

## Design and Analysis of Single-Factor Experiments

**Reference: Chapter 10 of Devore's 8<sup>th</sup> Edition**

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The objective of one-factor experiments is to determine if a factor (or an input) has a significant impact on a response variable (or an output)  $Y$ . This is accomplished through an ANOVA (analysis of variance) Table. To illustrate the procedure, consider the example 10.1 on pages 392-397 of Devore's 8<sup>th</sup> edition. Here the objective is to determine if the Compression Strength (CMPS measured in lbs), the output  $Y$ , is significantly affected by the input "different types of boxes". Note that the factor "Box Type" has  $a = 4$  levels (1, 2, 3, 4), which are qualitative. The ultimate objective is to determine which one of the 4 types of boxes, (if any), produces the most amount of CMPS.

Further, each level of the input (or each box type) represents a normal population with population mean  $E(Y_{ij}) = \mu_i$  ( $i = 1, 2, 3, 4 = a$ ) and common variance  $\sigma^2$ ; note that Devore uses the uncommon notation  $I$  for total number of levels. Our null hypothesis is  $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu$  VS the alternative  $H_1$ : at least two  $\mu_i$ 's are significantly different. Further, only under the null hypothesis do all the treatments have identical population means  $\mu$ . Let  $y_{ij}$  be the actual  $j^{\text{th}}$  observation at the  $i^{\text{th}}$  level of the factor (or the  $i^{\text{th}}$  treatment),  $j = 1, 2, \dots, n_i$ . When  $n_1 = n_2 = \dots = n_a = n$ , the design is called balanced. For the Table 10.1,  $n_1 = n_2 = n_3 = n_4 = n = 6$  so that the design is balanced; further,  $y_{34} = 671.7$ ,  $y_{26} = 774.8$  pounds, etc. Devore uses the uncommon notation  $J$  for  $n$ .

To obtain the ANOVA table, we 1<sup>st</sup> develop the linear additive model (LAM) as follows:

$$\text{LAM: } y_{ij} \equiv y_{ij} + \mu - \mu + \mu_i - \mu_i \equiv \mu + (\mu_i - \mu) + (y_{ij} - \mu_i) \equiv \mu + \tau_i + \epsilon_{ij} \quad (44)$$

In the LAM (44),  $\tau_i = \mu_i - \mu$  is called the effect of the  $i^{\text{th}}$  treatment ( $i = 1, 2, \dots, a$ ), and  $\epsilon_{ij} = y_{ij} - \mu_i$ ,  $j = 1, 2, \dots, n_i$ , is called the experimental error of the  $j^{\text{th}}$  observation at the  $i^{\text{th}}$  factor level, and  $\epsilon_{ij}$ 's are always assumed  $\text{NID}(0, \sigma^2)$ , i.e., in ANOVA one assumption is that different factor levels have the same population variance. Therefore, under model (44), our null hypothesis becomes  $H_0: \mu_i = \mu$ , or  $H_0: \tau_i = 0$  for all  $i$  VS  $H_1$ : at least one  $\tau_i \neq 0$ , i.e., under  $H_0$  all treatment effects are hypothesized to be zero, or put differently the factor has no impact (or effect) on

the response variable  $Y$  under  $H_0$ . Clearly,  $V(\epsilon_{ij}) = \sigma_\epsilon^2 = \sigma^2$ . Note that Devore uses  $X_{ij}$  for the dependent variable, but this notation is not as prevalent as using  $y_{ij}$  for the output (or the dependent variable). Recall that in calculus the dependent variable is generally denoted by  $y$ , while  $x$  represent the independent variable (or the abscissa).

Next, we replace the parameters in identity (44) by their corresponding point unbiased estimators in order to obtain the identity

$$y_{ij} \equiv \bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.}) \equiv \hat{\mu} + \hat{\tau}_i + e_{ij} = \hat{y}_{ij} + e_{ij} \quad (45)$$

For our example,  $\hat{y}_{ij} = \hat{\mu} + \hat{\tau}_i = \bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}) = \bar{y}_{i.}$ ,  $\hat{\mu} = \bar{y}_{..} = 16380.1 / (4 \times 6) = 682.50417$ ,  $\hat{\tau}_1 = 713.00 - 682.50417 = 30.49583, \dots$ ,  $\hat{\tau}_4 = 562.01667 - 682.50417 = -120.4875$ .

Note that  $e_{ij} = y_{ij} - \hat{y}_{ij}$  is called the residual for the  $j^{\text{th}}$  observation at the  $i^{\text{th}}$  level of the factor,

where  $\hat{y}_{ij} = \hat{\mu} + \hat{\tau}_i + \hat{\epsilon}_{ij} = \bar{y}_{i.}$ , because the best predictor of  $\epsilon_{ij}$  is zero. Further,  $\sum_{j=1}^{n_i} e_{ij} \equiv 0$  for

all  $i$ , while this last constraint does not apply to  $\epsilon_{ij}$ 's.

**Exercise 81.** Verify that equation (45) is indeed an identity, compute  $\hat{\tau}_2, \hat{\tau}_3$ , and

verify that  $\sum_{i=1}^4 \hat{\tau}_i \equiv 0$ . (b) Prove that for an unbalanced Design  $\sum_{i=1}^a n_i \hat{\tau}_i \equiv 0$ . In your

derivation, let  $n_T$  (T for total) =  $\sum_{i=1}^a n_i$  be the total number of all observations in the experiment.

You may also verify that the sum over  $j$  of the  $i^{\text{th}}$  residual  $e_{ij}$  is zero for all  $i$ , and that the double sum over  $i$  and  $j$  of all  $n_T$  residuals is always identically zero. (c) Assuming that  $\tau_i$ 's are constant, use model (44) to show that the  $V(y_{ij})$  is also equal to  $\sigma^2$ .

We now rewrite identity (45) as  $y_{ij} - \bar{y}_{..} \equiv (\bar{y}_{i.} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.}) \equiv \hat{\tau}_i + e_{ij}$ . This last identity breaks down the total deviation on the LHS into 2 separate orthogonal (i.e., additive) components: one among (or Between) factor levels ( $\hat{\tau}_i$ ) and the other Within the same factor levels ( $e_{ij}$ ). To obtain the test statistic for testing  $H_0$ , we first square both sides of this last identity and then sum over both  $i$  and  $j$ , i.e.,

$$\sum_{i=1}^a \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2 \equiv \sum \sum \left[ (\bar{y}_{i.} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.}) \right]^2 \quad (46)$$

**Exercise 82.** Show that in the above identity (46), the double sum of the cross-product terms vanishes and hence the identity reduces to

$$\begin{aligned} \sum \sum (y_{ij} - \bar{y}_{..})^2 &\equiv \sum_{i=1}^a \sum_{j=1}^{n_i} (\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^a \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 \\ \sum \sum (y_{ij} - \bar{y}_{..})^2 &\equiv \sum_{i=1}^a n_i (\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^a \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2. \end{aligned} \quad (47a)$$

$$SS(\text{Total}) \equiv SS(\text{Between Treatments}) + SS(\text{Within Treatments, or Subgroups})$$

The identity in (47a) is fundamental to the ANOVA of a one-factor experiment because it breaks down the total sum of squares on the LHS, denoted by  $SS(\text{Total}) = SS_T$  (Corrected Total Sum of Squares), into 2 orthogonal components: the first SS on the RHS describes the variability between treatments (or levels), and 2nd SS on the RHS describes the variability within treatments, i.e.,

$$SS(\text{Total}) = SS_{\text{Total}} = SS_T = SS(\text{Treatments}) + SS(\text{Experimental Error}) \quad (47b)$$

Some authors use  $SS_E = SS(\text{Error})$  for  $SS(\text{Experimental Error})$ , and my preference is to use  $SS(\text{Model})$  for  $SS(\text{Treatments})$ . Further,  $(\text{Total SS})/\sigma^2$  has a  $\chi^2$  distribution with  $n_T - 1 = (\sum n_i) - 1$  degrees of freedom (*df*),  $SS(\text{Treatments})/\sigma^2 \sim \chi_{a-1}^2$  and hence the distribution of  $(SS_{\text{Error}})/\sigma^2$  follows a  $\chi^2$  with  $(n_T - 1) - (a - 1) = (n_T - a)$  *df*, where  $N = n_T$  is also common and prevalent.

We next prove that under linear additive model (44),  $E(SS_E) = (N - a)\sigma^2$ .

$$\begin{aligned} E(SS_E) &= E \sum \sum (y_{ij} - \bar{y}_{i.})^2 = E \sum_{j=1}^{n_i} \left[ (\mu + \tau_i + \varepsilon_{ij}) - (\mu + \tau_i + \bar{\varepsilon}_{i.}) \right]^2 \\ &= \sum E \sum_{j=1}^{n_i} (\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2 = \sum_{i=1}^a (n_i - 1) \sigma^2 = (n_T - a) \sigma^2. \end{aligned} \quad \text{QED!}$$

This last result clearly shows that an unbiased estimator of  $\sigma^2$  is given by  $SS_E / (n_T - a)$ .

Before proceeding to the following Exercise, we must state that there are 2 types of the LAM (44): Fixed- and Random- Effects Models. Model (44) is fixed iff the levels of the factor are not selected at random (just like the example under consideration). When treatments are fixed, i.e., are not selected at random, then statistical conclusions pertain only to the levels under study. Since our example is one of fixed effects, then all conclusions must pertain only to box types 1, 2, 3, and 4. If the levels of a factor are selected at random from a population of treatments (like selecting 4 box types at random from a population of 40 available types), then model (44) represents the random effects model, in which case  $\tau_i$ 's are no longer constants (or fixed) but are rvs assumed to be  $NID(0, \sigma_\tau^2)$  and independent of  $\varepsilon_{ij}$ 's.

**Exercise 83.** For the fixed effects model, show that under model (44), the

$E(SS_{\text{Treatments}}) = \sum_{i=1}^a n_i \tau_i^2 + (a-1) \sigma^2$ . Therefore, under the null hypothesis  $H_0 : \tau_i = 0$  for all  $i$ , it

follows that another unbiased estimator of  $\sigma^2$  is given by  $SS(\text{Treatments}) / (a-1)$ . (b)

Show that for the fixed -effects model,  $\sum_{i=1}^a \tau_i \equiv 0$ , even if the design is unbalanced.

Thus far, we have established that Mean square (MS) of errors,  $MS(\text{Error}) = SS_E / (n_T - a)$ , is an unbiased estimator of  $\sigma^2$  even if  $H_0$  is false, and  $MS(\text{Treatments}) = SS(\text{Treatments}) / (a-1)$  is an unbiased estimator of  $\sigma^2$  only if  $H_0$  is true. Further, we have stated that  $SS(\text{Treatments}) / \sigma^2$  and  $SS(\text{Error}) / \sigma^2$  have independent  $\chi^2$  distributions with  $(a-1)$  and  $(n_T - a)$

$df$ , respectively. Recall from Chapter 9 that the random variable (rv),  $\frac{\chi_{v_1}^2 / v_1}{\chi_{v_2}^2 / v_2}$ , has Fisher's F

distribution with  $v_1$   $df$  for the numerator and  $v_2$   $df$  for the denominator. Applying this principal to ANOVA, it follows that the statistic

$$F_0 = \frac{SS(\text{Treatments}) / [\sigma^2(a-1)]}{SS(\text{Error}) / [\sigma^2(n_T - a)]} = \frac{MS(\text{Treatments})}{MS(\text{Error})} \quad (48)$$

has the Fisher's F distribution with  $v_1 = a - 1$  for the numerator, and  $v_2 = n_T - a$  denominator degrees of freedom. Note that the numerator of equation (48) is expected to exceed that of its denominator due to the fact that  $E(MS_{\text{Treatments}}) = \sigma^2 + \left(\sum_{i=1}^a n_i \tau_i^2\right) / (a - 1)$ .

When  $H_0$  is true, i.e.,  $\tau_i = 0$  for all  $i$ , then the value of  $F_0$  in (48) is expected to be close to 1 because  $E(MS_{\text{Error}}) = \sigma^2$  whether  $H_0$  is true or false. However, when  $H_0$  is false, the value of  $F_0$  is expected to far exceed 1. This constitutes a right-tailed test on  $H_0 : \tau_i = 0$  for all  $i$ , i.e.,  $H_0$  is rejected at the LOS  $\alpha$  only if the statistic  $F_0$  in (48) exceeds  $F_{\alpha}(a - 1, n_T - a) = F_{\alpha, a-1, n_T-a}$

**Exercise 84.** Show that the computing formulas for  $SS(\text{Total})$  and  $SS(\text{Treatments})$  are given by  $SS_T = SS(\text{Total}) = \sum \sum (y_{ij} - \bar{y}_{..})^2 = (\sum \sum y_{ij}^2) - y_{..}^2 / n_T = USS - CF$ ,  $SS(\text{treatments}) = \sum_{i=1}^a n_i (\bar{y}_{i.} - \bar{y}_{..})^2 = \left(\sum_{i=1}^a y_{i.}^2 / n_i\right) - CF$ , and hence  $SS(\text{Error}) = SS_T - SS(\text{Treatments})$ .

We are now in a position to develop the ANOVA table and conduct the F-test for the Example 10.1 on pages 392-397 of Devore.

**Solution:**  $CF = 16380.1^2/24 = 11179486.50417$  (with 1 *df*),  $USS = 655.5^2 + 788.3^2 + \dots + 520.0^2 = 11340700.23$  (with 24 *df*)  $\rightarrow SS_T = USS - CF = 161213.72958333$  (with 23 *df*).  $SS(\text{Model}) = (4278^2 + 4541.6^2 + 4188.4^2 + 3372.1^2)/6 - CF = 127374.754583$  (with 3 *df*). Hence  $SS(\text{Error}) = SS(\text{Total}) - SS(\text{Model}) = 33838.975$  (with 20 *df*). The ANOVA Table is given in Table 2 below. Note that the value of the F statistic is highly significant ( $P\text{-value} = 0.00000052545$ ), and thus the null hypothesis of no treatment effects (i.e.,  $\tau_i = 0$  for all  $i$ ) must be strongly rejected. It seems that level 2 of the factor (followed by Type 3) produces the highest CMPS.

**Exercise 85.** Show that for a balanced design  $E(e_{ij}) = 0$ , and  $V(e_{ij}) = (n - 1)\sigma^2/n$ , and thus the  $se(e_{ij}) = [(n - 1)MS_E/n]^{1/2}$ , where  $e_{ij}$  is the  $ij^{\text{th}}$  residual. (b) Use the results of part (a) to compute the values of the 3 Studentized residuals  $r_{21}$ ,  $r_{15}$  and  $r_{34}$ . (c) Redo the ANOVA table for the Example 10.1 coding the data by subtracting 650 from all  $y_{ij}$ 's.

**Exercise 86.** Work Exercises 3, 6 and 9 on pages 400-401 of Devore. Obtain a 95% CI

**Table 2 (The ANOVA Table for the Example 10.1 of Devore on p. 393)**

| SOURCE OF VARIATION  | <i>df</i> | SS           | MS = SS/ <i>df</i> | $F_0 =$<br>$MS_{BT}/MS_E$ | <i>PR LEVEL</i><br>$= P\text{-value}$ |
|----------------------|-----------|--------------|--------------------|---------------------------|---------------------------------------|
| Total                | 23        | 161213.72953 |                    |                           | $= \hat{\alpha}$                      |
| Model (OR Box Types) | 3         | 127374.75453 | 42458.25153        | 25.0943                   | 0.0 <sup>6</sup> 552545               |
| Rep ERROR            | 20        | 33838.975    | 1691.94875         | $F_{0.05,3,20} =$         | 3.0984                                |

for  $\mu_1$  of Exercise 6, and a 95% CI for  $\mu_1 - \mu_4$  of Exercise 9 on p. 401 of Devore.

## Tukey's Post-ANOVA For Fixed-Effects Only if $H_0$ is Rejected

### Example 10.1(pp. 392-397)

**Step 1:** Arrange the  $\alpha = 4$  level means in ascending order

$$\bar{y}_4 = 562.02 \quad \bar{y}_3 = 698.07 \quad \bar{y}_1 = 713.00 \quad \bar{y}_2 = 756.93$$

**Step 2:** Obtain the critical value of Tukey's (Studentized Range) from Table A.10 (p. Appendix A-20),  $Q_{0.05,4,20}$ , and use it to compute the 95% half confidence interval band as

$$w_{ij} = Q_{0.05,4,20} \times \sqrt{\frac{MS_{\text{Error}}(1/n_i + 1/n_j)}{2}} = 3.96 \times \sqrt{\frac{1691.94875(1/6 + 1/6)}{2}}$$

$$= 3.96 \times \sqrt{1691.94875/6} = 3.96 \times se(\bar{y}) = 66.4987$$

**Step 3:** Underline all the means in step 1 which do not differ by as much as  $w_{ij}$ .

$$\bar{y}_{4.} = 562.02 \quad \bar{y}_{3.} = 698.07 \quad \bar{y}_{1.} = 713.00 \quad \bar{y}_{2.} = 756.93$$


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**Step 4:** Use the underlined means to draw the conclusion that they came statistically from the same population from the standpoint of population means. Thus, Box types 3, 1, and 2 means are statistically the same; all the other pairs of means are significantly different at the 5% level.

**Exercise 87.** Work Exercises 11, 17 and 18 on page 407 of Devore.

### THE RANDOM-EFFECTS MODEL (SECTION 10-3, pp. 414-415)

Recall that this model is valid only when the levels of a factor are randomly selected from a population of levels (e.g., selecting 4 box types at random from a population of 40 available types). Box types would then form a random qualitative factor but the conclusions drawn from the ANOVA Table would pertain to the entire population of 40 types. Note that there will not be any interest in determining which 2 of the actually selected treatments (or box types) are significantly different (i.e., Tukey's procedure and orthogonal contrasts will not be applicable) but rather the objective is to determine if there is significant variation in the entire population of treatments (or box types). Therefore, our null hypothesis is  $H_0 : \sigma_{\tau}^2 = 0$  VS  $H_1 : \sigma_{\tau}^2 > 0$ , which again is clearly a right-tailed test. The ANOVA Table is obtained exactly as in the case of fixed-effects, but the post-ANOVA procedure is quite different. In the case of a random-effects model, if the F-test rejects the null hypothesis, then it is essential to estimate the components of variance in the LAM :  $y_{ij} = \mu + \tau_i + \epsilon_{ij}$ , where  $\mu = E(y_{ij})$  for all  $i = 1, 2, \dots, a$ , and  $j = 1, 2, \dots, n$ .

Note that we are considering only the simpler balanced case (i.e.,  $n_i = n$  for all  $i$ ) for the random-effects model. Further,  $\tau_i$ 's are rvs assumed  $NID(0, \sigma_{\tau}^2)$  so that the constraint

$\sum_{i=1}^a \tau_i = 0$  no longer applies to the random-effects model. The experimental errors,  $\epsilon_{ij}$ 's,

are as before NID  $(0, \sigma_{\varepsilon}^2)$  and independent of  $\tau_i$ 's. This leads to  $V(y_{ij}) = \sigma_{\tau}^2 + \sigma_{\varepsilon}^2 = \sigma_{\tau}^2 + \sigma^2$ , i.e., there are two variance components.

As an example of obtaining the ANOVA Table for a random-effects model see Example 10.11, on p. 414 of Devore. In order to estimate the 2 components of variance,  $\sigma_{\varepsilon}^2$  and  $\sigma_{\tau}^2$ , we must derive  $E(MS_E)$  and  $E(MS_{\text{Treatments}})$ . As in the case of fixed-effects, it can easily be verified that  $E(MS_E) = \sigma^2$ . We now prove that, for the balanced case, under the LAM (44),

$$E(MS_{\text{Treatments}}) = n \sigma_{\tau}^2 + \sigma^2. \quad \text{Proof:}$$

$$\begin{aligned} E(MS_{\text{Treatments}}) &= E \sum_{i=1}^a \sum_{j=1}^n (\bar{y}_{i.} - \bar{y}_{..})^2 / (a-1) = n E \sum_{i=1}^a (\bar{y}_{i.} - \bar{y}_{..})^2 / (a-1) \\ &= \frac{n}{a-1} E \sum [(\mu + \tau_i + \bar{\varepsilon}_{i.}) - (\mu + \bar{\tau} + \bar{\varepsilon}_{..})]^2 = \frac{n}{a-1} E \sum [(\tau_i - \bar{\tau}) + (\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})]^2 \\ &= \frac{n}{a-1} \left[ E \sum (\tau_i - \bar{\tau})^2 + E \sum (\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2 \right] = \frac{n}{a-1} \left[ (a-1) \sigma_{\tau}^2 + (a-1) \sigma_{\varepsilon}^2 \right] = \sigma^2 + n \sigma_{\tau}^2. \end{aligned}$$

$$\text{If the design were unbalanced, then } E(MS_{\text{Treatments}}) = \sigma^2 + \sigma_{\tau}^2 \left[ N - \sum_{i=1}^a (n_i^2 / N) \right] / (a-1).$$

As before, an unbiased estimator of  $\sigma^2$  is clearly  $MS_E$ , and the above proof shows that  $E(MS_{\text{Treatments}}) - \sigma^2 = n \sigma_{\tau}^2$ . Inserting  $E(MS_E)$  for  $\sigma^2$  in this last equation results in  $E(MS_{\text{Treatments}}) - E(MS_E) = n \sigma_{\tau}^2$ , or  $E[(MS_{\text{Treatments}} - MS_E)/n] = \sigma_{\tau}^2$ . This last equality clearly shows that an unbiased estimator of  $\sigma_{\tau}^2$  is indeed  $(MS_{\text{Treatments}} - MS_E)/n$ . We are now in a position to estimate the components of variance for the Example 10.11 on p. 415. From the ANOVA Table on page 415, the unbiased estimate of  $\sigma_{\varepsilon}^2$  is  $MS(\text{Error}) = 16.17$  and the unbiased estimate of  $\sigma_{\tau}^2$  is  $\hat{\sigma}_{\tau}^2 = (1862.1 - 16.17)/3 = 615.31$ . This implies that 97.44% of variation in travel times is attributed to the rails.

**Exercise 88.** Work Exercise 41 and on p. 417 of Devore.



### THE RANDOMIZED COMPLETE BLOCK DESIGN (RCBD)

Consider an experiment where it is desired to determine if there are significant differences among 3 computer languages (Java, VB, and C++) in coding a complex problem. Obviously we need human subjects for the DOE (or DOX) so that the 3 computer coding methods can be analyzed for efficiency. Suppose we select 10 programmers at random who are competent at all 3 languages to design 3 programs each, and as the response variable we measure the length of time that each programmer takes to complete a set of workable codes. Note that each programmer forms a Block and acts as his/her own control, and the experiment would be an Incomplete Block Design only if some of the programmers could not code in at least one of the 3 languages. The data are provided in Table 3 below. Further, randomization

**Table 3. (Experimental Layout)**

|                | programmer<br>1 | 2   | 3   | 4    | 5   | 6   | 7    | 8    | 9   | BLK<br>10 | y <sub>i</sub>         |
|----------------|-----------------|-----|-----|------|-----|-----|------|------|-----|-----------|------------------------|
| Java           | 2.5 Hrs         | 3.2 | 3.1 | 3.7  | 2.4 | 2.1 | 3.3  | 3.5  | 1.9 | 5.3       | 31.0                   |
| VB             | 3.1             | 2.5 | 2.8 | 4.7  | 2.9 | 2.9 | 4.4  | 4.4  | 2.5 | 4.6       | 34.8                   |
| C++            | 2.2             | 3.7 | 2.7 | 3.9  | 2.4 | 1.7 | 2.6  | 3.0  | 1.8 | 4.1       | 28.1                   |
| y <sub>j</sub> | 7.8 Hrs.        | 9.4 | 8.6 | 12.3 | 7.7 | 6.7 | 10.3 | 10.9 | 6.2 | 14.0      | y <sub>..</sub> = 93.9 |

takes place only within each block. Before developing the LAM and the corresponding E(SS) for the model, we first obtain the ANOVA Table and draw appropriate conclusions. The ANOVA calculations are: the CF =  $93.9^2 / 30 = 293.907$  (with 1 *df*), the USS =  $2.5^2 + 3.1^2 + \dots + 4.1^2 = 319.030$  (with 30 *df*) →  $SS_T = SS(\text{Total}) = 25.123$  (with 29 *df*). The treatment SS is computed using the subtotals pertaining to the 3 languages, i.e.,  $SS(\text{Treatments}) = (31^2 + 34.8^2 + 28.1^2) / 10 - CF = 2.2580$  (with 2 *df*);  $SS(\text{Blocks}) = (7.8^2 + 9.4^2 + \dots + 14.0^2) / 3 - CF = 18.749667$  (with 9 *df*), and thus  $SS(\text{Residuals}) = 4.115333$  (with 18 *df*). The ANOVA is given in Table 4.

**Table 4. (The ANOVA Table for the Data of Table 3)**

| Source of Variation        | df | SS       | MS      | F <sub>0</sub> | F <sub>0.05(2, 18)</sub> | P-value |
|----------------------------|----|----------|---------|----------------|--------------------------|---------|
| Total                      | 29 | 25.123   |         |                |                          |         |
| Computer Languages         | 2  | 2.258    | 1.129   | 4.9381         | 3.55                     | 0.01951 |
| Programmers (or Blocks)    | 9  | 18.74967 |         |                |                          |         |
| Residuals (or Left-Over's) | 18 | 4.115333 | 0.22863 |                |                          |         |

The ANOVA Table 4 clearly indicates that there are significant differences among the 3 programming languages (i.e., the Treatments) at the *Pr level*  $\hat{\alpha} = 0.01951$ , i.e., we must reject the null hypothesis  $H_0: \tau_1 = \tau_2 = \tau_3 = 0$  at levels of significance as small as 0.01951.

**Exercise 89.** Analyze the result of the above experiment as a CRD, i.e., the order of the 30 observations were completely randomized. ANS:  $F_0 = 1.3332$ .

We now develop the LAM for a RCBD. As before we start with an identity!

$$y_{ij} \equiv y_{ij} + \mu - \mu + \mu_i - \mu_i + \mu_j - \mu_j \equiv \mu + (\mu_i - \mu) + (\mu_j - \mu) + (y_{ij} - \mu_i - \mu_j + \mu) \equiv \mu + \tau_i + B_j + \epsilon_{ij} \quad (49)$$

Model (49) shows that  $\hat{y}_{ij} = \hat{\mu} + \hat{\tau}_i + \hat{B}_j + 0 = \bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}) + (\bar{y}_{.j} - \bar{y}_{..}) = \bar{y}_{i.} + \bar{y}_{.j} - \bar{y}_{..}$ . Thus, the  $ij^{\text{th}}$  residual is given by  $e_{ij} = y_{ij} - \hat{y}_{ij} = y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}$ . The term  $B_j = \mu_j - \mu$  in (49) is called the effect of the  $j^{\text{th}}$  block, and the definitions of the other 2 terms on the RHS of (49) are self-explanatory! Again, we replace the model parameters in equation Eq. (49) by their corresponding point unbiased estimators and obtain:

$$y_{ij} \equiv \bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}) + (\bar{y}_{.j} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})$$

$$y_{ij} - \bar{y}_{..} \equiv (\bar{y}_{i.} - \bar{y}_{..}) + (\bar{y}_{.j} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}) \quad (50)$$

We 1st square both sides of (50) and then sum from  $i = 1$  to " $a$ " (the number of treatments) and also sum from  $j = 1$  to " $b$ " (the no. of blocks), i.e.,

$$\sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \bar{y}_{..})^2 \equiv b \sum_{i=1}^a (\bar{y}_{i.} - \bar{y}_{..})^2 + a \sum_{j=1}^b (\bar{y}_{.j} - \bar{y}_{..})^2 + \sum \sum (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$$

$$SS_{\text{Total}} \equiv b \sum_{i=1}^a \hat{\tau}_i^2 + a \sum_{j=1}^b \hat{B}_j^2 + \sum \sum e_{ij}^2 \quad (51)$$

Note that the 1<sup>st</sup> term on the RHS of (51) gives  $SS_{\text{Treatments}} = SS_{\text{Tr}}$ , the 2<sup>nd</sup> term gives  $SS(\text{BLKS})$ , and the last term gives  $SS(\text{Residuals})$ .

**Exercise 90.** Show that equation (51) is indeed an identity because the double sum of the 3 cross-product terms on the RHS vanish.

**Exercise 91.** For the mixed-effects (i.e., fixed treatments but random blocks), show that

$$E(SS_{\text{Treatments}}) = \sigma^2(a-1) + b \sum_{i=1}^a \tau_i^2, \text{ and } E(SS_{\text{Blocks}}) = (b-1)(\sigma^2 + a\sigma_\beta^2), \text{ and } E(SS_E) =$$

$(a-1)(b-1)\sigma^2$ , where  $\sigma_\beta^2$  represents the variance of the random blocks. Further, show that for a RCBD  $V(e_{ij}) = [(a-1)(b-1)/ab]\sigma_\epsilon^2$ .

It should be quite clear that identity (51) is fundamental to the analysis of RCBD experiments because it breaks down the Total SS on the LHS into 3 orthogonal (i.e., additive) components: the 1st SS on the RHS is due to Treatments, the 2nd is due to Blocks, and the 3rd SS is due to Residuals. The *df* of Treatments is  $(a-1)$ , that of Blocks is  $(b-1)$ , and hence the *df* of Residuals is  $(ab-1) - (a-1) - (b-1) = (a-1)(b-1)$ , where  $ab = n_T$ .

**Exercise 92.** Work Exercise 36 on page 371-372 of Devore using ANOVA procedure and compare your result against that of the paired t-test. (b) Then work Exercise 38 on page 372 of Devore in 2 different ways: (i) using the paired t-test, (ii) using ANOVA for a RCBD, and show that your results are exactly identical.

## Bartlett's Test for Equality of More Than Two Treatment Variances

### This test is always right-tailed

The null hypothesis is  $H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \dots = \sigma_a^2$  versus the alternative  $H_1$ : at least 2 of the "a" variances are significantly different, where  $a$  stands for number of populations.

The testing procedure is to compute the Bartlett's statistic, given below, whose sampling distribution is approximately chi-squared with  $(a-1)$  degrees of freedom. Thus, if we are comparing 4 ( $= a$ ) treatments, e.g., 4 Box-types on p. 393 of Devore (8e), the degrees of freedom will be 3. The test

statistic is  $\chi_0^2 = q/c$ , where  $q = (N-a)\ln(S_p^2) - \sum_{i=1}^a (n_i - 1)\ln(S_i^2)$ , the constant  $c = 1 +$

$[\sum_{i=1}^a (n_i - 1)^{-1} - (N - a)^{-1}]/[3(a - 1)]$ ,  $S_p^2 = \sum_{i=1}^a \text{CSS}_i^2 / (N - a)$ , and  $N = n_T$ . If  $\chi_0^2 > \chi_{0.05, a-1}^2$ , reject

$H_0$  at the 5% LOS. When the design is balanced, i.e., all  $n_i$ 's =  $n$ , then  $S_p^2 = \sum_{i=1}^a S_i^2 / a$ ,  $c = 1 +$

$(a+1)/[3a(n-1)]$ , and  $q = (N-a)\ln(S_p^2) - (n-1)\sum_{i=1}^a \ln(S_i^2)$ . For the Example 10.1 on pp. 392-3 of

Devore (8e), because  $n_i$ 's are all equal to 6, then  $S_p^2 = \sum_{i=1}^a S_i^2 / a = \sum_{i=1}^4 S_i^2 / 4 = \text{MS}_{\text{RES}} = 1691.529366$ ,

and  $N = n_T = \sum_{i=1}^{a=4} n_i = 24$ . Thus,  $q = (24-4)\ln(S_p^2) - 5\sum_{i=1}^4 \ln(S_i^2) = 0.274591$ ,  $c = 1 + 5/[12(5)] =$

$1.08333333$ , and  $\chi_0^2 = q/c = 0.25347$ ;  $\chi_{0.05, 3}^2 = 7.81473 \rightarrow$  The Bartlett's statistic  $\chi_0^2 = 0.25347$

is not at all significant even at the 90% level because the posterior Pr level of the testing  $H_0$  is  $P\text{-value} = \Pr(\chi_3^2 \geq 0.25347) = 0.96852824 > 0.90$ . Thus, we conclude that the variances of the 4 box-types are quite homogeneous.