STOCHASTIC FORMULATIONS OF THE REDUNDANCY ALLOCATION PROBLEM

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ABSTRACT

Previous research on the redundancy allocation problem for series-parallel systems has focused on deterministic versions where component reliability is assumed to be an exact value. This paper discusses alternative formulations which allow system reliability and mean time to failure to be treated as random variables. Stochastic formulations allow for explicit consideration of risk aversion, and thus, are more reflective of actual design problems. A genetic algorithm is developed for these problem formulations and is demonstrated on an illustrative problem.

KEYWORDS

Redundancy Allocation Problem, Genetic Algorithms, Combinatorial Optimization.

INTRODUCTION

Determination of an optimal or near optimal design is very important to economically produce new systems which meet and exceed customers' expectations for reliability, quality and performance. When developing a new system, there are detailed engineering specifications which prescribe minimum levels of reliability, maximum weight, etc. The redundancy allocation problem involves the simultaneous evaluation and selection of components and a system-level design configuration which can collectively meet all design constraints, and at the same time, optimize some objective function, usually system cost or reliability. In this paper, the problem is formulated with the objective function as the lower-bound on system reliability (or system time to failure) and with uncertainty in the component reliability estimates.

There have been many successful attempts to determine optimal design configurations for different formulations of the redundancy allocation problem [1-12]. The majority of optimization methods for this problem use a deterministic value (or the expected value) for system reliability as a measure of system integrity or to determine compliance to a constraint. This implies that either there is no variability associated with reliability estimation, or that system designers and users are risk-neutral, which they are not. The availability of methods, such as those described in this paper, which specifically address variability in system reliability estimation and the associated risk-return trade-offs are beneficial and more realistic. They provide valuable insights to system designers who need to recognize the implications of variability within system designs and reject outdated notions of using conservative "safety factors" as a means to compensate for variability.

PROBLEM FORMULATION

A primary deficiency with existing formulations of the redundancy allocation problem is that component reliability values are assumed to be constant. It is either assumed that these values are not subject to variation (i.e., deterministic) or the analyses are conducted on the basis of the expected value of reliability at both the component and system level. Neither scenario is reasonable for most actual design problems. Reliability of a component (and system) is inherently probabilistic, and system designers and users will invariably be risk-averse and wish to conduct analyses and trade-offs with an associated degree of confidence (1 - α), not the expected value.

For this formulation of the redundancy allocation problem, there are m_i discrete component choices available for each subsystem (i=1,...,s). The m_i components have fixed cost (c_{ij}) and weight (w_{ij}) values. Component reliability (or time-to-failure) is a random variable and is described by a probability density function, $f_{ij}(r)$ (or $f_{ij}(t)$). System reliability is a function of the design configuration and the corresponding component reliabilities, and is therefore, also a random variable. Figure 1 presents a typical example of a series-parallel system with k-out-of-n subsystem reliabilities. For each subsystem i, a minimum of k_i components must be chosen from among the m_i available choices (with replacement). Additionally, the problem involves the consideration of adding more levels of redundancy (> k_i) to improve the system reliability as an alternative to using a more reliable, and more costly, component. It then becomes a combinatorial problem to select the optimal combination of components and levels of redundancy.

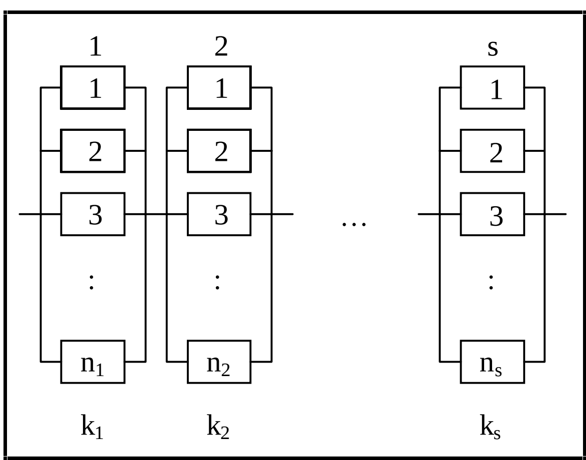

Figure 1: Series-parallel system configuration.

Problem formulations with probabilistic component reliability values and deterministic weight and cost, are as follows:

max $R_{1-\alpha}(\mathbf{x} \mid t')$ (or $T_{1-\alpha}(\mathbf{x})$)

subject to
$$\sum_{i=1}^{s} C_i(\mathbf{x}_i) \le C$$
$$\sum_{i=1}^{s} W_i(\mathbf{x}_i) \le W$$
$$k_i \le \sum_{j=1}^{m_i} x_{ij} \le n_{\max,i} \;\; \forall\, i$$
$$x_{ij} \in (0, 1, 2, ...)$$

where,

$R_{1-\alpha}(\mathbf{x} \mid t') = 100 \times (1 - \alpha)\%$: lower bound on system reliability (for a fixed t′ and specified α)
$T_{1-\alpha} = 100 \times (1 - \alpha)\%$: lower bound on system mean time to failure (for a specified α)
t′ = mission time
$\mathbf{x}$ = solution vector = $(\mathbf{x}_1,\mathbf{x}_2, ..., \mathbf{x}_n)$
$\mathbf{x}_i$ = solution vector for i^{th} subsystem = $(x_{i1},x_{12},...,x_{i,n_i})$
x_{ij} = number of the j^{th} available component used in subsystem i
C = system cost constraint
W = system weight constraint
$C_i(\mathbf{x}_i)$ = cost for subsystem i
$W_i(\mathbf{x}_i)$ = weight for subsystem i

An optimization routine requires the calculation of the lower bound on system reliability for each solution vector, $\mathbf{x}$, encountered. An interesting result is that, for non-trivial problems, the lower bound on system reliability depends only on the mean and the variance of the component reliabilities, and is independent of the specific distribution [13]. Pictorially, a lower bound on system reliability is presented in Figure 2.

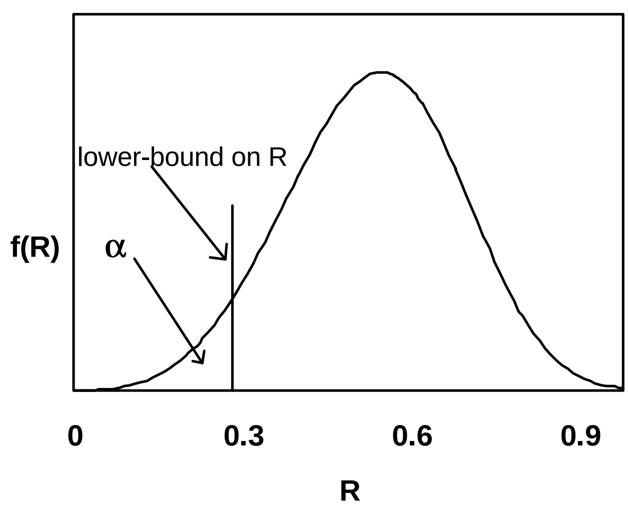

Figure 2: (1 - α) Lower bound on system reliability.

A lower bound on system reliability can be computed for a specified confidence (1 - α) using the following equations [13]. This bound is for series-parallel configurations with k = 1. Since it uses the central limit theorem, results are very accurate for s > 8, and still quite accurate for smaller numbers of subsystems.

$$R_{1\text{-}\alpha}(\mathbf{x}|t') = \exp(\mu(\mathbf{x}) - z_\alpha \sigma(\mathbf{x}))$$

where,

z_α = standard normal statistic (N(0,1)) for specified α

$$\mu(\mathbf{x}) = 2\ln E(R_{sys}) - \tfrac{1}{2}\ln\left((E(R_{sys}))^2 + Var(R_{sys})\right)$$

$$\sigma(\mathbf{x}) = \sqrt{\ln\left((E(R_{sys}))^2 + Var(R_{sys})\right) - 2\ln E(R_{sys})}$$

$$E(R_{sys}) = \prod_{i=1}^{s}\left(1 - \prod_{j=1}^{n_i}(1-\rho_{ij})\right)$$

$$Var(R_{sys}) = \prod_{i=1}^{s}\left(\prod_{j=1}^{n_i}(\sigma_{ij}^2 + (1-\rho_{ij})^2) - \prod_{j=1}^{n_i}(1-\rho_{ij})^2 + \left(1 - \prod_{j=1}^{n_i}(1-\rho_{ij})\right)^2\right) - \prod_{i=1}^{s}\left(1 - \prod_{j=1}^{n_i}(1-\rho_{ij})\right)^2$$

ρ_{ij} = mean reliability of component *j* in subsystem *i*
σ_{ij}^2 = variance of the reliability for component *j* in subsystem *i*

Of particular interest are analyses of user risk profiles, as characterized by the choice of α. Multiple comparisons have been conducted to determine how the optimal solution varies as the confidence (1-α) and the amount of variability increases. Changes in the user risk patterns often lead to entirely new optimal solutions even with the same constraint set and component choices. Designers with similar constraints and available parts may have significantly different solutions depending on the inherent risk associated with a particular design problem. For example, it is entirely logical for an electronic device which monitors critical human conditions to be designed quite differently than a device without serious repercussions of failure (i.e., less risk).

GENETIC ALGORITHM

A genetic algorithm (GA) was developed to solve this class of problems [12]. A GA is a stochastic optimization technique patterned after natural selection in biological evolution as initially described by Holland [14]. The GA approach is very flexible, can accommodate both discrete and continuous functions, and can investigate a larger search space than many corresponding formulations of this problem. The previous work of the authors relaxed the restriction that only identical components be placed in parallel within a given subsystem, allowing a heterogeneous mix of components [12]. Deficiencies with the GA approach are that there are several search parameters which must be found experimentally, and the GA cannot guarantee convergence to the optimal solution, although it has been consistently demonstrated that it provides good solutions to reliability design problems [12, 13].

The GA developed for this general problem is described in [12] and is briefly as follows. An initial population of solutions is randomly selected to form the first generation of the GA. Crossover and mutation operators are then performed on the population members to produce subsequent generations. Each member of the population is evaluated in accordance with its “fitness” (objective function plus a penalty for violated constraints), which is used as basis for selecting parent solutions and for culling inferior solutions from the population. An effective GA depends on complementary crossover and mutation operators. The crossover operator influences the rate and success of convergence to optimality, while the mutation operator prevents the algorithm from prematurely converging. Problem specific variants of the following operators were developed for this problem as described in [12]:

- encoding of solutions
- selection of an initial population
- crossover breeding operator
- mutation operator
- culling of inferior solutions.

ILLUSTRATIVE PROBLEM

The illustrative problem solved had a relatively small search space but was useful to demonstrate the effects of different user risk profiles. Component choices and associated component reliability, variance, cost and weight (indicated as R, V, C, W, respectively) are presented in Table 1. The system cost and weight constraint are 30 and 53 respectively, and the problem was solved for an α of 0.5, corresponding to a risk-neutral analyst, and for an α of 0.01 for a risk-averse analyst.

Table 1: Component selections for illustrative problem.

	subsystem 1				subsystem 2				subsystem 3			
j	R	V	C	W	R	V	C	W	R	V	C	W
1	.94	.003	7	4	.97	.001	6	5	.96	.002	7	6
2	.91	.001	6	6	.86	.004	3	7	.89	.022	6	8
3	.89	.004	6	8	.70	.052	2	4	.72	.003	4	5
4	.75	.016	3	7	.66	.014	2	7	.71	.015	3	4
5	.72	.013	3	8	.65	.170	2	6	.67	.130	2	4

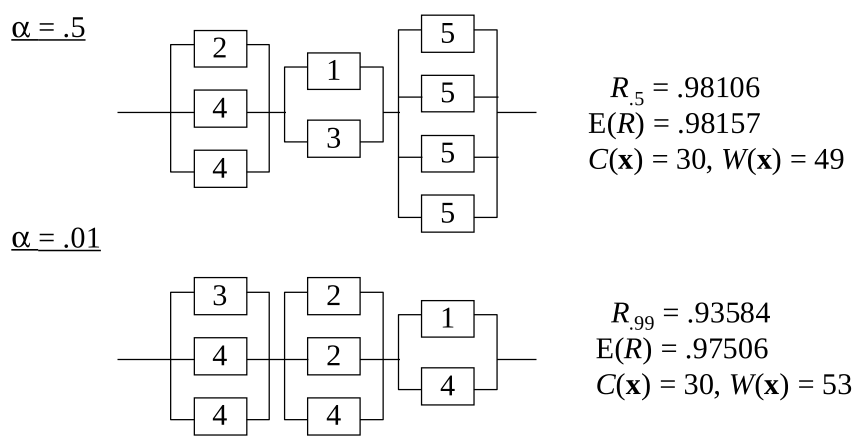

Figure 3: Solutions for illustrative problem.

The GA was run ten independent times for each α level using a population size of 40 for 1500 generations. For each generation, 22 children were created through crossover and 18 mutants were created through random component changes. This resulted in a total of 60,000 (non-unique) solutions considered for each α level. The two best configurations are presented in Figure 3. The results are interesting in several respects. Most notably, the results are very different. Components which were attractive in the risk-neutral case are not attractive in the risk-averse case. Second, the relative amounts of redundancy for the particular subsystems and the expected system reliability are also different.

To fully examine the power of the approach, it has also been applied to the standard test problem of Fyffe, Hines and Lee [2] with 33 problem variations from Nakagawa and Miyazaki [3]. Considering component mixing (more than one component type is allowed per subsystem), the size of the search space is greater than 7.6×10^{33}. There is a cost constraint of 130 and the weight constraint is varied incrementally from 191 to 159 to form the 33 variations. Each of the 33 problems has been solved for α = 0.10, 0.01 and 0.001. This problem was useful to thoroughly test the robustness of the developed algorithms for different risk profiles and degree of constraints. The complete results can be found in [13].

CONCLUSIONS

By relaxing two unrealistic restrictions of the redundancy allocation problem, solutions can fulfill functional and performance requirements at minimal cost (or maximum reliability). The first assumption of homogeneity among subsystem components had been relaxed in previous work by the authors [12], and resulted in configurations which could exploit mixing of components to provide lower cost solutions without sacrificing adherence to reliability and other design constraints.

The second assumption of deterministic reliability, and its companion assumption of risk neutrality in all design situations, were relaxed in this paper. Component reliability was expressed as a random variable specified by mean and standard deviation. The resulting system level mean and standard deviation of reliability could then be calculated for each configuration. These system level calculations allow optimization for system reliability over any percentile of the distribution, as shown for 0.5 and 0.01 in the example presented herein. It was shown that the choice of optimization objective can substantially influence the final design configuration. This matching of appropriate reliability objective with optimal configuration is a major advancement in solution approaches to the redundancy allocation problem.

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BIOGRAPHICAL SKETCHES

David W. Coit received a BS degree in mechanical engineering from Cornell University in 1980, an MBA degree from Rensselaer Polytechnic Institute in 1988 and an MS degree in industrial engineering from the University of Pittsburgh in 1993. He is currently a Ph.D. candidate at the University of Pittsburgh. From 1980 to 1992, he was a reliability engineer and project manager at IIT Research Institute, Rome NY where he established reliability programs, analyzed the reliability of engineering designs and developed statistical models to predict reliability of electronic components for client companies. His current research involves reliability optimization, stochastic optimization techniques and industrial applications for artificial neural networks. Mr. Coit is a student member of IEEE, IIE and INFORMS.

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