Reference: Chapter 5 of Devore (8e) **Maghsoodloo**

JOINT PROBABILITY DISTRIBUTION FUNCTIONS

Consider two production lines that manufacture a certain item. The production rates for both lines vary randomly from day to day. Line 1 has a capacity of 4 units per day while line II has a capacity of 3 units per day. Further, both lines produce at least one unit on any given day. Let X_1 = No. of units produced by line I/day, and X_2 = No. of units produced by line II per day. The joint probability (Pr) distribution (JPD) of the bivariate vector X = $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is given below:

X_1 \ X_2	1	2	3	$p_1(x_1)$
1	0.01	0.05	0.04	0.10
2	0.05	0.10	0.10	0.25
3	0.10	0.15	0.10	0.35
4	0.04	0.15	0.11	0.30
$p_2(x_2)$	0.20	0.45	0.35	

The above table implies that the joint Pr $P(X_1 = 2, X_2 = 3) = p(2, 3) = 0.10$, and $p(4, 2) = 0.15$, etc. Further, $p_1(x_1)$ and $p_2(x_2)$ are referred to as the marginal Pr distributions (mpds) of X_1 and X_2, respectively. Note that $p_1(x_1) = \sum_{R_2} p(x_1, x_2)$ and $p_2(x_2) = \sum_{R_{x1}} p(x_1, x_2) = \sum_{R_1} p(x_1, x_2)$. Further,

$\mu_1 = E(X_1) = 0.10 + 0.50 + 1.05 + 1.20 = 2.85$ units/day, and

$\mu_2 = E(X_2) = 0.20 + 0.90 + 1.05 = 2.15$ units/day.

Similarly,

$E(X_1^2) = 9.05 \longrightarrow \sigma_1^2 = \sigma_{11} = 0.9275 \longrightarrow \sigma_1 = 0.9631$

$E(X_2^2) = 5.15 \longrightarrow \sigma_2^2 = \sigma_{22} = 0.5275 \rightarrow \sigma_2 = 0.7263.$

The covariance between 2 random variables (rvs) is defined as:

$$\sigma_{12} = COV(X_1, X_2) = E\,[(X_1 - \mu_1)(X_2 - \mu_2)] = E\,(X_1X_2) - \mu_1\mu_2.$$

For the above example,

$$E(X_1X_2) = 0.01 + 2\times0.05 + 3\times0.04 + 2\times0.05 + 4\times0.10 + 6\times0.10 + 3\times0.10 + 6\times0.15 + 9\times0.10 + 4\times0.04 + 8\times0.15 + 12\times0.11 = 6.11$$

$$\rightarrow \quad \sigma_{12} = 6.11 - 2.85\,(2.15) = -\,0.0175$$

The covariance matrix of the bivariate random vector X is given by:

$$COV(X) = COV\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0.9275 & -0.0175 \\ -0.0175 & 0.5275 \end{bmatrix}$$

Note that the covariance matrix Σ is always symmetrical because $\sigma_{ij} = \sigma_{ji}$ for all $i \neq j$.

Further, covariance must be taken only between two rvs at a time (not 3 or more).

The correlation coefficient between X_1 and X_2 is defined as:

$$\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\,\sigma_{22}}} = \frac{\sigma_{12}}{\sigma_1\,\sigma_2} = \frac{-0.0175}{(0.9631)(0.7263)} = -\,0.02502\,.$$

It can be shown that $-1 \leq \rho \leq +1$, where $\rho = 0$ implies no correlation between X_1 and X_2 ($\rho = 0$ does not always imply that X_1 and X_2 are independent but shows that there is no linear relationship between X_1 and X_2). A value of $\rho = \pm 1$ implies perfect correlation between X_1 and X_2. A positive $0 < \rho \leq 1$ implies that the relationship between x_1 and x_2 is linearly increasing and vice a versa when $-1 \leq \rho < 0$. For example, there is a positive correlation between X_1 = the amount of irrigation, and X_2 = crop yield. While, there is a negative association between X_1 = width of road, and X_2 = accident rate.

CONDITIONAL PROBABILITY DISTRIBUTIONS

The conditional Pr distribution of X_2 given $X_1 = x_1$ is defined as:

$$p_2(x_2 \mid x_1) = \frac{p(x_1, x_2)}{p_1(x_1)}, \text{ and similarly, } p_1(x_1 \mid x_2) = \frac{p(x_1, x_2)}{p_2(x_2)}\,.$$

As an example, for the JPD on page 69, $p_2(x_2 | X_1 = 1)$ = $\frac{p(1, x_2)}{0.10}$, i.e.,

$p_2(x_2 | X_1 = 1)$ = $\begin{cases} 0.10, & x_2 = 1 \\ 0.50, & x_2 = 2 \\ 0.40, & x_2 = 3 \end{cases}$, while $p_1(x_1 | X_2 = 3)$ = $\begin{cases} 4/35, & x_1 = 1 \\ 10/35, & x_1 = 2, 3 \\ 11/35, & x_1 = 4. \end{cases}$

Exercise 36.

(a) Obtain $p_2(x_2 | X_1 = i)$, i = 2, 3, or 4.

(b) Obtain $p_1(x_1 | X_2 = i)$, i = 1 or 2.

CONDITIONAL EXPECTATIONS

These are defined as follows: E ($X_2 | x_1$) = $\sum_{R_2} x_2 p_2(x_2 | x_1)$, and

E ($X_1 | x_2$) = $\sum_{R_1} x_1 p_1(x_1 | x_2)$, where $R_1 = R_{x_1}$ and $R_2 = R_{x_2}$. For example,

E ($X_2 | X_1$ = 1) = 0.10 +1 + 1.20 = 2.30, and E ($X_1 | X_2$ = 3) = 2.80.

Exercise 36 (continued).

(c) Compute E ($X_2 | X_1$ = i), i = 2, 3, or 4 and E ($X_1 | X_2$ = i), i = 1 or 2. ANS: E ($X_1 | X_2$ = 1) = 2.850.

Note that for any bivariate random vector X, it is always true that $p(x_1, x_2) = p_1(x_1) \times p(x_2 | x_1) = p_2(x_2) \times p(x_1 | x_2)$. For the JPD on page 69, p(1, 3) = 0.04, $p_1(1) \times p_2(3 | X_1 = 1)$ = 0.10 (4/10) = 0.04, or p_2 (X_2 = 3) = 0.35 , $p_1(X_1 = 1 | X_2 = 3)$ = 4/35, p_2 (X_2 = 3) × $p_1(X_1 = 1 | X_2 = 3)$ = 0.35 (4/35) = 0.04 = p(1,3).

Exercise 36 (continued). (d) Verify that p(3,2) = $p_1(3) \times p_2(X_2 = 2 | X_1 = 3) = p_2(2) \times p_1(X_1 = 3 | X_2 = 2)$. (e) Compute the $P(X_1 > 1 | X_2 > 2)$. ANS: (e) $P(X_1 > 1 | X_2 = 3)$ = 31/35.

INDEPENDENCE OF TWO RANDOM VARIABLES

Two random variables, X_1 and X_2, are independent iff (if and only if) $p(x_1, x_2) = p_1(x_1)\times p_2(x_2)$. If X_1 and X_2 are independent, then always $\sigma_{12} = 0$ and hence $\rho = 0$. Note that the converse of this last claim is not necessarily true (see Exercise 38 below) unless the random vector $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has a bivariate normal density function. In short, two rvs are independent iff their JPDF factors out into the product of the individual mpds.

Exercise 37. A shop has 2 machines M_1 and M_2. Let the rv X_i = Number of defective units produced per hour on M_i ($i = 1, 2$). The JPDF of random vector $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is given below. **(a)** Obtain the mpdfs of X_1 and X_2 and the covariance matrix Σ = COV($\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$). Then compute the correlation coefficient ρ to 5 decimals.

X_1 \ X_2	1	2	3	4	$p_1(x_1)$
0	0.02	0.08	0.08	0.02	
1	0.03	0.12	0.12	0.03	
2	0.03	0.12	0.12	0.03	
3	0.02	0.08	0.08	0.02	
$p_2(x_2)$					

(b) Compute $E(X_2 | X_1 = 3)$, $E(X_2 | X_1 = 2)$ and $E(X_2)$. **(c)** Compute the $P(X_2 > 2 | X_1 = 1)$ **(d)** Determine if X_1 and X_2 are independent and why.

Exercise 38. Repeat all parts of Exercise 37 for the following JPDF.

X_1 \ X_2	0	1	2	3	$p_1(x_1)$
0	1/12	1/12	1/12	1/12	
1	1/12	0	0	1/12	
2	1/12	0	0	1/12	
3	1/12	1/12	1/12	1/12	
$p_2(x_2)$					

CONTINUOUS BIVARIATE RANDOM VARIABLES

Suppose X_1 represents surface tension and X_2 represents the acidity of the same sampling unit of a chemical product. The joint probability density function (jpdf) of the random vector X = $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is given by

$$f(x_1, x_2) = C(6 - x_1 - x_2), \qquad 0 \le x_1 \le 2, \quad 2 \le x_2 \le 4.$$

Example 33. **(a)** Determine the value of the above constant C such that $f(x_1, x_2)$ is a jpdf, i.e., find C such that the volume under $f(x_1, x_2)$ and rectangular region $R_X = [0 \le x_1 \le 2,$ and $2 \le x_2 \le 4]$ is equal to 1 (or 100% probability). That is,

$$C \int_{x_2=2}^{4} \int_{x_1=0}^{2} (6 - x_1 - x_2)\,dx_1 dx_2 = C\int_2^4 \left[6x_1 - \frac{x_1^2}{2} - x_2 x_1\right]_0^2 dx_2 \ \underline{\underline{\text{Set to}}}\ 1$$

$$C\int_2^4 (12 - 2 - 2x_2)\,dx_2 = C\left[10x_2 - x_2^2\right]_2^4 = C(24 - 16) = 8C = 1 \ \rightarrow C = 0.125.$$

Thus $f(x_1, x_2) = 0.125(6 - x_1 - x_2)$ is a jpdf over R_X: $0 \le x_1 \le 2$, $2 \le x_2 \le 4$ because the volume under $f(x_1, x_2)$ is identically equal to 100%.

(b) Compute the joint Pr that a randomly selected unit has a surface tension less than 1 and an acidity not exceeding 3.

$$P(X_1 \le 1, X_2 \le 3) = \frac{1}{8}\int_0^1 \int_2^3 (6 - x_1 - x_2)dx_2 dx_1 = 3/8 = 0.3750 = F_{X1,X2}(1, 3)$$

It can be shown that the Joint-cdf of the above joint-pdf is given by

$$F(x_1, x_2) = 0.125\int_0^{x_1} \int_2^{x_2} (6 - x_1 - x_2)dx_2 dx_1 = 0.125(x_1^2 + 6x_1x_2 - 10x_1 - x_1x_2^2/2 - x_1^2x_2/2),$$

$0 \le x_1 \le 2$, $2 \le x_2 \le 4$. Note that $\frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} = f(x_1, x_2)$.

(c) We next compute the $P(X_1 + X_2 \le 4)$, or the $P(X_2 \le 4 - X_1)$.

$$P(X_1 + X_2 \le 4) = 0.125 \int_{x_1=0}^{2} \int_{x_2=2}^{4-x_1} (6 - x_1 - x_2)dx_2\, dx_1 = 2/3.$$

Exercise 39. **(a)** Re-compute the above $P(X_1 + X_2 \le 4)$ by integrating with respect to (wrt) x_1 first followed by x_2.

MARGINAL PROBABILITY DENSITY FUNCTIONS (mpdf)

Analogous to the discrete case, the mpdf of the continuous rv X_1 is defined as

$$f_1(x_1) = \int_{R_2} f(x_1, x_2)\,dx_2 = \int_{x_2=2}^{4} 0.125(6 - x_1 - x_2)\,dx_2 = \frac{3 - x_1}{4}, \quad 0 \le x_1 \le 2.$$

$$\text{Therefore, } E(X_1) = \int_0^2 x_1 f_1(x_1)\,dx_1 = \int_0^2 x_1 \frac{3 - x_1}{4} dx_1 = 5/6.$$

Exercise 39 (b). Obtain the mpdf of X_2 for the Example 33 and verify that both $f_1(x_1)$ and $f_2(x_2)$ are indeed probability density functions. **(c)** Compute the $\Pr(X_2 \le 3)$ and $E(X_2)$.

CONDITIONAL PROBABILITY DENSITY FUNCTIONS

The conditional pdf of X_2 given x_1 is defined as $f(x_2|x_1) = \dfrac{f(x_1, x_2)}{f_1(x_1)} = \dfrac{6 - x_1 - x_2}{2(3 - x_1)}$, $2 \le x_2 \le 4$. Since the expression for $f(x_2|x_1) = \dfrac{6 - x_1 - x_2}{2(3 - x_1)}$ is not free of x_1, then the rv X_2 is not independent of X_1.

<u>Exercise 39(d).</u> Verify that $f(x_2|x_1)$ is indeed a pdf over the range $R_2 = [2, 4]$. Then obtain $f(x_1|x_2)$ and determine if X_1 is independent of X_2. Verify your answer over the range $R_1 = [0, 2]$.

CONDITIONAL EXPECTATIONS

The conditional expectation of X_2 given the value of x_1 is defined as

$$E(X_2|x_1) = \int_{R_2} x_2 f(x_2|x_1)\,dx_2 = \int_2^4 x_2\left(\frac{6 - x_1 - x_2}{6 - 2x_1}\right)dx_2 = \frac{26 - 9x_1}{3(3 - x_1)}.$$

Note that because X_2 is not independent of X_1, then the $E(X_2|x_1)$ is a function of x_1 over the range space $R_1 = [0, 2]$.

<u>Exercise 39(e)</u>. Compute $E(X_2|X_1 = 0.50)$ and obtain $E(X_1|x_2)$ and use it to re-compute the unconditional expectation $E(X_1)$. Use $E(X_2|x_1)$ and $f_1(x_1)$ to re-compute the unconditional $E(X_2)$. **(f)** Obtain the covariance matrix Σ. (ANS: $\sigma_{11} = 11/36$, $\rho = -1/11$). **(g)** Obtain the $V(X_2|x_1)$.

<u>Exercise 40.</u> **(a)** Show that $-1 \le \rho \le 1$ for all bivariate random vectors. Hint: Expand $V(c_1X_1 + c_2X_2)$ and use the fact that $V(c_1X_1 + c_2X_2) \ge 0$ for all choices of real constants c_1 and c_2. **(b)** Show that $\rho = +1$ if $X_2 = a + bX_1$, but $\rho = -1$ when $X_2 = a - bX_1$, where the constant $b > 0$.

<u>Exercise 41.</u> Consider the uniform joint-pdf

$$f(x_1,x_2) = \begin{cases} 1 , & 0 \le x_1 \le 1 , \quad -x_1 \le x_2 \le x_1 \\ 0 , & \text{elsewhere.} \end{cases}$$

(**a**) Draw the triangular region $R_X = [0 \le x_1 \le 1, -x_1 \le x_2 \le x_1]$ and obtain the covariance matrix Σ. **(b)** Verify that $\rho = 0$ but yet X_1 and X_2 are not independent. **(c)** Show that the joint-cdf is given by $F(x_1,x_2) = \begin{cases} x_1x_2 + 0.5(x_1^2 + x_2^2), & -1 \le x_2 \le 0 \\ x_1x_2 + 0.5(x_1^2 - x_2^2), & 0 \le x_2 \le 1 \end{cases}$. Again, note that $\dfrac{\partial^2 F(x_1,x_2)}{\partial x_1 \partial x_2} = f(x_1, x_2)$. **(d)** Work Exercises 9, 13, and 17 on pp. 204-205 of Devore (8e).

Note that a necessary (but not sufficient) condition for two rvs to be independent is that their range space, or SPUS, R_X must be rectangular.

LINEAR COMBINATIONS (WHEN INDIVIDUAL COMPONENTS of the LC MAY BE CORRELATED)

Suppose $X_1, X_2, ..., X_n$ are random variables with known means μ_1, $\mu_2, ..., \mu_n$ and known variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_2^2$, respectively, and covariances σ_{ij} $(i \ne j)$. Then the rv $Y = \sum_{i=1}^{n} c_i X_i$, where c_i's are known constants, is called a linear combination (LC). In other words, we have complete information about the 1st two moments of the n inputs X_i's, and the objective is to use them to compute E (Y) and V (Y), i.e., the 1st two moments of the linear output Y, as shown below.

$$\mu_Y = E\,(Y) = E\,[\sum_{i=1}^{n} c_i X_i\,] = \sum_{i=1}^{n} c_i E(X_i) = \sum_{i=1}^{n} c_i\, \mu_i \qquad (31a)$$

Note that the E(Y) is the same LC of μ_i's as Y is of X_i's! We next compute the σ_y^2 by applying the nonlinear variance operator V.

$$\sigma_y^2 = V(Y) = E(Y-\mu_y)^2 = E\left[(\sum_{i=1}^{n} c_i X_i - \sum_{i=1}^{n} c_i \mu_i)^2\right] = E\left[\sum c_i (X_i - \mu_i)\right]^2$$

$$= E\left[\sum_{i=1}^{n} c_i^2 (X_i-\mu_i)^2 + \sum_{j\neq i}^{n}\sum_{i=1}^{n-1} c_i c_j (X_i-\mu_i)(X_j-\mu_j)\right]$$

$$= \sum_{i=1}^{n} c_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_i c_j E\left[(X_i-\mu_i)(X_j-\mu_j)\right] =$$

$$= \sum_{i=1}^{n} c_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_i c_j \sigma_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \sigma_{ij} \qquad (31b)$$

If the rvs $X_1, X_2, ..., X_n$ are independent, then σ_{ij}'s in equation (31b) are all zero for any i ≠ j and as a result the V(Y) reduces to $\sum_{i=1}^{n} c_i^2\sigma_i^2$, as before. Further, if X_i's are also normally distributed (besides being jointly independent), then Y ~ $N(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2)$. For example, the sample mean $\bar{\mathbf{x}}$ from a normal universe is a LC whose c_i = 1/n for all i = 1, 2, …, n so that $\bar{x}$ ~ N(μ, $\sum_{i=1}^{n}(1/n)^2\sigma_x^2$), or $\bar{x}$ ~ N(μ, σ_x^2 / n). However, if X_i's are correlated (i.e., $\sigma_{ij} \neq 0$) and normally distributed, then the linear combination Y = $\sum_{i=1}^{n} c_i X_i$ is also Gaussian with E(Y) = $\sum_{i=1}^{n} c_i \mu_i$ and V(Y) = $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \sigma_{ij}$.

SIMPLE RANDOM SAMPLING

Suppose X is a continuous random variable with pdf f(x; μ, σ^2) and let a random sample of size n be drawn from this population. Denote the n sample values by $x_1, x_2, ...,$

x_n ; then $X_1, X_2, ..., X_n$ are random variables with pdfs $f_1(x_1), f_2(x_2), ..., f_n(x_n)$. The method of sampling, which possesses the following two properties, is called random sampling:
(1) $X_1, X_2, ..., X_n$ are mutually independent. (2) $f(x_i) = f(x)$ for all i.
Therefore, if $X_1, X_2, ..., X_n$ are elements of a random sample, then $E(X_i) = \mu$ and $V(X_i) = \sigma^2$ for all i because all X_i's are identically distributed like the parent pdf $f(x; \mu, \sigma^2)$.

Exercise 42. Let $\bar{X}$ be the mean of a random sample of size n from a population with mean μ and variance σ^2. **(a)** Show that $E(\bar{X}) = \mu$ and $V(\bar{X}) = \sigma^2/n$. **(b)** Further, if the population is normal, then $\bar{X}$ is also $N(\mu, \sigma^2/n)$. **(c)** Now consider the LC: $Y = 2X_1 – 3X_2 – 4X_3 + 5X_4$, where $\mu_1 = 50$, $\mu_2 = \mu_3 = 25$, $\mu_4 = 35$, $\sigma_1^2 = \sigma_4^2 = 1.25$, $\sigma_2^2 = \sigma_3^2 = 1.95$, $\sigma_{12} = 1.40$, $\sigma_{34} = 1.20$ and all other covariances are 0. Assuming that Y is normally distributed, compute the $Pr(Y > 110)$. Part(c) ANS for $\sigma_{34} = 1.20$: 0.013042

Exercise 43. Work Exercises 1, 3, 15, 37, 39, 42, 46, 47, 50, 53, 56, 58, 59, 60, 65, 73, 76, 77, and 78 on pages 203-236 of Devore's 8th Edition.

Exercise 44. The smog content of air in a certain area is monitored daily. The acceptable content of a particular constituent is at 7.7%. If the actual content, X, of this constituent is N (7.6, 0.0016), and the measuring instrument has an error ε which is N (0, 0.0009), compute: **(a)** The Pr that a single measurement will exceed 7.7%, **(b)** The Pr that the mean of 5 measurements is less than 7.55. ANS: (a) 0.02275, (b) 0.012674.

Exercise 45. Suppose X, Y and Z are NID (normally and independently distributed) with means 100, 48, 48 and variances 10, 13 and 13, respectively. Compute the $Pr(X > Y + Z)$. ANS: 0.74751.