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The late Dr. Perry of AU Math Dept. was my Ph.D. major prof

Eccentricities for which Ellipsoidal Probabilities are Good Approximations to Spherical Probabilities

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Please see the last 2 pages

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When launching a vehicle with space guidance systems, the error coordinates at mid-course and on the surface of the moon have a multi-variate normal distribution.

On the surface of the moon, where the problem is two-dimensional, the probability that the vehicle comes within a distance R of the theoretical point of impact can be well approximated with probabilities over associated ellipses having eccentricities in the interval [0.60, 1.0].

At mid-course, where the problem is three-dimensional, the probability that the vehicle comes within a distance R relative to where it ought to be can be well approximated with probabilities over associated ellipsoids having eccentricities in the interval [0.70, 1.0].

1. INTRODUCTION

Consider the theoretical trajectory, sketched below, which indicates the passage from the earth to the moon of a space vehicle.

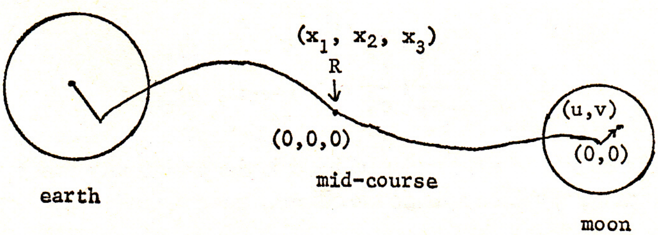

FIGURE 1

Due to small errors of thrust and angle at launching the vehicle has mid-course error coordinates (x_1, x_2, x_3) relative to where it ought to be. Similarly, it has error coordinates (u, v) on the plane tangent to the moon at the theoretical point of impact. Experience with space guidance systems (Noton, 1960) indicates that the set (x_1, x_2, x_3) possesses an approximate three-dimensional multivariate normal distribution (Cramer, 1946), and that similarly the set (u, v) has an approximate two-dimensional multivariate normal distribution.

The objective of this paper is to approximate the probability that the vehicle comes within a distance R of the origin† using the probabilities for conveniently related ellipsoids.

2. APPROXIMATION OF PROBABILITIES IN TWO DIMENSIONS

We first state the following theorem from Anderson (1958).

THEOREM *If the n-component vector X is distributed according to $N(0, \Sigma)$, where Σ is nonsingular, then the quadratic form $X^T\Sigma^{-1}X$ has a χ^2 distribution with n degrees of freedom.*

With the theoretical point of impact at the origin, we rotate the axes so that the covariance elements of Σ become zero, then the joint *pdf*‡ of (u, v) is

$$f(u, v) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\left(\frac{u}{\sigma_1}\right)^2 + \left(\frac{v}{\sigma_2}\right)^2\right]\right\}, \tag{1}$$

where now

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$

Therefore at a plane tangent to the moon, by the theorem from Anderson, the equation

$$\frac{u^2}{\sigma_1^2} + \frac{v^2}{\sigma_2^2} = \chi^2_{1-\alpha;\,2} \tag{2}$$

represents a family of ellipses for which α is the probability that a randomly chosen point (u, v) will be outside the ellipse.

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†Hereafter we refer to (0, 0) and (0, 0, 0) as the origin.
‡Pdf stands for probability density function.

For the sake of illustration, let us see what is meant by the 95% ellipse ($\alpha = 0.05$). The origin of coordinates is taken to be the point where the vehicle is intended to land, and by the 95% ellipse we mean an elliptical region for which the probability is 0.95 that the vehicle will land within such a region.

The interest lies in obtaining the probability of landing within a distance R of (0, 0), i.e., to land within a circle about the origin of radius R. In order to approximate this probability (Noton, 1960) it is assumed that the probability for the circle is nearly the same as that of an ellipse of the same area. Since

$$P\left(\frac{u^2}{\sigma_1^2}+\frac{v^2}{\sigma_2^2} \leqq \chi^2_{1-\alpha;\,2}\right) = \alpha,$$

then for given α, σ_1 and σ_2 it is possible to obtain the corresponding value of R, by equating the area of the circle to that of the associated ellipse in the following manner:

$$\pi R^2 = \pi(\sigma_1\sqrt{[\chi^2_{1-\alpha;\,2}]})\,(\sigma_2\sqrt{[\chi^2_{1-\alpha;\,2}]}) \tag{3}$$

Eq. (3) then yields:

$$R = \sqrt{[\sigma_1\sigma_2\chi^2_{1-\alpha;\,2}]}. \tag{4}$$

The given value of α will determine the approximate probability, i.e., the *pdf* integrated over the ellipse, to be

$$\iint_{\text{ellipse}} f(u, v)\, du\, dv = 1-\alpha. \tag{5}$$

In order to find the exact probability, say P_2, over the corresponding circle of radius R, it is necessary to evaluate

$$P_2 = \frac{1}{2\pi\sigma_1\sigma_2} \iint_{\text{circle}} \exp\left\{-\frac{1}{2}\left[\left(\frac{u}{\sigma_1}\right)^2+\left(\frac{v}{\sigma_2}\right)^2\right]\right\} du\, dv \tag{6}$$

The objective is now to determine the range of eccentricities for the ellipse where the integral in (5) will closely approximate the integral in (6). Before the evaluation of the double integral in (6), the following lemma is in order; the use of the lemma will be valid to the order of approximation used in this manuscript.

LEMMA *The double integral* $\iint_{\text{circle}} f(u, v)\, du\, dv$ *depends only on* α *and the quantity* $E = \sigma_2/\sigma_1$, *where without loss of generality we assume that* $\sigma_2 \leqq \sigma_1$.

Proof Letting $Z_1 = u/\sigma_1$, $Z_2 = v/\sigma_2$, and recalling that $R^2 = \sigma_1\sigma_2\chi^2_{1-\alpha;\,2}$, we have:

$$P_2 = \frac{1}{2\pi\sigma_1\sigma_2} \iint_{u^2+v^2 \leqq R^2} \exp\left\{-\frac{1}{2}\left[\left(\frac{u}{\sigma_1}\right)^2 + \left(\frac{v}{\sigma_2}\right)^2\right]\right\} du\, dv$$

$$= \frac{1}{2\pi} \int\int_{E^{-1}Z_1^2 E Z_2^2 \leqq \chi_1^2{}_{-\alpha;2}} \exp\{-\tfrac{1}{2}(Z_1^2+Z_2^2)\} \quad dZ_1\, dZ_2. \tag{7}$$

The double integral in (7) clearly depends only on α and E; this completes the proof.

In order to evaluate the integral in (6), we make the transformation $u = r \cos\theta$ and $v = r \sin\theta$ which on integrating out r results in

$$P_2 = \frac{1}{2\pi\sigma_1\sigma_2} \int_0^{2\pi} g(\theta)\, d\theta, \tag{8}$$

where

$$g(\theta) = \frac{1-\exp\{-\frac{1}{2} R^2[(\cos^2\theta/\sigma_1^2+\sin^2\theta/\sigma_1^2)]\}}{(\cos^2\theta/\sigma_1^2+\sin^2\theta/\sigma_2^2)}.$$

Since $g(\theta)$ has no convenient antiderivative, the integration in (8) was carried out using Simpson's rule and as a result

$$P_2 = \frac{1}{3m\sigma_1\sigma_2} [g(0)+4g(\theta_1)+2g(\theta_2)+4g(\theta_3)+2g(\theta_4)$$
$$+\ldots+2g(\theta_{m-2})+4g(\theta_{m-1})+g(2\pi)].$$

A Fortran IV program was written for the evaluation of the integral in (8) and is available upon request. The program is simple and requires as input the values of σ_2, σ_1, $\chi^2_{1-\alpha;\,2}$, and m. It was tested for accuracy by computing P_2 for $E = 1.0$, $\alpha = 0.05$, $m = 30$, 40, and 50; the respective values of P_2 were 0.94998592, 0.94998699, and 0.94998813. Furthermore, the values of P_2 for $m = 30$, 40, and 50 were compared, and since for $m = 40$ and 50 the values of P_2 were not significantly different, a mesh of length $2\pi/50$ was used. The results of programming, correct to five significant figures, are given in Table I.

From Table I, it is concluded that $(1-\alpha)$ is a reasonable approximation to P_2 for eccentricities satisfying the range $0.60 \leqq E \leqq 1.0$.

3. APPROXIMATION OF PROBABILITIES IN THREE DIMENSIONS

The situation in mid-course is somewhat different, since here the approximation of probability that the vehicle will stay within a certain distance, R, of the point where it ought to be means that we are seeking to approximate the probability that the vehicle stays within a sphere of radius R about the point (0, 0, 0) (See Figure 1). The distance R is limited to the amount of correction which it is possible to make using signals from the earth.

TABLE I

The results of computer programming for $n = 2$ and $m = 50$

$\alpha = 0.50$		$\alpha = 0.05$		$\alpha = 0.01$	
E	P_2	E	P_2	E	P_2
1.00000	0.50000	1.00000	0.95000	1.00000	0.99000
0.90000	0.49932	0.90000	0.94867	0.90000	0.98931
0.80000	0.49699	0.80000	0.94424	0.80000	0.98708
0.70000	0.49237	0.70000	0.93555	0.70000	0.98256
0.60000	0.48450	0.60000	0.92086	0.60000	0.97447
0.50000	0.47189	0.50000	0.89760	0.50000	0.96049
0.40000	0.45202	0.40000	0.86149	0.40000	0.93625
0.30000	0.42054	0.30000	0.80503	0.30000	0.89313
0.20000	0.36903	0.20000	0.71321	0.20000	0.81237
0.10000	0.27875	0.10000	0.54819	0.10000	0.64453

After rotation of axes, the *pdf* of the variate $X = (x_1, x_2, x_3)$ becomes

$$f(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} \exp\left[-\tfrac{1}{2} \sum_{i=1}^{3} (x_i/\sigma_i)^2\right]. \qquad (9)$$

By the Theorem from Anderson the quadratic form $X^T\Sigma^{-1}X$ is distributed as chi-square with 3 degrees of freedom, and thus the ellipsoid

$$\sum_{i=1}^{3} (x_i/\sigma_i)^2 = \chi^2_{1-\alpha;\,3}$$

contains all but α of the three-dimensional distribution. The value of R is obtained from the following equation, where the volume of the sphere is taken to be the same as that of the associated ellipsoid.

$$\frac{4}{3}\pi R^3 = \frac{4}{3}\pi(\sigma_1\sqrt{[\chi^2_{1-\alpha;\,3}]})(\sigma_2\sqrt{[\chi^2_{1-\alpha;\,3}]})(\sigma_3\sqrt{[\chi^2_{1-\alpha;\,3}]}) \qquad (10)$$

Eq. (10) yields

$$R = (\sigma_1\sigma_2\sigma_3)^{1/3}(\chi^2_{1-\alpha;\,3})^{1/2} \tag{11}$$

The given value of α will determine the approximate probability, i.e., the *pdf* integrated over the ellipsoid, to be

$$\iiint_{\text{ellipsoid}} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 1-\alpha. \tag{12}$$

The exact probability, say P_3, over the sphere is obtained from

$$P_3 = \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} \iiint_{\text{sphere}} \exp[-\tfrac{1}{2}\sum_{i=1}^{3}(x_i/\sigma_i)^2\, dx_1\, dx_2\, dx_3. \tag{13}$$

Defining $E_1 = \sigma_1/\sigma_2$ and $E_2 = \sigma_1/\sigma_3$, we assume without loss of generality that $\sigma_1 \leqq \sigma_2$ and $\sigma_1 \leqq \sigma_3$ so that $0 \leqq E_1 \leqq 1$ and $0 \leqq E_2 \leqq 1$. Since the Lemma can easily be extended to the case $n = 3$, it is concluded that P_3 depends only on E_1, E_2, and α.

The objective now is to study the range of values of E_1 and E_2 for which $(1-\alpha)$ will closely approximate P_3.

To evaluate the triple integral in (13) the following cylindrical transformation is made

$$\begin{aligned} x_1 &= r \sin\phi \cos\theta \\ x_2 &= r \sin\phi \sin\theta \\ x_3 &= r \cos\phi. \end{aligned} \tag{14}$$

Under this transformation, P_3 becomes

$$P_3 = \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{R} \exp[-r^2 h(\phi, \theta)/2]\, r^2 \sin\phi\, dr\, d\theta\, d\phi \tag{15}$$

where

$$h(\phi, \theta) = \frac{\sin^2\phi\cos^2\theta}{\sigma_1^2} + \frac{\sin^2\phi\sin^2\theta}{\sigma_2^2} + \frac{\cos^2\phi}{\sigma_3^2}.$$

In contrast to the two-dimensional case, the integrand in (15) has no convenient antiderivative with respect to r, but after integration by parts with respect to r, we obtain

$$P_3 = \frac{1}{2^{3/2}\pi\sigma_1\sigma_2\sigma_3}\int_0^{\pi}\int_0^{2\pi} G(\phi, \theta)\, d\phi\, d\theta \tag{16}$$

where

$$G(\phi, \theta) = g(\phi, \theta) \frac{\sin\phi}{h(\phi, \theta)},$$

$$g(\phi, \theta) = \frac{1}{\sqrt{[2h(\phi, \theta)]}} ERF(R\sqrt{[h(\phi, \theta)/2)]} - \frac{R}{\sqrt{\pi}} \exp[-R^2 h(\phi, \theta)/2],$$

and

$$ERF(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Due to the fact that $ERF(x)$ is a built-in function for the IBM 360 computer, the triple integral in (15) reduces to the double integral in (16).

TABLE II

The results of computer programming for $n = 3$, $\alpha = 0.50$, $N = 48$ and $K = 48$

E_2	$E_1 = 1.0000$	$E_1 = 0.8000$	$E_1 = 0.6000$	$E_1 = 0.4000$	$E_1 = 0.2000$
	P_3	P_3	P_3	P_3	P_3
1.00000	0.50000	0.49545	0.47722	0.43318	0.33269
0.90000	0.49897	0.49653	0.48069	0.43910	0.34020
0.80000	0.49544	0.49530	0.48220	0.44364	0.34736
0.70000	0.48859	0.49093	0.48092	0.44606	0.35369
0.60000	0.47723	0.48220	0.47568	0.44529	0.35843
0.50000	0.45964	0.46735	0.46471	0.43971	0.36034
0.40000	0.43318	0.44364	0.44529	0.42675	0.35736
0.30000	0.39352	0.40648	0.41278	0.40209	0.34592
0.20000	0.33270	0.34736	0.35843	0.35736	0.31899
0.10000	0.23340	0.24717	0.26153	0.27175	0.25877

E_2	$E_1 = 0.9000$	$E_1 = 0.7000$	$E_1 = 0.5000$	$E_1 = 0.3000$	$E_1 = 0.1000$
	P_3	P_3	P_3	P_3	P_3
1.00000	0.49897	0.48859	0.45964	0.39351	0.23340
0.90000	0.49895	0.49084	0.46435	0.40047	0.24011
0.80000	0.49652	0.49093	0.46735	0.40648	0.24717
0.70000	0.49084	0.48802	0.46785	0.41090	0.25443
0.60000	0.48069	0.48092	0.46471	0.41278	0.26153
0.50000	0.46435	0.46785	0.45623	0.41063	0.26777
0.40000	0.43910	0.44606	0.43971	0.40209	0.27175
0.30000	0.40047	0.41090	0.41063	0.38312	0.27069
0.20000	0.34020	0.35369	0.36034	0.34592	0.25878
0.10000	0.24011	0.25443	0.26777	0.27068	0.22129

TABLE III

The results of computer programming for $n = 3$, $\alpha = 0.05$, $N = 48$ and $K = 48$

E_2	$E_1 = 1.0000$	$E_1 = 0.8000$	$E_1 = 0.6000$	$E_1 = 0.4000$	$E_1 = 0.2000$
	P_3	P_3	P_3	P_3	P_3
1.00000	0.95000	0.94359	0.91474	0.83540	0.63777
0.90000	0.94861	0.94544	0.92118	0.84754	0.65355
0.80000	0.94359	0.94432	0.92569	0.85931	0.67073
0.70000	0.93319	0.93860	0.92689	0.86992	0.68940
0.60000	0.91474	0.92569	0.92247	0.87775	0.70942
0.50000	0.88422	0.90146	0.90839	0.87963	0.73003
0.40000	0.83541	0.85931	0.87775	0.86921	0.74856
0.30000	0.75861	0.78847	0.81853	0.83391	0.75704
0.20000	0.63776	0.67068	0.70930	0.74826	0.73266
0.10000	0.44154	0.47016	0.50730	0.55705	0.60732
E_2	$E_1 = 0.9000$	$E_1 = 0.7000$	$E_1 = 0.5000$	$E_1 = 0.3000$	$E_1 = 0.1000$
	P_3	P_3	P_3	P_3	P_3
1.00000	0.94861	0.93319	0.88421	0.75861	0.44174
0.90000	0.94869	0.93714	0.89345	0.77326	0.45527
0.80000	0.94543	0.93859	0.90146	0.78848	0.47053
0.70000	0.93714	0.93601	0.90713	0.80389	0.48794
0.60000	0.92118	0.92689	0.90839	0.81855	0.50800
0.50000	0.89345	0.90713	0.90150	0.83026	0.53132
0.40000	0.84754	0.86992	0.87962	0.83395	0.55828
0.30000	0.77326	0.80388	0.83022	0.81816	0.58740
0.20000	0.65352	0.68932	0.72983	0.75671	0.60798
0.10000	0.45499	0.48742	0.53038	0.58606	0.56609

To calculate the double integral in (16) numerically, we use a mechanical cubature (Scarborough, 1955) which combines Weddle's and Simpson's rule. Using Weddle's rule, we first integrate over θ, reducing the double integral to a single integral of the type

$$P_3 = \int_0^\pi A(\phi)\, d\phi \tag{17}$$

to which Simpson's rule is applied.

The integration over θ yields

$$A(\phi) = \frac{1}{2^{3/2}\pi\sigma_1\sigma_2\sigma_3}\left(\frac{3}{10}\right)\left(\frac{2\pi}{N}\right)\sum_{\theta=0}^{2\pi} C_\theta G(\phi, \theta)\,,$$

where

$$\theta = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, 2\pi,$$

$$C_\theta = 1, 5, 1, 6, 1, 5, 2, 5, 1, 6, 1, 5, 2, 5, 1, 6, 1, 5, 2, \ldots$$

and N must be a positive integer multiple of 6.

Finally, the application of Simpson's rule to (17) gives

$$P_3 = \frac{\pi}{3K}\{A(0)+4A(\pi/K)+2A(2\pi/K)+4A(3\pi/K)+2A(4\pi/K)$$
$$+\ldots+2A\left[\frac{(K-2)\pi}{K}\right]+4A\left[\frac{(K-1)\pi}{K}\right]+A(\pi)\}$$

where K must be a positive interger.

TABLE IV

The results of computer programming for $n = 3$, $\alpha = 0.01$, $N = 48$ and $K = 48$

E_2	$E_1 = 1.0000$	$E_1 = 0.8000$	$E_1 = 0.6000$	$E_1 = 0.4000$	$E_1 = 0.2000$
	P_3	P_3	P_3	P_3	P_3
1.00000	0.99000	0.98701	0.97119	0.91580	0.73546
0.90000	0.98938	0.98797	0.97512	0.92527	0.75149
0.80000	0.98701	0.98759	0.97812	0.93457	0.76891
0.70000	0.98168	0.98492	0.97952	0.94328	0.78786
0.60000	0.97119	0.97813	0.97790	0.95055	0.80842
0.50000	0.95159	0.96362	0.97031	0.95437	0.83034
0.40000	0.91580	0.93456	0.95053	0.94989	0.85218
0.30000	0.85125	0.87787	0.90549	0.92521	0.86796
0.20000	0.73530	0.76861	0.80786	0.85124	0.85488
0.10000	0.52334	0.55570	0.59774	0.65534	0.72774

E_2	$E_1 = 0.9000$	$E_1 = 0.7000$	$E_1 = 0.5000$	$E_1 = 0.3000$	$E_1 = 0.1000$
	P_3	P_3	P_3	P_3	P_3
1.00000	0.98938	0.98168	0.95159	0.85127	0.52413
0.90000	0.98944	0.98387	0.95791	0.86435	0.53953
0.80000	0.98796	0.98492	0.96362	0.87793	0.55689
0.70000	0.98387	0.98402	0.96814	0.89184	0.57665
0.60000	0.97512	0.97952	0.97031	0.90562	0.59943
0.50000	0.95792	0.96814	0.96761	0.91798	0.62601
0.40000	0.92527	0.94327	0.95434	0.92540	0.65723
0.30000	0.86431	0.89176	0.91780	0.91789	0.69311
0.20000	0.75127	0.78745	0.82959	0.86709	0.72608
0.10000	0.53856	0.57522	0.62409	0.69208	0.69622

The program for evaluation of P_3 requires as input σ_1, σ_2, σ_3, $\chi^2_{1-\alpha;\,3}$, N, and K, and is available upon request. The program was tested for $E_1 = 1.0$, $E_2 = 1.0$, $\alpha = 0.50$, and N, $K = 24$, 36, and 48; the respective values of P_3 were 0.50000016, 0.49999950, and 0.49999936. Since the values of P_3 for N and $K = 36$ were the same as those for N and $K = 48$ to 6 significant figures, N and $K = 48$ were used.

The results of programming, correct to 5 significant figures, are given in Tables, II, III and IV. These Tables show that $(1-\alpha)$ is a reasonable approximation to P_3 for eccentricities satisfying the range $0.70 \leqq E_1 \leqq 1.0$ and $0.70 \leqq E_2 \leqq 1.0$.

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My Memoirs of the Late Dr. Norman C. Perry (1951-1975) of the Auburn University (AU) Mathematics Department

Please note that I am writing these Memoirs based on my long-term memory; some date and time inaccuracies exit 09/08/2020

Dr. Perry was a unique Statistics Teacher; in fact the supreme master of Geometrical Statistics-Teaching (almost like Jascha Heifetz who was the "God" of Violin, Glenn Gould (or Wanda Landowska) playing the music of Johan Sebastian Bach, Van Cliburn playing Rachmaninoff's Concerto No. 3 during 1959 in Moscow, David Oistrakh and Yehudi Menuhin playing J. S. Bach's Double Concerto in D Minor in Paris 1958, Sviatoslav Richter playing Mussorgsky's Pictures at an Exhibition, Arthur Rubinstein playing Brahms Concerto No. 2 with Boston symphony, etc.). We will not have many better Geometrical-Statistic teachers like him; please understand that I am talking about only the teaching of Statistics not accomplishments in my field. Please Google the well-known Statisticians; you will see John W. Tukey, Karl Pearson (named the Laplace-Gaussian distribution as the normal curve around 1894), Sir R. A. Fisher (I am awed), Sir Maurice G. Kendall (the creator of Rank Correlation; I am one of his disciples even if I do not qualify), the late George E. P. Box (who created fractional replicates in base-2 designs in early 1960's, wrote the "books" on Time-Series, Respond Surfaces, and so many more), W. Edwards Deming (the Quality Guru of the 20th century who revolutionized Japanese Industrial Quality Control and Management approximately during 1948-1954), and at least 20 other notable 20th century Statisticians.

Dr. Perry who graduated from the USC, Los Angeles (I am guessing around 1949 and was hired by our own late Dr. W. Van Parker who was then the department head of our Math Dept. located in Brown Hall) was so unique that he could draw the pictures of all foundational concepts in my field; that is exactly how I got my statistical academic background from him. I was the only Ph.D. student he ever had. He was very opposed to the academic system of Publish-or-Perish; this is exactly why he did not publish his works. I was the luckiest student in the Auburn University Math Department during 1966-1968 to be his only PhD student; his spirit is always with me when I do any statistical work (such as teaching, writing articles, directing Ph.D. students, Extension and consulting, etc.). Recall that our Stat Department did not exist until after the year 1999.

In the Spring Quarter 1966 the Nasa Space Flight Center called the AU Mathematics Department and asked the Dept. head (the late Dr. Pat Burton) if our Math Department had a statistician who could solve problems in the 3-dimensional space for the Moon-Flight that eventuated in the summer of 1969. The department head answered the only person is Dr. Norman C. Perry; so early June 1966 I took him to the Auburn Andrew's gas and also bus station (he did not know how to drive); Nasa Engineers picked him up at their bus station. He stayed in Huntsville, AL for the Summer Quarter 1966 and returned to continue teaching at the Auburn University Mathematics Department.

During the Fall Quarter 1966, Winter and Spring Quarters 1967 NASA Engineers made couple of trips (I do not remember how many) to the Auburn University Mathematics Department in order to ask questions about the complexity of the above article; of course, Dr. Perry drew statistical pictures on the blackboard, while I sat in his office (awed) to watch him in action. Clearly, the idea behind the above article was given to me only by our own Dr. Norman C. Perry in its entirety, although I had to write a Fortran Program to do all computations. I must also state that the field of Applied Multivariate Analysis was not fully developed before 1966. I hope everybody understands that Dr. Perry's contributions to our Moon-Flight of 1969 was minimally needed, but of course, it was not at all sufficient. Also, please understand that our NASA Space Center in Huntsville, AL could have sought another statistician, but I am not certain that another statistician could have drawn the 3-dimentional pictures like our own Dr. Norman C. Perry during 1966.

Further, I learned everything that I know about the field of Applied Mathematics during the years 1962-1965 originated from the late Dr. Ernest Ikenberry (the internationally well-known Quantum Physicist of our AU Math department). Dr. Perry was so gracious in the Summer Quarter of 1962 to ask Dr. Ikenberry to be my MS degree advisor; he relayed to Dr. Ikenberry that "Saeed" will be a good graduate student for you. Dr. Ikenberry also served on my PhD committee and sadly retired from AU in June of 1975.

I also owe an immense amount of gratitude to our own late Dr. Howard E. Carr, the head of AU Physics Department during the years 1945-1980; I am not sure about this last date). Dr. Carr was an internationally well-known Physicist but amazingly modest! In the Fall Quarter 1960 when somehow my Father's support had not yet arrived (I do not remember why) he said something to me in his office (as I was in tears) that I have always remembered, and I quote "Saeed please do not worry b/c you are now part of us and I will take care of it", end of quote. His wife Carolyn was also very nice and kind to me. During the 1990's I often saw them in our Kroger shopping center, and ASA I would see them I ran to their automobile and helped Dr. Carr out of their automobile. It has always seemed that he thought of me as one of his own decedents! I do not have the words to describe how great a professor, mentor, teacher, and fatherly he was to me (Unbelievable!). Other AU professors (of Physics and Math. Departments) who had great influence on my academic career were: Drs. James D. Louck (AU professor 1961-64) who sadly left Auburn for the Los Alamos Laboratory as a research Physicist, Ray Askew, John French, Paul Budenstein, Ben Fitzpatrick, the well-known Emily Haynsworth, Dr. Taylor Littleton (the AU VP for academic affairs) and other notables.

Last but not least, it behooves me to thank the late Dr. George H. Brooks (the head of our IE department during the years 1966-1979). After leaving AU for ETSU in Johnson City, TN during the late August 1968, Dr. Brooks (in consultation with Dean Cox) rehired me as an assistant professor starting the Summer Quarter 1969. Drs. Brooks and Hool were the two witnesses at my citizenship ceremony on May 2nd, 1975 in Montgomery, AL. Further, Dr. Brooks continually encouraged me to succeed academically.

Sincerely, Saeed Maghsoodloo, Professor Emeritus of the ISE Department (Summer Quarter 1966-Spring Semester 2015; I was at the ETSU located in Johnson City, TN, for the Fall Quarter of 1968 thru Spring Quarter of 1969), Auburn University, AL 36849

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