Sample (or Outcome) Space: This is the universe or the set of all possible outcomes of an experiment (denoted by U). An experiment is a test, an action or a process that generates observation(s). Experiments can be deterministic or random. Random experiments are also called nondeterministic or stochastic. All experiments in this course are stochastic (or random) in nature.

Example 3. Two fair dice, red and white, are tossed and the "up" faces are observed. The sample space is given by the set: U = {(1, 1), 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, (6,6)}, where 32 = (3 on red, 2 on white) for brevity. Or U = {(i, j) | i, j = 1, 2, 3, 4, 5, 6}. Note that the combination "12" = (1, 2) means that the red die comes up 1 and the white die shows 2, while the ordered pair "21" = (2, 1) implies just the opposite. Further, the universe U contains N(U) = 36 outcomes because $6 \times 6 = 36$ and all outcomes are equally likely because both dice are fair and balanced.

Example 4. A lot of 10 items has 2 nonconforming units (NCUs) and 8 good units. One unit is drawn after another at random (and W/O replacement) until the last defective unit is identified. The drawing is stopped ASA the 2nd defective is removed from the lot. Let C denote a conforming unit and D denote a NCU (or defective). Then, the set of all possible outcomes is U = {DD, (DCD, CDD), (DCCD, CDCD, CCDD), (DCCCD, CDCCD, CCDCD, CCCDD),…, CCCCCCCDCD, CCCCCCCCDD}, where N(U) =

$$\sum_{i=2}^{10} (i-1) = \sum_{i=1}^{9} i = 9\,(10)\,/\,2 = 45 \text{ outcomes.}$$

Example 5. Three coins (nickel, dime and quarter) are tossed and the 3 faces are observed. The sample space for this experiment is U = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}, where N(U) = 8.

Definition: An event, A, is a set of possible outcomes or simply a subset of the universe U. If an event contains no outcome, it is said to be null or empty, denoted by ϕ; if an event has exactly one outcome, it is called simple (or elementary); if it has more than one outcome, then it is said to be a compound event.

Example 6. Refer to Example 3 above: Define the event A = {the 2 faces of the dice sum to a total of 7}. Then A = {16, 25, 34, 43, 52, 61} is a compound event for which N(A) = 6.

Example 7. Refer to Example 4. Define B = {At most 5 units are drawn at random in order to remove both NCUs from the lot} = {DD, DCD, CDD, DCCD, CDCD, CCDD, DCCCD, CDCCD, CCDCD, CCCDD}, where N(B) =10.

Example 8. Refer to Example 5: Define the event C = {Exactly 2 heads come up} = {HHT, HTH, THH}, where N(C) = 3.

Since events are sets, then we define and conclude that : (i) A′ = The compliment of event A = The event that A does not occur, and is also sometimes denoted by $\bar{A}$. (ii) $A \cup B$ = At least one of the 2 events occur = either A or B occurs (or both A and B occur). (iii) $A \cap B$ occurs iff Both A and B occur simultaneously. Events A and B are mutually exclusive (MUEX) iff $A \cap B = \phi$ (the null set).

Example 9. Consider the Exercise 4 on page 55 of Devore(8e). (a) U = {FFFF, FFFV, FFVF, FVFF, VFFF, FFVV, FVFV, FVVF, VFFV, VFVF, VVFF, FVVV, VFVV, VVFV, VVVF, VVVV}, where N(U) =16. (b) The event B ={FFFV, FFVF, FVFF, VFFF}, where N(B) = 4. (c) C = {FFFF, VVVV}. (d) The event D = {FFFF, FFFV, FFVF, FVFF, VFFF }, where N(D) = 5. (e) $C \cup D$ = {FFFF, FFFV, FFVF, FVFF, VFFF, VVVV } and $C \cap D$ = {FFFF}.

Definition: The occurrence probability (Pr, or P) of an event A is defined as the function $P(A) = \frac{N(A)}{N(U)}$, iff all the elementary outcomes in U are equally likely.

Example 10. Refer to the Example 6 above: P(A) = 6/36 = 1/6. In Example 7, P(B) = 10/45 because all the outcomes in U are equally likely. For the Example 8 above, P(C) = 3/8.

The function P(A) = Pr(A) must satisfy the following axioms:

(1) $0 \le P(A) \le 1$ because $0 \le N(A) \le N(U)$.

(2) $P(U) \equiv 1$, and $P(\phi) \equiv 0$, i.e., the Pr of a null set is identically zero.

(3) If A and B are MUEX, then P(A ∩B) = P(∅) = 0, and P(A∪B) = P(A) + P(B); further, if A_1, A_2, A_3, ... are a accountably infinite jointly MUEX events, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Because A and A′ = $\bar{A}$ are MUEX, then A∪A′ = U and as a result P(A∪A′) = 1 = P(A) + P(A′) → P(A′) = 1 – P(A), or P($\bar{A}$ = A′) = 1 – P(A) = Q(A).

How do we assign probabilities to an event A?

(1) By experimentation, **(2)** Estimates from prior data, **(3)** Analytical consideration of experimental conditions, **(4)** Subjective assumptions. For example, If the odds in favor of an event A are 4 to 3, then its occurrence probability is P(A) = 4/(4+3) = 4/7, i.e., we are willing to bet \$4.00 in order to win \$3.00. In general, if odds in favor of an event A are n_1 to n_2, then we are making the subjective assumption that its occurrence Pr is P(A) = $\frac{n_1}{n_1 + n_2}$ $\rightarrow (n_1 + n_2)P(A) = n_1 \rightarrow n_2 P(A) = n_1[1 - P(A)] \rightarrow \frac{n_1}{n_2} = \frac{P(A)}{1 - P(A)} = \frac{P(A)}{P(\bar{A})}$. Thus, the odds that A will occur is stated as the ratio of P(A) to 1 – P(A) = $P(\bar{A} = A')$ = Q(A), i.e., odds in favor of A are P(A) to 1 – P(A), while the odds against A are stated as P(A′) to P(A). For Example, If P(A) = 0.80 = 4/5, then the odds in favor of A are 4 to 1; if P(A) = 0.60 = 3/5, then the odds in favor of the event A are 3 to 2. On the other hand, if P(A) = 0.05 = 1/20, then the odds against its occurrence are 19 to 1, i.e., we are willing to bet one dollar against \$19.00 (in order to net \$19, i.e., get back a total of \$20.00). The common notation *O*(A) used in statistics is counterintuitive because it always describes the odds against the event A, i.e., *O*(A) = $n_2 / n_1 = \frac{n_2/(n_1 + n_2)}{n_1/(n_1 + n_2)} = P(A' = \bar{A}) / P(A)$ = Q(A)/P(A) and is read as odds against A are n_2 to n_1. So, I will make the counterintuitive notation that the odds in favor of event A is given by $\bar{O}(A) = O'(A) = P(A)/P(A')$. For Example, if a bookie gives you a 2 to 1 odds that Auburn beats Bama this year, then you must bet \$1.00 (i.e., pay upfront) and if Auburn wins you will receive a total of \$3.00. On the other hand, if the same bookie gives a Bama fan a 1 to 3 odds, then a Bama fan must bet \$3.00 and if Bama wins, s/he will receive \$4.00. Note that bookies must give different odds in order to guarantee a net profit after the game is played. Thus, in this scenario, if 12000 Auburn fans bet the minimum of $1.00, and 10000 Bama fans bet the minimum of $3.00, the bookie takes in $12000.00 + 10000$\times$3 = $42,000.00. If Auburn wins, the bookie pays off $36000.00, and if Bama wins, the bookie pays off $40,000.00. If we define the event A = {Auburn beats Bama in 2014}, then in the above scenario the bookie's odds against A is O(A) = 2/1 = $\mathrm{P(A')} / \mathrm{P(A)} \rightarrow$ P(A) = 1/3, while if we define the event B = {Bama beats Auburn in 2014}, then the bookie's odds against B is O(B) = 1/3 $\rightarrow$ the bookie's Pr that Bama wins is equal to P(B) = 3/4. Finally, bookies always take somewhere within 17-20% off the top (the most common being 17%). So in the above scenario if Auburn wins, the pay-off is 0.83$\times$$36000.00 = $29880.00; if Bama wins, the pay-off is 33200.00.

<u>A Result that will be shown in class using a Venn Diagram.</u> Let A and B belong to the same sample space U. Then, in general $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for all events A and B in the same universe U.

<u>Example 11.</u> A repairman claims that the Pr is P(A) = 0.85 that the air conditioning compressor is all right, and P(B) = 0.64 that the fan motor is all right, and 0.45 that both are all right. Can his claim be true? $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.85 + 0.64 - 0.45 = 1.04$. Clearly, his claim is false! If he revises his claim to P(B) = 0.55, would his assignment of Prs be correct then? Had he claimed that $P(A \cap B) = 0.60$, could his claim be true? Answers: Yes!

<u>Exercise 2.</u> In a department of 340 ISE UG students, 125 are enrolled in statistics (event A), 80 in OR (Event B = enrolled in Operation Research) and 50 in both. An ISE student is selected at random from a frame that lists all ISE students in alphabetical order. What is the Pr that s/he is not enrolled in either course?

<u>Exercise 3.</u> A plant has 400 employees, 300 of whom got a raise during the last year, 250 got an increase in pension benefits and 40 got neither. An employee is selected at random from a frame that lists all plant employees in an alphabetical order. Compute the Pr that s/he got both. ANS: 0.475.

<u>Exercise 4.</u> Among 200 persons (selected at random) interviewed as part of an urban mass transportation study, some lived more than 5 miles from the city center (event A), some regularly drove their own car (event B), and some would gladly switch to a public mass transportation if it were made available (event C). Further, it was determined that $N(A) = 113$, $N(B) = 130$, $N(C) = 117$, $N(A \cap B) = 100$, $N(A \cap C) = 84$, $N(B \cap C) = 86$ and $N(\bar{A} \cap \bar{B} \cap \bar{C}) = 35$. One person is selected at random from a frame that has listed the names of the 200 persons. (a) Draw a Venn diagram and use it to compute the Pr that the randomly selected person fits all 3 profiles. (b) Compute the Pr that at least one of the 3 events occur. (c) Compute the $P(B \cup C)$ and $P(B' \cup C)$. (d) Compute the $P(\bar{A} \cap B \cap \bar{C})$, $P[A' \cup B' \cup C]$, $P[A \cup (B \cap C)]$, $P[(A \cup B) \cap C]$, and $P[(A \cup C) \cap B]$. ANS: (a) 0.375, (b) 0.825, (c) 0.805, 0.78, (d) 0.095, 0.875, 0.620, 0.475, 0.555.

Multiplication Principle: If an event A_1 can occur in n_1 different ways, event A_2 can occur in n_2 ways, ..., and event A_K in n_K different ways, then the event A_1, followed by A_2, ..., followed by A_K can occur one after another in $n_1 \times n_2 \times \ldots \times n_K$ different ways.

Example 12. In how many different ways can a local union with 40 members select a president, followed by a VP and then a secretary treasurer? ANS : $40 \times 39 \times 38 = 59280$ different ways.

Example 13. A test consists of 20 T/F questions. If a student flips a coin to select his answers, in how many different ways can s/he answer all 20 questions? ANS: $2 \times 2 \times \ldots \times 2 = 2^{20} = 1048576$ different ways. If 60% performance is passing, it will be shown in Chapter 3 the Pr that he/she will pass the test is Pr = 0.2517223.

Exercise 5. A dime is tossed, then a quarter is tossed followed by a die. Determine the total number of outcomes for this compound experiment. (ANS: 24)

Permutations: The total number of different ways that n distinct objects can be permuted is given by ${}_nP_n = n!$ As an example, consider the 3 distinct objects A, B, C. The permutations of these objects are ABC, ACB, BAC, BCA, CAB, CBA, of which there are six. Therefore, ${}_3P_3 = 3! = 6$. The total number of ways that r ($\leq$ n) objects can be selected from n objects and then permuted is given by ${}_nP_r = \dfrac{n!}{(n-r)!}$.

For example, consider 4 objects A, B, C and D. Then, the elements of ${}_4P_2$ are: AB,

BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC, of which there are 12 = 4!/(4-2)! = 24/2, while the elements of ${}_4P_3$ are: ABC, ACB, BAC, BCA, CAB, CBA, ABD, ADB, BAD, BDA, DAB, DBA, …, DCB, of which there are 4!/(4 – 3)! = 24.

Combinations: The total number of different ways that r distinct objects can be selected from n ($r \le n$) is given by ${}_nC_r = \binom{n}{r} = {}_nP_r / r! = \dfrac{n!}{(n-r)!\,r!}$.

For example, consider the combinations of the 4 objects A, B, C and D, taken 2 at a time:

AB, AC, DA, BC, DB, CD,

of which there are 6 = ${}_4C_2$ = 4!/(2!×2!) of them.

Exercise 6. **(a)** Five cards are drawn completely at random from a well-shuffled deck of 52 cards. What is the Pr of getting exactly 3 kings and 2 aces? $0.0^5 92345$.

(b) Thirteen cards are drawn at random from a deck of 52 cards. What is the Pr that all 4 aces will be drawn? ANS: 0.0026411

(c) Five cards are drawn at random from a well-shuffled deck of 52 cards W/O replacement. What is the Pr that the 5 cards form a straight (exclude straight-flush)? What is this Pr for the case of with replacement? ANS: 0.00392465, 0.0032193241

(d) Compute the Pr of a full-house for a 5-card draw of a poker hand.

(e) A production lot contains 80 units of which 6 are NC. A random sample of size n = 10 is drawn from the lot and the number of NCU's is counted. What is the Pr that the sample contains exactly 2 NCU's? What is the Pr that the sample contains at least one NCUs? ANS: 0.13731, 0.5637

(f) Five cards are drawn at random with replacement from well-shuffled deck of 52 cards. Compute the Pr that there are exactly 2 aces amongst the 5 cards.
ANS: P(exactly 2 aces) = 0.0465401.

CONDITIONAL PROBABILITY

Examples 14. Consider 100 individuals in a ballroom. Sixty of the 100 are college graduates, 55 are married and 35 are married college graduates. The names of the 100 individuals are written on 100 different slips (in order to form a frame) and put in a box. One slip is drawn completely at random. Now define the following events.

Let event A = {The slip randomly drawn is an individual who is a college graduate}. Let event B = {The randomly selected person is married}. Clearly, *O*(A) = 40/60, P(A) = 60/100 = 0.60, *O*(B) = 45/55, P(B) = 55/100 = 0.55, and P(A∩B) = 0.35.

(a) What is the Pr that the selected individual is an unmarried college graduate?

$$P(A \cap B') = P(A) - P(A\cap B) = 0.25$$

Similarly, $P(A' \cap B)$ = 0.20 gives the Pr that the selected person is W/O a college degree but married.

(c) Suppose you are told that the selected person is a college graduate, i.e., we know that the event A has already occurred. Now what is the Pr that the selected person is also married? The Pr of the event B given that A has already occurred is given by

$$P(B|A) = \frac{N(A \cap B)}{N(A)} = \frac{35}{60} = \frac{N(A \cap B)/N(U)}{N(A)/N(U)} = \frac{P(A \cap B)}{P(A)} = 0.5833333\overline{3}.$$

The event A in part (c) above is called the reduced universe, R_U, to account for the conditional information provided by the slip that was drawn at random. In general, we deduce from the above that $P(A\cap B) = P(A)\times P(B|A)$, or $P(A\cap B) = P(B)\times P(A|B)$ holds true for any 2 events in the same universe U.

Two events A and B in the same universe, U, are stochastically independent iff $P(A|B) = P(A)$ and also $P(B|A) = P(B)$, i.e., iff $P(A\cap B) = P(A)\times P(B)$.

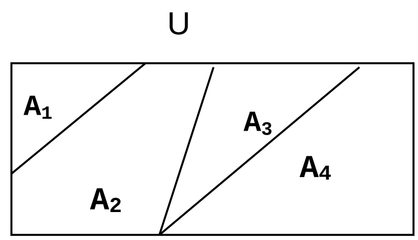
As an example, suppose a 2nd person is selected at random from the above ballroom (W/O replacing the 1st slip back in the box). Let the event C = {the 2nd person is also a college grad}. Then, $P(A\cap C) = P(A)\times P(C|A) = (60/100)\times(59/99) = 0.3576$, and events C and A would not be independent because the occurrence Pr of C depends on A. Note that $P(C|A') = 60/99$.

Had the 2nd drawing been performed with replacement, then $P(A\cap C) = P(A)P(C) = (60/100)\times(60/100) = 0.36$ so that events A and C would be independent because P(C) would not depend on whether A occurred or not.

<u>Exercise 7.</u> **(a)** Compute the Pr that the 1st person selected at random from the ballroom is neither a college grad nor married. **(b)** Compute the Pr that if 2 persons are selected at random (and W/O replacement from the box with 100 slips), one is a CG and the other is married (or both). **(c)** Repeat part **(b)** if the 2nd random selection is done with replacement. ANS: 0.20, $0.5393\overline{93}$, 0.5375.

Bayes' Theorem

The events $A_1, A_2, ..., A_K$ form a partition of a sample space U iff (1) $A_i \cap A_j = \phi$ for all $i \neq j$ (i.e., A_i and A_j must be MUEX for all $i \neq j$), and (2) $A_1 \cup A_2 \cup ... \cup A_k = \bigcup_{i=1}^{k} A_i = U$. Let B denote any event in the universe U. Then $B = \bigcup_{i=1}^{k} (B \cap A_i)$ because the events $B \cap A_i$ (i = 1, 2, ..., k) are also jointly MUEX. Thus,



$$P(B) = P[\bigcup_{i=1}^{k} (B \cap A_i)] = \sum_{i=1}^{k} P(B \cap A_i) = \sum_{i=1}^{k} \left[P(A_i) \times P(B | A_i) \right].$$

In the Bayes' theorem, the objective is to compute the conditional Pr, $P(A_j | B)$, for a specified j = 1, 2, 3, ..., k. Then

$$P(A_j | B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(A_j) \times P(B | A_j)}{\sum_{i=1}^{k} \left[P(A_i) \times P(B | A_i) \right]}.$$

<u>Example 15.</u> A job shop has 3 machines M_1, M_2 and M_3 where the FNC from them are 0.02, 0.05, and 0.03, respectively. An item is drawn at random from the combined well-mixed daily outputs of the 3 machines and found to be defective. What is the Pr that the randomly selected item was manufactured by machine 2 if the relative production rates (PRs) of the 3 machines are 0.40, 0.30 and 0.30, respectively?

Let A_i = {the randomly selected item came from M_i}, i = 1, 2, 3, and event B = {the item is defective or NC}. Note that the event B in the Bayes' formula is always the one that has already occurred. (In fact in this case, the event B represents Average Outgoing Quality, denoted by AOQ.)

$$P(B) = \sum_{i=1}^{k} \left[P(A_i) \times P(B | A_i) \right] = 0.40 \times 0.02 + 0.30 \times 0.05 + 0.30 \times 0.03 = 0.032 = AOQ$$

$$P(A_2 | B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{P(A_2) \times P(B | A_2)}{0.032} = \frac{0.30 \times 0.05}{0.032} = 0.46875.$$

Exercise 8. Repeat the above example 15 for the case when 2 items are drawn from the daily output and both found to be defective. Assume that the proportions 0.40, 0.30 and 0.30 stay roughly the same for the 2nd randomly selected unit. What is the Pr that both units came from machine M_2? ANS: 0.21973 (approximately).

Exercise 9. Let U = {0,1, 2, 3, 4, 5, 6, 7, 8, 9}, event A = {1, 2, 3, 4}, B = {2, 6}, and Event C = {1, 3, 5, 7}. (a) Determine if A, B and C are MUEX. (b) If a number is selected at random from base-10, compute the P(A), P(B), P(C) and the Pr that both A and B occur. Determine if any two of the 3 events are stochastically independent.

Exercise 10. Suppose an applicant for a job feels that the odds are 7 to 4 of getting an offer after the interview, i.e., the subjective Pr that s/he is assigning to receiving an offer is 7/11. (a) Suppose now a businessman feels that the odds are 3 to 2 that a certain venture will succeed, i.e., s/he is willing to bet 300 against 200 so that upfront s/he will put up $300.00, and if the venture succeeds s/he will get back a grand total of $500.00. What is the subjective Pr that s/he is assigning to the success of the venture? (b) Suppose a student is STAT3600 is willing to bet $30.00 against 10.00 that s/he will pass the course but not $40.00 against 10.00; what is the subjective Pr that Kristen is assigning to passing my course? (c) Suppose an expert in a horse-racing event claims that for horse A the odds against are *O*(A) = 2/1 of wining the race and *O*(B) = 3/1; the expert also feels that there is less than even-Steven chance that either horse would win the race. Is this expert consistent, or s/he is a half-whit?

Exercise 11(Borrowed from Miller et al 2011, p.66, Prentice Hall, 8e). The supplier of an optical equipment feels that the odds are 7/5, read as 7 to 5, against a shipment arriving late (event A= a shipment arriving late), and 11/1 against not arriving at all (event B = not arriving at all). Further, s/he feels that the odds are 1/1 that such a shipment will either arrive late or not at all. Is s/he accurate in the above odds assignments, or again full-of-it? Put differently, is the glass half-full, or half-empty?

My Chapter 2 notes are mostly edited by Mohammad-Ali Alamdar-Yazdi (mza0052), and also by Rong Huangfu (rzh0024).