The Convolutions of the U[a, b] Distribution by Saeed Maghsoodloo 10/15/2018

Let the rv X be uniformly distributed over the interval [0, c], c > 0, i.e., X~ U(0,c); then its pdf is given by

$$f(x) = 1/c,\ 0 \le x \le c, \text{ and 0 elsewhere.}$$

Then, the density of $T = X_1 + X_2$, where X_1 and X_2 are iid as U[0, c] is called the 2-fold convolution of U[0,c] and is given by

$$f_{(2)}(t) = \begin{cases} t/c^2, & 0 \le t \le c \\ (-t+2c)/c^2, & c \le t \le 2c \end{cases}, \text{ and zero elsewhere.}$$

In order to obtain the 2-fold convolution of U(a, b), b > a, c = b − a, we have to make a transformation such that the lower limit of the interval $0 \le t \le c$ will change to 2a, i.e., we must let y = t+2a. At t = c, y = c+2a = a+ (c+a) = a+b , and at t = 2c, y = 2c +2a =2(a+c) = 2b. Letting c = b − a > 0 be the Uniform-base, then

$$f_{(2)}(y) = \begin{cases} (y-2a)/c^2, & 2a \le y \le a+b \\ (2b-y)/c^2, & a+b \le t \le 2b \end{cases}, \text{ and zero elsewhere.}$$

The 3-fold convolution of U[0, c] is given by

$$f_{(3)}(t) = \begin{cases} t^2/(2c^3), & 0 \le t \le c \\ (-t^2+3ct-1.5c^2)/c^3, & c \le t \le 2c \\ (t^2/2-3ct+4.5c^2)/c^3, & 2c \le t \le 3c \end{cases}$$

In order to obtain the 3-fold convolution of U[a, b], let y = t+3a; at t = c, y = 2a+(a+c) = 2a+b. At t = 2c, y= 2c+3a = a +2(a+c) = a+2b, etc.

$$f_{(3)}(y) = \begin{cases} (y-3a)^2/(2c^3), & 3a \le y \le 2a+b \\ [-(y-3a)^2+3c(y-3a)-1.5c^2]/c^3, & 2a+b \le y \le a+2b \\ [(y-3a)^2-6c(y-3a)+9c^2)/(2c^3), & a+2b \le y \le 3b \end{cases}$$

where c = b −a is the base of the U[a, b].

The 4-fold convolution of U[0, c] is given by

$$f_{(4)}(t) = \begin{cases} t^3/(6c^4), & 0 \le t \le c \\ (-t^3/2 + 2ct^2 - 2c^2 t + 2c^3/3)/c^4, & c \le t \le 2c \\ (t^3/2 - 4ct^2 + 10c^2 t - 22c^3/3)/c^4, & 2c \le t \le 3c \\ (-t^3/6 + 2ct^2 - 8c^2 t + 32c^3/3)/c^4, & 3c \le t \le 4c \end{cases}$$

In order to obtain the 4-fold convolution of U[a, b], let y = t+4a; at t = c, y = 3a+(a+c) = 3a+b. At t = 2c, y= 2c+4a = 2a +2(a+c) = 2a+2b, at t = 3c, y= 3c+4a= a+3(a+b)= a+3b; at y= 4c, y= 4(a+c)=4b.

The resulting 4-fold convolution of U[a, b] is given by

$$f_{(4)}(y) = \begin{cases} (y-4a)^3/(6c^4), & 4a \le y \le 3a+b \\ [-(y-4a)^3/2 + 2c(y-4a)^2 - 2c^2(y-4a) + 2c^3/3]/c^4, & 3a+b \le y \le 2a+2b \\ [(y-4a)^3/2 - 4c(y-4a)^2 + 10c^2(y-4a) - 22c^3/3]/c^4, & 2a+2b \le y \le a+3b \\ [-(y-4a)^3/6 + 2c(y-4a)^2 - 8c^2(y-4a) + 32c^3/3]/c^4, & a+3b \le y \le 4b \end{cases}$$

The 5-fold convolution of U[0, c], also denoted as $U_{(5)}[0, c]$, is given by

$$f_{(5)}(t) = \begin{cases} t^4/(24c^5), & 0 \le t \le c \\ (-t^4/6 + 5ct^3/6 - 5c^2 t^2/4 + 5c^3 t/6 - 5c^4/24)/c^5, & c \le t \le 2c \\ (t^4/4 - 5ct^3/2 + 35c^2 t^2/4 - 25c^3 t/2 + 155c^4/24)/c^5, & 2c \le t \le 3c \\ (-t^4/6 + 5ct^3/2 - 55c^2 t^2/4 + 65c^3 t/2 - 655c^4/24)/c^5, & 3c \le t \le 4c \\ (t^4/24 - 5ct^3/6 + 25c^2 t^2/4 - 125c^3 t/6 + 625c^4/24)/c^5, & 4c \le t \le 5c \end{cases}$$

In order to obtain the 5-fold convolution of U[a, b], let y = t+5a; at t = c, y = 4a+(a+c) = 4a+b. At t = 2c, y= 2c+5a = 3a +2(a+c) = 3a+2b, at t = 3c, y= 3c+4a= a+3(a+b)= a+3b; at y= 4c, y= a + 4(a+c)= a+4b, and at t= 5c, y= 5b.

The 6-fold convolution, $U_{(6)}[0, c]$, is given by

$$f_{(6)}(t) = \begin{cases} t^5/(120c^6), & 0 \le t \le c \\ (-t^5/24 + ct^4/4 - c^2 t^3/2 + c^3 t^2/2 - c^4 t/4 + c^5/20)/c^6, & c \le t \le 2c \\ (t^5/12 - ct^4 + 9c^2 t^3/2 - 19c^3 t^2/2 + 39c^4 t/4 - 79c^5/20)/c^6, & 2c \le t \le 3c \\ (-t^5/12 + 3ct^4/2 - 21c^2 t^3/2 + 71c^3 t^2/2 - 231c^4 t/4 + 731c^5/20)/c^6, & 3c \le t \le 4c \\ (t^5/24 - ct^4 + 19c^2 t^3/2 - 89c^3 t^2/2 + 409c^4 t/4 - 1829c^5/20)/c^6, & 4c \le t \le 5c \\ (-t^5/120 + ct^4/4 - 3c^2 t^3 + 18c^3 t^2 - 54c^4 t + 324c^5/5)/c^6, & 5c \le t \le 6c \end{cases}$$

In order to obtain the 6-fold convolution of U[a, b], let y = t+6a in $f_{(6)}(t)$ and obtain the limits as shown above.

The 7-fold convolution of U[0, c] is given by

$f_{(7)}(t) =$

$$\begin{cases} t^6/(720c^7), & 0 \le t \le c \\ (-t^6/120 + 7ct^5/120 - 7c^2t^4/48 + 7c^3t^3/36 - 7c^4t^2/48 + 7c^5t/120 - 7c^6/720)/c^7, & c \le t \le 2c \\ (t^6/48 - 7ct^5/24 + 77c^2t^4/48 - 161c^3t^3/36 + 329c^4t^2/48 - 133c^5t/24 + 1337c^6/45)/c^7, & 2c \le t \le 3c \\ (-t^6/36 + 7ct^5/12 - 119c^2t^4/24 + 196c^3t^3/9 - 1253c^4t^2/24 + 196c^5t/3 - 12089c^6/360)/c^7, & 3c \le t \le 4c \\ (t^6/48 - 7ct^5/12 + 161c^2t^4/24 - 364c^3t^3/9 + 3227c^4t^2/24 - 700c^5t/3 + 59591c^6/360)/c^7, & 4c \le t \le 5c \\ (-t^6/120 + 7ct^5/24 - 203c^2t^4/48 + 1169c^3t^3/36 - 6671c^4t^2/48 + 7525c^5t/24 - 208943c^6/720)/c^7, & 5c \le t \le 6c \\ (t^6/720 - 7ct^5/120 + 49c^2t^4/48 - 343c^3t^3/36 + 2401c^4t^2/48 - 16807c^5t/120 + 117649c^6/720)/c^7, & 6c \le t \le 7c \end{cases}$$

As before, to obtain the 7-fold convolution over the interval [a, b], let y = t+7a and c= b–a, b > a.

The 8-fold convolution of U[0, c] of is given by $f_{(8)}(t) =$

$$\begin{cases} t^7/(5040c^8), & 0 \le t \le c \\ (-t^7/720 + ct^6/90 - c^2t^5/30 + c^3t^4/18 - c^4t^3/18 + c^5t^2/30 - c^6t/90 + c^7/630)/c^8, & c \le t \le 2c \\ (t^7/240 - ct^6/15 + 13c^2t^5/30 - 3c^3t^4/2 + 55c^4t^3/18 - 37c^5t^2/10 + 233c^6t/90 - 149c^7/210)/c^8, & 2c \le t \le 3c \\ (-t^7/144 + ct^6/6 - 5c^2t^5/3 + 9c^3t^4 - 256c^4t^3/9 + 53c^5t^2 - 488c^6t/9 + 2477c^7/105)/c^8, & 3c \le t \le 4c \\ (t^7/144 - 2ct^6/9 + 3c^2t^5 - 199c^3t^4/9 + 96c^4t^3 - 737c^5t^2/3 + 344c^6t - 64249c^7/315)/c^8, & 4c \le t \le 5c \\ (-t^7/240 + ct^6/6 - 17c^2t^5/6 + 53c^3t^4/2 - 2647c^4t^3/18 + 967c^5t^2/2 - 15683c^6t/18 + 139459c^7/210)/c^8, & 5c \le t \le 6c \\ (t^7/720 - ct^6/15 + 41c^2t^5/30 - 31c^3t^4/2 + 1889c^4t^3/18 - 4237c^5t^2/10 + 84881c^6t/90 - 187133c^7/210)/c^8, & 6c \le t \le 7c \\ (-t^7/5040 + ct^6/90 - 4c^2t^5/15 + 32c^3t^4/9 - 256c^4t^3/9 + 2048c^5t^2/15 - 16384c^6t/45 + 131072c^7/315)/c^8, & 7c \le t \le 8c \end{cases}$$

In order to obtain the 8-fold convolution of the U[a, b], replace t with x–8a in the above 8-fold convolution and change limits of the first interval to 8a ≤ x ≤ 7a+b, 2nd interval to 7a+b ≤ x ≤ 6a+2b, 6a+2b ≤ x ≤ 5a+3b, 5a+3b ≤ x ≤ 4a+4b, 4a+4b ≤ x ≤ 3a+5b, 3a+5b ≤ x ≤ 2a+6b, 2a+6b ≤ x ≤ a+7b, and the last interval limits to a+7b ≤ x ≤ 8b. The following table gives the exact cdfs of the $U_{(8)}[0,1]$ at x =1, 2, 3, 4, 5, 6, 7, 8, and the corresponding normal approximation.

The exact cdf	The normal approximation
0.000024801587302	0.000119281727014
0.006150793650794	0.007152939217715
0.112624007936508	0.110335680959923

0.112624007936508	0.110335680959923
0.500000000000000	0.500000000000000
0.887375992063492	0.889664319040077
0.993849206349254	0.992847060782285
0.999975198412748	0.999880718272986
1.000000000000049	0.999999518321496

The above table shows that at the left-tail the normal approximation underestimates the quantiles of $U_{(8)}[0,1]$ and vice versa at the upper tail. For example the exact $Q_{0.006151} = 2$ while that of the normal approximation $q_{0.006151} = 1.956$. While at the right tail, $Q_{0.99385} = 6$ while the corresponding normal approximation is $q_{0.99385} = 6.043995$.