vertex $v \in (F_1 \triangle F_2)$ such that $(u, v) \in E$ (see Fig. 3).*

The following lemma follows from Definition 2.4.5 and Theorem 4.1.

Lemma 4.2. *A system G is g-good-neighbor conditional t-diagnosable under the PMC model if and only if there is an edge $(u, v) \in E(G)$ with $u \in V - (F_1 \cup F_2)$ and $v \in (F_1 \triangle F_2)$ for each distinct pair of g-good-neighbor conditional faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$.*

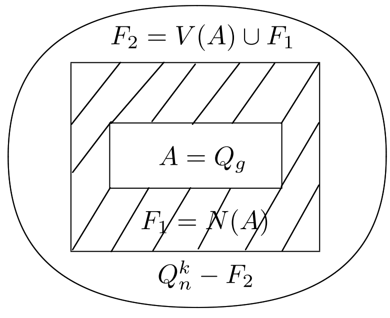


Fig. 4. An illustration about the proofs of Theorem 4.3 and Lemma 5.5.

Let g be a positive integer with $0 \leq g \leq n$. To find the g-good-neighbor conditional diagnosability $t_g(Q_n^k)$ under the PMC model, we first show that $t_g(Q_n^k)$ is no more than $(2n-g+1)2^g - 1$ for $k \geq 5, n \geq 3$ and $0 \leq g \leq n$.

Theorem 4.3. *Assume that $k \geq 5, n \geq 3$ and $0 \leq g \leq n$. Then the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the PMC model, $t_g(Q_n^k) \leq (2n-g+1)2^g - 1$.*

Proof. Let $A \cong Q_g$ be a subgraph of Q_n^k and let $F_1 = N_{Q_n^k}(A), F_2 = C_{Q_n^k}(A)$ (See Fig. 4). Then by Lemma 3.1, $|F_1| = (2n-g)2^g, |F_2| = (2n-g+1)2^g$, and the minimum degree of $Q_n^k - F_2$ is not less than $2n-2$, i.e., F_1 and F_2 are two g-good-neighbor conditional faulty sets of $V(Q_n^k)$ with $F_1 \leq (2n-g+1)2^g$ and $F_2 \leq (2n-g+1)2^g$. On the other hand, since $V(A) = F_1 \Delta F_2$ and $N_{Q_n^k}(A) = F_1 \subset F_2$, there is no edge $(u, v) \in E(Q_n^k)$ with $u \in V - (F_1 \cup F_2)$ and $v \in (F_1 \triangle F_2)$. By Lemma 4.2 and Definition 2.4.5, the g-good-neighbor conditional diagnosability $t_g(Q_n^k) \leq (2n-g+1)2^g - 1$ under the PMC model. The proof is complete. □

Next, we show that $t_g(Q_n^k)$ is no less than $(2n-g+1)2^g - 1$ for $k \geq 5, n \geq 3$ and $0 \leq g \leq n$.

Theorem 4.4. *Assume that $k \geq 5, n \geq 3$ and $0 \leq g \leq n$. Then the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the PMC model, $t_g(Q_n^k) \geq (2n-g+1)2^g - 1$.*

Proof. By Definition 2.4.6, it is sufficient to show Q_n^k is g-good-neighbor conditional $(2n-g+1)2^g - 1$-diagnosable. By Definition 2.4.5, to prove Q_n^k is g-good-neighbor conditional $(2n-g+1)2^g - 1$-diagnosable, it is equivalent to prove that F_1 and F_2 must be distinguishable for every two distinct g-good-neighbor conditional faulty sets F_1 and F_2 of Q_n^k with $|F_1| \leq (2n-g+1)2^g - 1$ and $|F_2| \leq (2n-g+1)2^g - 1$.

We prove this statement by contradiction. Suppose that there are two distinct g-good-neighbor conditional faulty sets F_1 and F_2 with $|F_1| \leq (2n-g+1)2^g - 1$ and $|F_2| \leq (2n-g+1)2^g - 1$, but (F_1, F_2) is indistinguishable. Now we consider all the possible cases such that F_1 and F_2 are indistinguishable. By Theorem 4.1, there are two cases such that F_1 and F_2 are indistinguishable: $V(Q_n^k) = F_1 \cup F_2$ or $V(Q_n^k) \neq F_1 \cup F_2$ but there is no edge from $V(Q_n^k) - (F_1 \cup F_2)$ to $F_1 \Delta F_2$. Without loss of generality, assume that $F_2 - F_1 \neq \emptyset$. We shall show that each of the above cases has a contradiction with our assumption.

Case 1. $V(Q_n^k) = F_1 \cup F_2$.

Since $k \geq 5, n \geq 3, g \leq n$ and $V(Q_n^k) = F_1 \cup F_2$, we deduce that

$$5^n \leq k^n = |V(Q_n^k)| = |F_1 \cup F_2| \leq |F_1| + |F_2| \leq 2 \times [(2n-g+1)2^g - 1] < (n+1)2^{n+1},$$

which is a contradiction.

Case 2. $V(Q_n^k) \neq F_1 \cup F_2$.

Since F_1 and F_2 are indistinguishable, by Theorem 4.1 (See Fig. 3), there are no edges between $V(Q_n^k) - (F_1 \cup F_2)$ to $F_1 \Delta F_2$. By the assumption that $F_2 - F_1 \neq \emptyset$ and F_1 is a g-good-neighbor conditional faulty set, any vertex in $F_2 - F_1$ has at least g good neighbors in the subgraph induced by $F_2 - F_1$. By Lemma 3.3, we have $|F_2 - F_1| \geq 2^g$. Since F_1 and F_2 are both g-good-neighbor conditional faulty sets, $F_1 \cap F_2$ is also a g-good-neighbor conditional faulty set. In addition, since there are no edges between $V(Q_n^k) - (F_1 \cup F_2)$ to $F_1 \Delta F_2$, $Q_n^k - F_1 \cap F_2$ is disconnected, that is $F_1 \cap F_2$ is a R_g-cut of Q_n^k. By Theorem 3.9, the cardinality of the minimum R_g-cut of Q_n^k is $(2n-g)2^g$. Thus, we obtain that $|F_1 \cap F_2| \geq (2n-g)2^g$. As a result, $|F_2| = |F_2 - F_1| + |F_1 \cap F_2| \geq 2^g + (2n-g)2^g$, which contradicts with $|F_2| \leq 2^g + (2n-g)2^g - 1$.

Based on these discussions above, we conclude that $t_g(Q_n^k) \geq (2n-g+1)2^g - 1$. The proof of this theorem is complete. □

Recently, the g-good-neighbor conditional diagnosability of hypercube $t_g(Q_n)$ under the PMC model is shown by Peng et al. [36].

Theorem 4.5 [36]. *The g-good-neighbor conditional diagnosability of Q_n under the PMC model is*

$$t_g(Q_n) = \begin{cases} 2^g(n-g+1) - 1, & \text{if } g \leq n-3; \\ 2^{n-1} - 1, & \text{if } n-2 \leq g \leq n-1. \end{cases}$$

Note that $Q_n^4 = Q_{2n}$. So, by Theorem 4.5, we can deduce the following result.

Corollary 4.6. *The g-good-neighbor conditional diagnosability of Q_n^4 under the PMC model is*

$$t_g(Q_n^4) = \begin{cases} 2^g(2n-g+1) - 1, & \text{if } g \leq 2n-3; \\ 2^{2n-1} - 1, & \text{if } 2n-2 \leq g \leq 2n-1. \end{cases}$$

Combining Theorems 4.3, 4.4 and Corollary 4.6, we can obtain the g-good-neighbor conditional diagnosability of Q_n^k for $0 \leq g \leq n, n \geq 3$ and $k \geq 4$.

Theorem 4.7. *Assume that $k \geq 4, n \geq 3$ and $0 \leq g \leq n$. Then the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k, under the PMC model, $t_g(Q_n^k) = (2n-g+1)2^g - 1$.*

5 The g-Good-Neighbor Conditional Diagnosability of k-Ary n-Cube Q_n^k under the MM* Model

In this section, we shall show the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the MM* model.

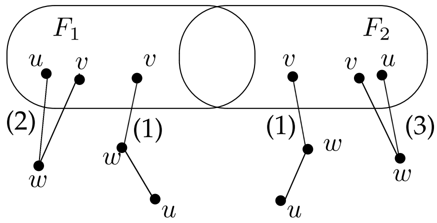


Fig. 5. Illustration of a distinguishable pair (F_1, F_2) under the MM* model.

Let $G = (V, E)$ be an undirected graph of a system G. In 1992, Dahbura and Masson [39] proposed a sufficient and necessary condition for two distinct subsets F_1 and F_2 to be a distinguishable pair under the MM* model.

Theorem 5.1 [39]. *For any two distinct subsets F_1 and F_2 of $V(G)$, (F_1, F_2) is a distinguishable pair under the MM* model if and only if one of the following conditions is satisfied (see Fig. 5):*

(1) There are two vertices $u, w \in V(G) - F_1 - F_2$ and there is a vertex $v \in F_1 \Delta F_2$ such that (u, w) and $(v, w) \in E$.

(2) There are two vertices $u, v \in F_1 - F_2$ and there is a vertex $w \in V(G) - F_1 - F_2$ such that (u, w) and $(v, w) \in E$.

(3) There are two vertices $u, v \in F_2 - F_1$ and there is a vertex $w \in V(G) - F_1 - F_2$ such that (u, w) and $(v, w) \in E$.

The following lemma follows from Definiton 2.4.5 and Theorem 5.1.

Lemma 5.2. *A system G is g-good-neighbor conditional t-diagnosable under the MM* model if and only if for each distinct pair of g-good-neighbor conditional faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions:*

(1) There are two vertices $u, w \in V(G) - F_1 - F_2$ and there is a vertex $v \in F_1 \Delta F_2$ such that (u, w) and $(v, w) \in E$.

(2) There are two vertices $u, v \in F_1 - F_2$ and there is a vertex $w \in V(G) - F_1 - F_2$ such that (u, w) and $(v, w) \in E$.

(3) There are two vertices $u, v \in F_2 - F_1$ and there is a vertex $w \in V(G) - F_1 - F_2$ such that (u, w) and $(v, w) \in E$.

Let g be a positive integer with $0 \leq g \leq n$. To find the g-good-neighbor conditional diagnosability $t_g(Q_n^k)$ under the MM* model, we first show that $t_g(Q_n^k)$ is no more than $(2n-g+1)2^g - 1$ for $k \geq 5, n \geq 3$ and $0 \leq g \leq n$.

Theorem 5.3 [45]. *Let $g \leq n, n \geq 3$, Q_g be a g-dimensional subcube of Q_n, $C_{Q_n}(Q_g) = N_{Q_n}(Q_g) \cup V(Q_g)$. Then $Q_n[N_{Q_n}(Q_g)]$ is the union of $n-g$ disjoint g-dimensional subcubes of Q_n and $Q_n - C_{Q_n}(Q_g)$ is connected and the minimum degree of $Q_n - C_{Q_n}(Q_g)$ is not less than $n-2$.*

Corollary 5.4. *Let the g-dimensional hypercube Q_g be a subgraph of Q_n^k, and $C_{Q_n^k}(Q_g) = N_{Q_n^k}(Q_g) \cup V(Q_g)$, where $0 \leq g \leq n$, $k \geq 4$ and $n \geq 3$. Then $|N_{Q_n^k}(Q_g)| = (2n-g)2^g$ and the minimum degree of $Q_n^k - C_{Q_n^k}(Q_g)$ is not less than $2n-2$.*

Proof. When $k = 4$, then by Theorem 5.3, we have $|N_{Q_n^4}(Q_g)| = |N_{Q_{2n}}(Q_g)| = (2n-g)2^g$ and the minimum degree of $Q_n^4 - C_{Q_n^4}(Q_g) = Q_{2n} - C_{Q_{2n}}(Q_g)$ is not less than $2n-2$. When $k \geq 5$, by Lemma 3.1, the result is also true. □

Lemma 5.5. *Assume that $k \geq 4, n \geq 3$ and $0 \leq g \leq n$. Then the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the MM* model, $t_g(Q_n^k) \leq (2n-g+1)2^g - 1$.*

Proof. Let $A \cong Q_g$ be a subgraph of Q_n^k and let $F_1 = N_{Q_n^k}(A), F_2 = C_{Q_n^k}(A)$ (see Fig. 4). Then by Corollary 5.4, $|F_1| = (2n-g)2^g, |F_2| = (2n-g+1)2^g$, and the minimum degree of $Q_n^k - F_2$ is not less than $2n-2$. That is F_1 and F_2 are two g-good-neighbor conditional faulty sets of $V(Q_n^k)$ with $|F_1| \leq (2n-g+1)2^g$ and $|F_2| \leq (2n-g+1)2^g$. On the other hand, by the definitions of F_1 and F_2, neither one of the three conditions of Lemma 5.2 is satisfied. By Lemma 5,2, k-ary n-cube Q_n^k is not g-good-neighbor conditional $(2n-g+1)2^g$-diagnosable. The proof is complete. □

Next, we show that under the MM* model, $t_g(Q_n^k)$ is no less than $(2n-g+1)2^g - 1$ for $k \geq 4, n \geq 3$ and $0 \leq g \leq n$.

Lemma 5.6. *Assume that $k \geq 4, n \geq 3$ and $0 \leq g \leq n$. Then the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the MM* model, $t_g(Q_n^k) \geq (2n-g+1)2^g - 1$.*

Proof. Suppose, by contradiction, that $t_g(Q_n^k)$ is less than $(2n-g+1)2^g - 1$ under the MM* model. By Lemma 5.2, there are two distinct g-good-neighbor conditional faulty sets F_1 and F_2 with $|F_1| \leq (2n-g+1)2^g - 1$ and $|F_2| \leq (2n-g+1)2^g - 1$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 5.2. Without loss of generality, assume that $F_2 - F_1 \neq \emptyset$. We shall discuss the following two cases.

Case 1. $V(Q_n^k) = F_1 \cup F_2$.

Since $k \geq 4, n \geq 3, g \leq n$ and $V(Q_n^k) = F_1 \cup F_2$, we obtain the following inequality:

$$\begin{aligned} 4^n \leq k^n = \left|V\left(Q_n^k\right)\right| &= |F_1 \cup F_2| \leq |F_1| + |F_2| \\ &\leq 2 \times [(2n-g+1)2^g - 1] \leq (n+1)2^{n+1} - 2 \\ &\leq 4^n - 2, \end{aligned}$$

which is a contradiction.

Case 2. $V(Q_n^k) \neq F_1 \cup F_2$.

In this case, we first prove a useful claim.

Claim 3. $Q_n^k - F_1 - F_2$ has no isolated vertex.

We show the claim by considering the following two subcases.

Subcase A. $g = 1$.

Suppose, by contradiction, that $Q_n^k - F_1 - F_2$ has at least one isolated vertex. Let w be an isolated vertex in $Q_n^k - F_1 - F_2$. Since F_1 is an 1-good neighbor condition faulty set, there is a vertex $u \in F_2 - F_1$ such that u is adjacent to w. On the other hand, since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 5.2, by Lemma 5.2(3) (see Fig. 5), there is at most one vertex $u \in F_2 - F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 - F_1$ such that u is adjacent to w. Similarly, we can deduce there is just a vertex $v \in F_1 - F_2$ such that v is adjacent to w. Let $W \subseteq V(Q_n^k) - F_1 - F_2$ be the set of isolated vertices, and let H be the induced subgraph by the vertex set $V(Q_n^k) - F_1 - F_2 - W$. Then for any vertex $w \in W$, there are $2n-2$ neighbors in $F_1 \cap F_2$. By $|F_2| \leq (2n-g+1)2^g - 1$ and $g = 1$, we have $|F_2| \leq 4n - 1$. Therefore,

$$\begin{aligned} \sum_{w \in W} |N_{F_1 \cap F_2}(w)| = |W|(2n-2) &\leq \sum_{v \in F_1 \cap F_2} d_{Q_n^k}(v) \\ &\leq |F_1 \cap F_2| 2n \leq (|F_2| - 1)2n \\ &\leq (4n-2)2n. \end{aligned}$$

It follows that $|W| \leq 4n + 3$. Assume $H = \emptyset$. Then

$$\begin{aligned} 4^n \leq k^n = \left|V(Q_n^k)\right| &= |F_1 \cup F_2| + |W| \\ &\leq |F_1| + |F_2| - |F_1 \cap F_2| + |W| \\ &\leq 2[(4n-1) - 1] + 4n + 3 = 12n - 1. \end{aligned}$$

It follows that $n < 3$, contradicts $n \geq 3$. So $H \neq \emptyset$. Since the vertex set pair (F_1, F_2) is not satisfied with the condition (1) of Lemma 5.2 (see Fig. 5), and any vertex of $V(H)$ is not isolated, we have there is no edge between H and $F_1 \Delta F_2$. Thus, $F_1 \cap F_2$ is a cut of Q_n^k and $\delta(Q_n^k - (F_1 \cap F_2)) \geq 1$, i.e., $F_1 \cap F_2$ is an 1-good-neighbor conditional cut of Q_n^k. By Theorem 3.9, $|F_1 \cap F_2| \geq 4n - 2$. Note that $|F_1| \leq (2n-g+1)2^g - 1 = 4n - 1, |F_2| \leq (2n-g+1)2^g - 1 = 4n - 1$ and neither $F_1 - F_2$ nor $F_2 - F_1$ is empty. Thus, $|F_1 - F_2| = |F_2 - F_1| = 1$. Say $F_1 - F_2 = \{v_1\}$, $F_2 - F_1 = \{v_2\}$. Then for any vertex $w \in W$, w are adjacent to v_1 and v_2. Since there are at most two common neighbors for any pair of vertices in Q_n^k, it follows that there are at most two isolated vertices in $V - F_1 - F_2$.

Assume there is exactly one isolated vertex v. Then v_1, v_2 are adjacent to v in $V - F_1 - F_2$. Clearly, $N_{Q_n^k}(v) - \{v_1, v_2\} \subseteq F_1 \cap F_2$. Since Q_n^k contains no triangle, it follows that $N_{Q_n^k}(v_1) - \{v\} \subseteq F_1 \cap F_2, N_{Q_n^k}(v_2) - \{v\} \subseteq F_1 \cap F_2$, $[N_{Q_n^k}(v) - \{v_1, v_2\}] \cap [N_{Q_n^k}(v_1) - \{v\}] = \emptyset$, and $[N_{Q_n^k}(v) - \{v_1, v_2\}] \cap [N_{Q_n^k}(v_2) - \{v\}] = \emptyset$. Since there are at most two common neighbors for any pair of vertices in Q_n^k, it follows that $|[N_{Q_n^k}(v_1) - \{v\}] \cap [N_{Q_n^k}(v_2) - \{v\}]| \leq 1$. Therefore,

$$\begin{aligned} |F_1 \cap F_2| &\geq \left|N_{Q_n^k}(v) - \{v_1, v_2\}\right| + \left|N_{Q_n^k}(v_1) - \{v\}\right| \\ &\quad + \left|N_{Q_n^k}(v_2) - \{v\}\right| - 1 \\ &= 2n - 2 + 2n - 1 + 2n - 1 - 1 \\ &= 6n - 5. \end{aligned}$$

It follows that

$$\begin{aligned} |F_2| = |F_2 - F_1| + |F_1 \cap F_2| &\geq 1 + 6n - 5 = 6n - 4 \\ &> 4n - 1, \end{aligned}$$

contradicting $|F_2| \leq (2n-g+1)2^g - 1 = 4n - 1$.

Assume there is another isolated vertex $v' \neq v$ in $V - F_1 - F_2$. Then v_1, v_2 are adjacent to v'. Similarly, since Q_n^k contains no triangle and there are at most two common neighbors for any pair of vertices in Q_n^k, it follows that the four vertex sets $N_{Q_n^k}(v) - \{v_1, v_2\}$, $N_{Q_n^k}(v') - \{v_1, v_2\}, N_{Q_n^k}(v_1) - \{v, v'\}$ and $N_{Q_n^k}(v_2) - \{v, v'\}$ do not intersect pairwise. Therefore,

$$\begin{aligned} |F_1 \cap F_2| &\geq \left|N_{Q_n^k}(v) - \{v_1, v_2\}\right| + \left|N_{Q_n^k}(v') - \{v_1, v_2\}\right| \\ &\quad + \left|N_{Q_n^k}(v_1) - \{v, v'\}\right| + \left|N_{Q_n^k}(v_2) - \{v, v'\}\right| \\ &= 2n - 2 + 2n - 2 + 2n - 2 + 2n - 2 \\ &= 8n - 8. \end{aligned}$$

It follows that

$$|F_2| = |F_2 - F_1| + |F_1 \cap F_2| \geq 1 + 8n - 8 \\ = 8n - 7 > 4n - 1,$$

a contradiction.

Subcase B. $g \geq 2$.

Since F_1 is a g-good-neighbor condition faulty set, $|N_{Q_n^k - F_1}(x)| \geq g$ for any $x \in V - F_1$. Note the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 5.2. By Lemma 5.2(3) (see Fig. 5), for any pair of vertices $u, v \in F_2 - F_1$, there is no vertex $w \in V - F_1 - F_2$ such that $(u, w), (v, w) \in E(Q_n^k)$. Thus, any vertex w in $V - F_1 - F_2$ has at most one neighbor in $F_2 - F_1$. Therefore, for any vertex $w \in V - F_1 - F_2$, $|N_{Q_n^k - F_1 - F_2}(w)| \geq g - 1 \geq 1$, i.e., every vertex of $Q_n^k - F_1 - F_2$ is not an isolated vertex. The proof of Claim 3 is complete.

Let u be a vertex in $Q_n^k - F_1 - F_2$. By Claim 3, u has at least one neighbor in $Q_n^k - F_1 - F_2$. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 5.2, by Lemma 5.2(1) (see Fig. 5), for any pair of adjacent vertices $u, w \in V - F_1 - F_2$, there is no vertex $v \in F_1 \Delta F_2$ such that (u, v) or $(v, w) \in E(Q_n^k)$. It follows that u has no neighbor in $F_1 \Delta F_2$. By the arbitrariness of u, there is no edge between $V - F_1 - F_2$ and $F_1 \Delta F_2$.

Since $F_2 - F_1 \neq \emptyset$ and F_1 is a g-good-neighbor conditional faulty set, $\delta(Q_n^k[F_2 - F_1]) \geq g$. By Lemma 3.3, $|F_2 - F_1| \geq 2^g$. Since both F_1 and F_2 are g-good-neighbor conditional faulty sets and there is no edge between $V - F_1 - F_2$ and $F_1 \Delta F_2$, $F_1 \cap F_2$ is a g-good-neighbor conditional cut of Q_n^k. By Theorem 3.9, we have $|F_1 \cap F_2| \geq (2n - g)2^g$.

Therefore,

$$|F_2| = |F_2 - F_1| + |F_1 \cap F_2| \geq 2^g + (2n - g)2^g \\ = (2n - g + 1)2^g,$$

contradicting $|F_2| \leq (2n - g + 1)2^g - 1$ The proof is complete. □

Combining Lemmas 5.5 and 5.6, the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k shows below.

Theorem 5.7. *Assume that* $k \geq 4, n \geq 3$ *and* $0 \leq g \leq n$. *Then the g-good-neighbor conditional diagnosability of k-ary n-cube* Q_n^k *under the MM* model,* $t_g(Q_n^k) = (2n - g + 1)2^g - 1$.

Proof. On the one hand, by Lemma 5.5, the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the MM* model, $t_g(Q_n^k) \leq (2n - g + 1)2^g - 1$. On the other hand, by Lemma 5.6, the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the MM* model, $t_g(Q_n^k) \geq (2n - g + 1)2^g - 1$. Therefore, the g-good-neighbor conditional diagnosability of k-ary n-cube Q_n^k under the MM* model, $t_g(Q_n^k) = (2n - g + 1)2^g - 1$. The proof is complete. □

Table 1 shows the g-good-neighbor conditional diagnosability of five-ary n-cube $t_g(Q_n^5)$ of small $n(\geq 3)$ where $0 \leq g \leq n$.

TABLE 1
The $t_g(Q_n^5)$ of Small n

n	g	$\lvert V(Q_n^5)\rvert$	$t_g(Q_n^5)$	ratio
3	0	5^3	6	0.045
3	1	5^3	11	0.088
3	2	5^3	19	0.152
3	3	5^3	31	0.248
4	0	5^4	8	0.0128
4	1	5^4	17	0.0272
4	2	5^4	27	0.0432
4	3	5^4	47	0.0752
4	4	5^4	79	0.1264
5	0	5^5	10	0.0032
5	1	5^5	19	0.00608
5	2	5^5	35	0.0112
5	3	5^5	63	0.02016
5	4	5^5	111	0.03552
5	5	5^5	191	0.06112
6	0	5^6	12	0.000768
6	1	5^6	23	0.001472
6	2	5^6	43	0.002752
6	3	5^6	79	0.005056
6	4	5^6	143	0.009152
6	5	5^6	255	0.01632
6	6	5^6	447	0.0286
7	0	5^7	14	0.0001792
7	1	5^7	27	0.0003456
7	2	5^7	51	0.0006258
7	3	5^7	95	0.001216
7	4	5^7	175	0.00224
7	5	5^7	319	0.0040832
7	6	5^7	575	0.00736
7	7	5^7	1023	0.0130944

6 Conclusions

The g-good-neighbor conditional diagnosability can measure diagnosability for a large-scale processing system more accurately than classical diagnosability because the classical diagnosability always assumes that all neighbors of each processor in a system can potentially fail at the same time regardless of the probability. In fact, if there are exactly n faulty processors in a system of minimum degree n, however, the probability of the faulty set containing all the neighbors of any vertex is statistically low for large multiprocessor systems. Therefore, it is worthy to the determining the g-good-neighbor conditional diagnosability of interconnection network for multiprocessor systems.

In the area of diagnosability, the PMC model and the MM* model are two well-known and widely chosen fault diagnosis models. In this paper, we study the g-good-neighbor conditional diagnosability of k-ary n-cube under the these models, and demonstrate the g-good-neighbor conditional diagnosabilitis of k-ary n-cube Q_n^k under the PMC model and MM* model are both $(2n - g + 1)2^g - 1$ for $k \geq 4, n \geq 3$ and $0 \leq g \leq n$. Observing that when $g = 0$, there is no restriction on the faulty sets and we have the traditional diagnosability on the hypercube as n. In addition, in the special case of $g = 1$, our result is slightly different from the measure of conditional diagnosability given by Lai et al. [28]. The difference between these two measures is that we only consider the condition of the fault-free vertices in the network. The generalization of conditional diagnosability by requiring every vertex to have at least g good neighbors is also an interesting problem to investigate in the future. For further discussion, it is an attractive work to develop different measures of these conditional diagnosabilities based on application environment, network topology, network reliability, and statistics related to fault patterns.

ACKNOWLEDGMENTS

This work was supported by the the National Natural Science Foundation of China (11026163, 61302020, 61070229, 61272263) and the Natural Science Foundation of Shanxi Province (2011021004). Xiao Qin's research was supported by the US National Science Foundation (NSF) under Grants CCF-0845257 (CAREER), CNS-0917137 (CSR), CNS-0757778 (CSR), CCF-0742187 (CPA), CNS-0831502 (CyberTrust), CNS-0855251 (CRI), OCI-0753305 (CITEAM), DUE-0837341 (CCLI), and DUE-0830831 (SFS). Jun Yuan is the correpond-ing author.

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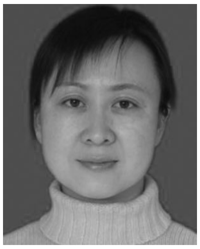
Jun Yuan received the BS, MS, and PhD degrees from Shanxi University, China, in 2002, 2005, and 2008, respectively, all in mathematics. Currently, he is an associate professor in the School of Applied Sciences, Taiyuan University of Science and Technology. His research interests include graph theory and interconnection networks, computational complexity, and algorithms.



Aixia Liu received the BS degree in mathematics from Taiyuan Normal University, China, in 2002, and the MS degree in mathematics from Shanxi University, China, in 2006. Currently, she is a faculty member of the School of Applied Sciences, Taiyuan University of Science and Technology. Her research interests include graph theory and interconnection networks, and parallel and distributed systems.

Xue Ma received the BS degree in mathematics from Tangshan Normal University, China, in 2011, and the MS degree in mathematics from the Taiyuan University of Science and Technology, China, in 2014. Her research interests include graph theory and interconnection networks, and parallel and distributed systems.

Xiuli Liu received the BS degree in geographic information system from Harbin Normal University, China, in 2012. Currently, she is working toward the postgraduate degree at the Taiyuan University of Science and Technology. Her research interests include complex networks system and geographic information system.

Xiao Qin received the BS and MS degrees in computer science from the HUST, China, and the PhD degree in computer science from the University of Nebraska-Lincoln in 1992, 1999, and 2004, respectively. Currently, he is an associate professor in the Department of Computer Science and Software Engineering, Auburn University. His research interests include parallel and distributed systems, storage systems, fault tolerance, real-time systems, and performance evaluation. He received the US National Science Foundation (NSF) Computing Processes and Artifacts Award, the NSF Computer System Research Award in 2007, and the NSF CAREER Award in 2009. He is a senior member of the IEEE.

Jifu Zhang received the BS and MS degrees in computer science and technology from the Hefei University of Technology, China, and the PhD degree in pattern recognition and intelligence systems from the Beijing Institute of Technology, in 1983, 1989, and 2005, respectively. He is currently a professor in the School of Computer Science and Technology at TYUST. His research interests include data mining and artificial intelligence, and parallel and distributed systems.