A Fast Greedy Algorithm for Routing Concurrent Video Flows

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Abstract— **Real-time multimedia communication is an important service that should be supported in wireless ad hoc networks. In this paper, we consider the problem of how to optimally support multiple concurrent video communication sessions in a wireless ad hoc network. Our problem formulation follows an *application-centric cross-layer approach* with the objective of minimizing the average distortion of all video sessions via finding optimal paths for each session. Since this network-wide optimization problem is shown to be NP-complete, we pursue to develop competitive heuristic algorithms to address this problem. Specifically, we describe a greedy algorithm based on the key characteristics of the end-to-end video distortion model, and use numerical results to demonstrate its performance as compared to the global optima. This greedy heuristic algorithm can be used to quickly compute a set of good paths for the video sessions. It can also be used to speed up the Genetic Algorithm-based algorithm proposed in our previous work.**

I. Introduction

With the recent advances in digital video technology and wireless ad hoc networking, there is a compelling need to support real-time multimedia communications in ad hoc networks, in addition to simple data communications. However, at present, there remain significant problems across different layers that need to be addressed in order to successfully deploy such services in ad hoc networks. In particular, issues such as interference, mobility, frequent link failures, and topology changes have all made this research much more challenging. As a result, wireless links in ad hoc networks are much more diverse in terms of quality (e.g., available bandwidth, loss, and delay) than links in wireline networks: any link in an ad hoc network could be highly fragile with dynamic state conditions. Consequently, optimized routing is an important mechanism for enabling multimedia services in wireless ad hoc networks, and it is important to investigate new methodologies for routing multimedia sessions in such networks.

In this paper, we consider the problem of supporting multiple concurrent video sessions in wireless ad hoc networks. This problem is important since it captures the scenario that there are typically more than one real-time multimedia communication sessions sustained by an ad hoc network. These sessions may share the same network resources (e.g., link bandwidth, buffer) and might interact with each other (i.e., MAC layer contentions or Physical layer interference). In our previous work [1], we formulated a combinatorial optimization problem from a cross-layer perspective by considering the application layer performance metric (i.e., average video distortion) as a function of network layer behavior (routing of each session). In particular, our constraints at the network layer address not only packet losses due to frequent node/link failures, which are unique to ad hoc networks, but also traditional network problems such as delays due to congestion. In other words, our application-centric problem formulation seamlessly unifies video distortion with packet loss (due to node/link failures) and delay via routing for each session.

It should be clear that the problem addressed in this paper differs from the network-layer QoS-routing problems for ad hoc networks [2]–[6]. In these efforts, the focus has been on addressing *network-layer* routing problems from various perspectives (e.g., associativity [2] of wireless links, differentiated link state updates [3], selecting node or edge disjoint paths [6], and end-to-end resource guarantees [4], [5]). In contrast to the present paper, most of these prior efforts do not explicitly formulate the objective function with an application layer metric via a cross-layer approach. Consequently, although these approaches could obtain optimal network layer performance, they may not yield optimal performance at the application layer.

The formulated problem exhibits a highly complex objective function and constraints, which renders this problem substantially more difficult than traditional network-centric (single network layer) QoS routing problems. In [1], we presented a Genetic Algorithm (GA)-based algorithm [7] for solving this cross-layer multimedia-centric routing problem. The GA-based algorithm produces near-optimal solutions, but with a relatively high computational complexity. In this paper, we describe an efficient greedy heuristic algorithm for the cross-layer routing problem. The proposed algorithm is based on the observation of the key characteristics of the video distortion model. It computes low loss and low congestion paths for the video sessions using an empirical compound routing metric. The computational complexity of the greedy algorithm is extremely low. We also show that the greedy algorithm gives very good solutions for practical range of network conditions through extensive simulation studies.

The remainder of this paper is organized as follows. In Section II, we present the network-wide optimal routing problem formulation for completeness. In Section III, we propose a fast greedy heuristic algorithm for the concurrent routing problem. We use extensive numerical results to demonstrate the efficacy of the greedy heuristic algorithm in Section IV. Section V concludes the paper.

II. Problem Formulation

A. Network Model

In order to model a wireless ad hoc network as an associated graph, we assume that a wireless link exists between nodes i and j if nodes i and j can communicate with each other. For example, a link may exist if nodes i and j are within reachable distance of their radio transmitter. Consequently, the wireless ad hoc network can be modeled as a time-varying directed graph $\mathcal{G}(\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of vertices, representing mobile nodes, and $\mathcal{L}$ is the set of wireless links in the network. In the graph, we characterize each wireless (directed) link $\{i,j\} \in \mathcal{L}$ by the following two parameters:

- c_{ij}: The capacity, or available bandwidth of link $\{i,j\}$;
- p_{ij}: The mean packet loss probability of link $\{i,j\}$, due to transmission errors or link failures.

Other characterizations of a wireless link in ad hoc networks can be incorporated into our model as well, although we intend to explore these in our future effort.

In this network, we assume that there exists a set of concurrent video sessions, denoted by $\mathcal{S}$. Each session $\sigma \in \mathcal{S}$ has a source node s_σ and a destination node d_σ. The rate of a video stream, R_σ, is bounded as $\underline{R}_\sigma \le R_\sigma \le \overline{R}_\sigma, \forall \sigma \in \mathcal{S}$. The lower and upper bounds of R_σ are determined by the specific video encoder and video sequence used at the source node s_σ, $\forall \sigma \in \mathcal{S}$. Our objective is to find optimal paths for the concurrent sessions such that the overall video distortion is minimized.

B. Video Rate-Distortion Modeling

For video coding and communications, a rate distortion model describes the relationship between the bit rate and the corresponding video distortion achieved. In the following, we introduce an empirical rate-distortion model that links the packet overdue and loss probabilities to video distortion, which is an important application layer video quality measure.

In [8], Stuhlmuller *et al.* developed an empirical rate-distortion model for a hybrid motion compensated video encoder. For a video sequence encoded at a target coding rate R_σ, the average end-to-end distortion D^e_σ consists of the encoding distortion caused by the lossy video coder, D^{enc}_σ, and the distortion due to transmission errors, which is the sum of the distortion caused by overdue video packets (i.e., packets experiencing large delay due to congestion in the network), D^{cg}_σ, and the distortion caused by lost video packets (i.e., due to link failure or other transmission errors), D^{ls}_σ. That is,

$$D^e_\sigma = D^{enc}_\sigma + D^{cg}_\sigma + D^{ls}_\sigma. \tag{1}$$

From [8] and [1], we have

$$D^e_\sigma = D_0 + \frac{\omega}{R_\sigma - R_0} + \kappa(1-p_\sigma)Pr(T_\sigma > \Delta_\sigma) + \kappa p_\sigma, \tag{2}$$

where D_0, ω, R_0, and κ are constants for a specific video codec (with fixed encoding parameters) and video sequence, which can be determined by training and curve matching. Since the model in (1) takes into account the effects of INTRA coding and spatial loop filtering, it matches simulation results closely [8].

For a given set of paths $\bar{x}$, the end-to-end packet loss probability p_σ and the end-to-end packet overdue probability $Pr(T_\sigma > \Delta)$ for a session $\sigma \in \mathcal{S}$ is determined by the corresponding link parameters p_{ij} and c_{ij}, $\{i,j\} \in \bar{x}$, as well as the correlation of the paths (i.e., jointness or disjointedness). We showed that these two parameters can be computed based on the Chernoff bound. Interested readers can refer to our previous work [1] for details.

C. The Optimal Routing Problem

For delineating an end-to-end path $\mathcal{P}_\sigma$ from s_σ to d_σ, $\sigma \in \mathcal{S}$, we define the following index variables:

$$x^\sigma_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \in \mathcal{P}_\sigma, \quad \forall\ \{i,j\} \in \mathcal{L} \\ 0, & \text{otherwise}, \qquad \forall\ \{i,j\} \in \mathcal{L}. \end{cases} \tag{3}$$

Using such index variables, the choice of path $\mathcal{P}_\sigma$ can be represented by a routing vector $\mathbf{X}_\sigma = \{x^\sigma_{ij}\}_{\{i,j\}\in\mathcal{L}}$ having $|\mathcal{L}|$ elements, each of which corresponds to a link and has a binary value.

We are now ready to mathematically formulate the problem of application-centric optimal routing for multiple concurrent video sessions:

OPT-CLR

Minimize: $D = \sum_{\sigma\in\mathcal{S}} D^e_\sigma$ (4)

subject to:

$$\underline{R}_\sigma \le R_\sigma \le \overline{R}_\sigma, \text{ for } \sigma \in \mathcal{S} \tag{5}$$

$$\rho_{ij} \le 1-\epsilon, \quad \forall\{i,j\} \in \mathcal{L}, \text{for some stability tolerance } \epsilon \tag{6}$$

$$\sum_{j:\{i,j\}\in\mathcal{L}} x^\sigma_{ij} - \sum_{k:\{k,i\}\in\mathcal{L}} x^\sigma_{ki} = \begin{cases} 1, & \text{if } i = s_\sigma \\ -1, & \text{if } i = d_\sigma \\ 0, & \text{otherwise} \end{cases}, \quad \forall i \in \mathcal{N}, \sigma \in \mathcal{S} \tag{7}$$

$$x^\sigma_{ij} \in \{0,1\}, \quad \forall\{i,j\} \in \mathcal{L}, \sigma \in \mathcal{S}. \tag{8}$$

We now provide an interpretation for the above problem formulation. The objective function (4) is the sum of the average distortion of all the concurrent video sessions. Minimizing (4) achieves the best overall quality for the video sessions, as well as the best utilization of network resources. The set of inequalities in (5) gives the range of feasible rates for each video stream, which is determined by the video sequence and encoder parameters. The inequality in (6) is the stability condition, which ensures that the link utilization (i.e., the ratio of the average aggregate traffic load on the link, and the link capacity) is less than 1. The remaining constraint (7) guarantees that each path $\mathcal{P}_\sigma$ from s_σ to d_σ is a valid path and is loop-free.[1]

In Problem OPT-CLR, there are two sets of tunable variables that form the search (optimization) space of feasible solutions. They are (i) the set of routing vectors: $\{\mathbf{X}_\sigma\}_{\sigma\in\mathcal{S}}$; and (ii) the set of rates of video sessions: $\{R_\sigma\}_{\sigma\in\mathcal{S}}$. The objective function (4) is a highly complex ratio of high-order polynomials of the x-variables. The objective evaluation of a set of feasible paths involves identifying the joint and disjoint links of the paths (in order to compute the traffic load on each link), which is only possible when all the paths are completely determined. Wang and Crowcroft [9] proved that QoS routing problems having multiple additive and/or multiplicative metrics are NP-complete. Our problem has an additive delay metric and a multiplicative loss metric. In addition, our problem has much more complex relationships pertaining to the contribution of any link to the objective function, as well as time-varying and coupled session delays (rather than constant link delays as in [9]). As a result, we conjecture that problem OPT-CLR is NP-complete and it is futile to pursue exact solutions. Therefore, efficient heuristic algorithms are desirable for obtaining near-optimal solutions to Problem OPT-CLR.

In [1], we present a Genetic Algorithm (GA)-based approach for OPT-CLR. In the following, we describe a fast greedy heuristic algorithm that also computes highly competitive solutions. Compared with the GA-based algorithm in [1], the greedy heuristic algorithm is faster in computation, but its solutions are generally slightly inferior to the GA solutions. We can use the greedy heuristic solutions for applications having stringent delay constraints, or use it to initialize the GA population in order to speed up the GA convergence.

III. The Greedy Heuristic Algorithm

Before describing the greedy heuristic algorithm, we first examine the total end-to-end distortion D^e_σ of a session $\sigma \in \mathcal{S}$ (see (1) and (2)). The first term D^{enc}_σ, the distortion caused by the encoder is a monotonically decreasing function of video rate R_σ, $\forall\sigma \in \mathcal{S}$. The second term D^{cg}_σ, the distortion caused by congestion, on the other hand, is a monotonically increasing function of the video rate R_σ, as well as the rates of all other sessions R_i, $i \ne \sigma$, that share one or more links with session σ, $\forall i, \sigma \in \mathcal{S}$. Both of these two terms are constrained by the stability constraint (6) and are thus determined by the available bandwidths of the used links. The third term D^{ls}_σ, the distortion caused by lost packets, is simply a function of the link loss probabilities. In order to minimize the video distortion for session σ, we need to find the paths having the highest end-to-end bandwidth, the minimal congestion, and the lowest end-to-end loss rate.

Furthermore, we plot D^{enc}_σ in Figure 1 for an H.263 coder with typical settings (e.g., Intra Rate 1/15, GOP length 15, and frame rate 30 fps) using the Quarter Common Intermediate Format (QCIF) formatted "Foreman" sequence. We observe that the curve is concave: when R_σ increases beyond a certain threshold, further increasing R_σ will only cause marginal reduction in D^{enc}_σ. For example, when R_σ

[1]Note that a feasible solution to these constraints could admit circuits whose edges are disconnected from the produced loop-free paths. However, the objective function would automatically prohibit this occurrence.

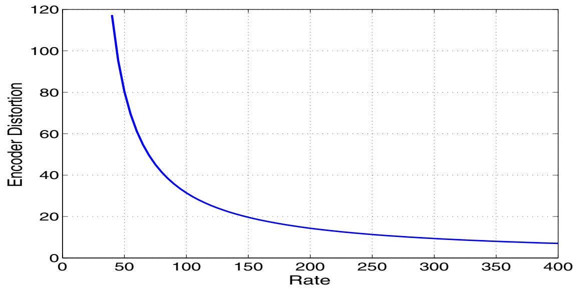

Fig. 1. Rate distortion curve for the video codec, i.e., D_σ^{enc} versus R_σ.

1. Set the cost of each link $\{i,j\}$ to $c_{ij} \cdot (1 - p_{ij})$, $\forall \{i,j\} \in \mathcal{L}$;
2. For every video session $\sigma \in \mathcal{S}$;
3. Use the algorithm in [10] and use the costs defined in 1. to find the path having the maximum end-to-end cost. Let this path be $\mathcal{P}_\sigma$;
4. Decrease the bandwidth of every link on $\mathcal{P}_\sigma$ by $\underline{R}_\sigma$, i.e., setting the link costs as $(c_{ij} - \underline{R}_\sigma) \cdot (1 - p_{ij})$, $\forall \{i,j\} \in \mathcal{P}_\sigma$;
5. After the paths for all sessions are found, apply OPT-Rate to determine the optimal rates for each session.

Fig. 2. GH: A fast greedy heuristic for routing concurrent video flows.

increases from $150Kbps$ to ∞, there is only a decrease of about 20 in D_σ^{enc}. However, such high rate will cause congestion, resulting in a much larger increase in D_σ^{cg}. Therefore, for practical R_σ values, such as within $[100Kbps, 400Kbps]$, reducing congestion conditions in the network would be more effective than increasing video rates in improving overall video quality.

In Figure 2, we describe a greedy heuristic GH for Problem OPT-CLR. In GH, an empirical compound link cost $c_{ij} \cdot (1 - p_{ij})$, which we call the *effective available bandwidth*, is used. For a given path, its end-to-end effective available bandwidth is the minimum among those of its links. By computing the path with the maximum effective available bandwidth, GH finds the currently "widest" path for session σ, which has the potential of supporting higher video rates and having less congestion. Since both link capacity and loss probability are considered in the compound link cost, GH may produce near-optimal solutions to Problem OPT-CLR, as will be shown in Section IV. For each session, the maximum-effective-available-bandwidth path can be found using the algorithm presented in [10], with a time complexity of $O(|\mathcal{L}| \cdot \log^* |\mathcal{N}|)$, where $\log^* n$ is the *iterated logarithm function*. Therefore, the overall time complexity of GH is $O(|\mathcal{S}| \cdot |\mathcal{L}| \cdot \log^* |\mathcal{N}|)$.

Finally, for the set of computed path set $x_k = \{P_\sigma\}_{\sigma \in \mathcal{S}}$ (which potentially has the minimal congestion and path losses), Problem OPT-CLR reduces to the following nonlinear optimization problem OPT-Rate, which further reduces the overall video distortion by finding the optimal video rates for the sessions.

OPT-Rate

$$\textbf{Minimize:} \quad D(x_k) = \sum_{\sigma \in \mathcal{E}} D_\sigma^e \tag{9}$$

subject to:

$$\underline{R}_\sigma \leq R_\sigma \leq \overline{R}_\sigma, \text{ for } \sigma \in \mathcal{S} \tag{10}$$

$$\rho_{ij} \leq 1 - \epsilon, \quad \forall \{i,j\} \in \mathcal{L}, \text{for some stability tolerance } \epsilon. \tag{11}$$

OPT-Rate is a nonlinear optimization problem with nonlinear constraints. It can be efficiently solved using an iterative procedure based on the *Sequential Quadratic Programming (SQP) method* [11].

For the path set found by the above procedure, we have the following proposition holding true.

Proposition 1: The distortion achieved by the path set computed by GH as depicted in Figure 2 is an upper bound of the total distortion D defined in (4).

Proof: From Figure 2, it can be easily verified that the path set is realizable, i.e., it satisfies all the constraints of Problem OPT-CLR. Therefore, the resulting distortion is an upper bound of the optimal distortion which is optimized over all feasible solutions. ∎

IV. Numerical Results

In this section, we present the simulation results for Problem OPT-CLR in order to examine the GH performance. In each experiment, an ad hoc network was generated by placing a number of nodes at random locations in a rectangular region. Each video session had a rate bounded by 100Kbps and 400Kbps. We used an H.263+ codec and the 400-frame "Foreman" trace in the QCIF format. The video was encoded with an Intra Rate of $1/15$, GOP length of 15, and a frame rate of 30 fps. A decoding deadline of 100 ms was used for the results reported in Table I, Figure 3, and Figure 4. The rate-distortion parameters were found to be $D_0 = 0.38$, $R_0 = 18.3$, $\omega = 2537$, and $\kappa = 750$ [8].

A. GH versus Exhaustive Search

First, we examine the optimality of the GH solutions by comparing them with those found by a brute force, exhaustive search (ES) over the entire solution space. For these experiments, link loss probabilities were randomly chosen from [1%, 10%]; link bandwidths were randomly chosen from [100Kbps, 400Kbps]. There were three concurrent video sessions having randomly chosen source and destination nodes. The total distortion values found by GH and ES are presented in Table I for six randomly generated networks with various topologies and parameter settings. We find that the GH solutions are quite competitive. Specifically, for networks III and V, GH actually found the exact optimal paths, yielding the minimum total distortion. In most of the cases, the GH distortions are within 10% of the global optimum, while in the worst case (network IV), the GH distortion is within 31.8% of the global optimum. The corresponding average PSNR value for the sessions is also presented in the table, which is computed as $10 \cdot \log(255^2 / D_\sigma^e)$. Except for network IV, the GH average PSNR is within 0.7 dB of the global optimal. For all the cases, GH terminates in a couple of hundred milliseconds, while ES takes about 20 minutes to run with a Pentium-4 2.4GHz computer (512 MB memory). This clearly illustrates the time efficiency of GH.

TABLE I

Optimality of Solutions Found by GH

Network	I	II	III	IV	V	VI
Network Size	9	9	9	11	11	11
GH Distortion	690.6	229.2	245.3	264.9	345	232.6
ES Distortion	641.8	201.7	245.3	201.0	345	201.6
GH PSNR (dB)	24.5	29.3	29.0	28.7	27.5	29.2
ES PSNR (dB)	24.8	29.9	29.0	29.9	27.5	29.9

B. Impact of Link Statistics

In order to further examine the performance of the proposed greedy heuristic algorithm, we present the impact of link statistics in this subsection. In the following experiments, we used an 11-node ad hoc network with three video sessions and a fixed topology, while varying either the average link loss probabilities or the average link bandwidths. More specifically, we fixed the link bandwidths, while increasing the link loss probabilities in Figure 3; and fixed the link loss probabilities, while increasing the link bandwidths in Figure 4. Then, for each resulting network, we computed the minimal achievable total distortion using an exhaustive search, and then computed the near-optimal routes and the corresponding total distortion using

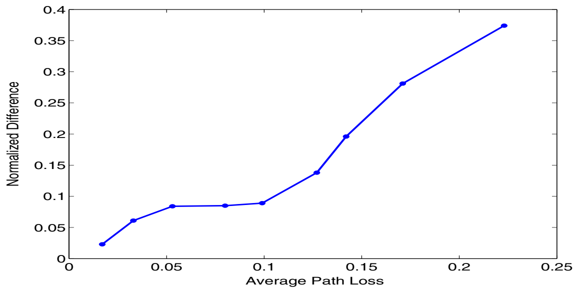

Fig. 3. Impact of link statistics on the GH performance: normalized difference versus average path loss.

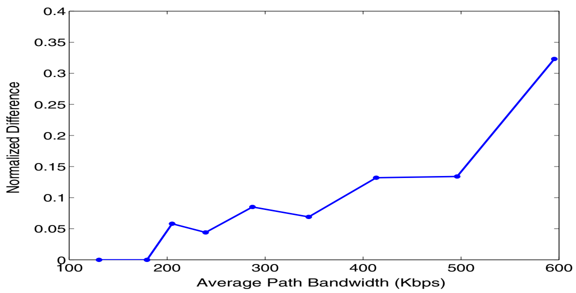

Fig. 4. Impact of link statistics on the GH performance: normalized difference versus average path bandwidth.

GH. The normalized difference (defined to be $(GH - ES)/ES$) versus average path loss probability and average path bandwidth are plotted in Figure 3 and Figure 4, respectively.

From Figure 3, it can be seen that the normalized difference is low (less than 10%) and relatively constant for low path loss probabilities. However, the normalized difference increases over 20% when the average path loss probability exceeds 15%. For such cases, the network can hardly provide satisfactory video quality, due to the large amount of lost packets (note that more packets will be delayed beyond their decoding deadline due to congestion, which further degrade the video quality). Clearly, for cases where the end-to-end loss rates are less than 15% (which are practical cases in general), GH can provide quite competitive paths for concurrent video sessions. We observe similar trend in Figure 4, where the normalized difference gradually increases with the average path bandwidth. This implies that GH is more accurate for bandwidth limited networks where congestion is more persistent. When average link bandwidth gets large, congestion will be rare and the video quality is mainly determined by link loss probabilities. We may deploy a pure loss-based routing algorithm for such cases (e.g., setting link cost to $-\log(1 - p_{ij})$ and then using Dijkstra's algorithm to compute the minimum cost paths).

C. GH versus Shortest Path Routing

We further examined the GH performance using a 50-node network with 10 concurrent video sessions. Since ES was not feasible for such large-sized network, we compared GH with shortest path routing (SP). Specifically, we considered two SP algorithms: (i) SP-Hop: using hop count; and (ii) SP-Loss: using $-\log(1 - p_{ij})$ as routing metric for link $\{i, j\}$ (i.e., finding the minimum loss path). The link bandwidth c_{ij} was randomly chosen from [100Kbps, 1Mbps] and the link loss probability p_{ij} was randomly chosen from [1%, 10%]. The decoding deadline for the video packets was 450 ms.

The distortion values for the sessions computed by the three algorithms are plotted in Figure 5. We find that for many sessions, GH achieves a much lower distortion than the two SP-based algorithms.

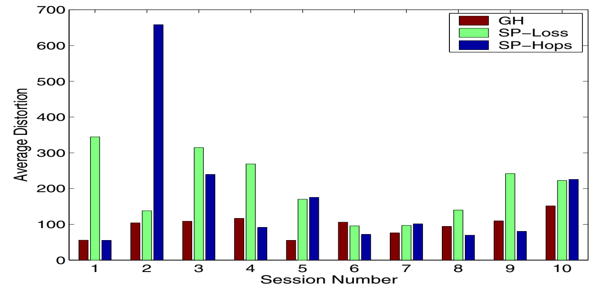

Fig. 5. Distortion values for each video session in a 50-node network obtained by different algorithms.

Although for several sessions, GH distortion is higher that SP-Hop or SP-Loss, the difference is quite small in all of these cases. The total distortion achieved by GH is 976.8, which is much lower that that achieved by SP-Hop (1767.7) and SP-Loss (2032.8). Such gain is due to the effective available bandwidth used in GH routing, which takes into consideration both link capacity and loss probability.

V. CONCLUSIONS

In this paper, we have studied the important problem of supporting multiple concurrent video sessions in wireless ad hoc networks. We have developed a fast greedy heuristic algorithm for the cross-layer optimization problem. Through extensive numerical results, we show that the greedy algorithm performs very well for practical network and video system settings.

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