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RENEWAL AND AVAILABILITY FUNCTIONS

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Abstract

The renewal and availability functions for some common failure and repair underlying distributions are explored. Exact results for the renewal functions and availability of normal time to failure and time to repair, and gamma time to failure and exponential time to repair are provided. Because obtaining the *n*-fold convolutions of time between cycles for general classes of failure and repair distributions is intractable, we obtained the availability functions for some commonly-encountered failure distributions but at a constant repair-rate. A MATLAB program was devised to perform all calculations.

1. Introduction

This article generalizes the work in [1] by the same authors, where we now assume the MTTR (mean time to repair) is not negligible and that TTR has a pdf (probability density function) denoted as $r(t)$. Let the variates $X_1, X_2, X_3, \ldots$ represent the *i*th time to failure (TTF_i) be independently and identically distributed (iid) with the same underlying failure density $f(x)$ having mean $\text{MTTF} = \mu_x$ and variance σ_x^2; further, $Y_1, Y_2, Y_3, \ldots$ represent

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the ith time to restore (TTR_i), $i = 1, 2, 3, 4, \ldots$ with the same pdf $r(y)$ having mean $\text{MTTR} = \mu_y$ and variance σ_y^2. Then $T_i = X_i + Y_i$ represents the time between cycles (TBCs) which are also iid whose density is given by the convolution $g(t) = f(t) * r(t)$, and whose Laplace transform (LT) is given by $\mathcal{L}\{g(t)\} = \bar{g}(s) = \bar{f}(s) \times \bar{r}(s)$. Clearly, the mean and variance of the cycle-times T_i's are $\mu_x + \mu_y$ and $\sigma_x^2 + \sigma_y^2$. As described by [2] there will be two types of renewals:

(1) A transition from a Y-state (i.e., when system is under repair) to an X-state (at which the system is operating reliably).

(2) A transition from an X-state (or operating-reliably state) to a Y-state (where system will go under repair or restoration).

Let $M_1(t)$ represent the expected number of cycles (or number of renewals of type 1), and $M_2(t)$ represent the mean number of failures (or renewals of type 2). Then, as proven by [3] and later by [2], the LTs of the two renewal functions (RNFs), respectively, are given by

$$\overline{M}_1(s) = \frac{\bar{g}(s)}{s[1 - \bar{g}(s)]} = \frac{\bar{f}(s) \times \bar{r}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}, \tag{1a}$$

$$\overline{M}_2(s) = \frac{\bar{f}(s)}{s[1 - \bar{g}(s)]} = \frac{\bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}. \tag{1b}$$

The corresponding LTs of RNIFs (renewal-intensity functions) are given by

$$\bar{\rho}_1(s) = \frac{\bar{f}(s) \times \bar{r}(s)}{1 - \bar{f}(s) \times \bar{r}(s)} \text{ and } \bar{\rho}_2(s) = \frac{\bar{f}(s)}{1 - \bar{f}(s) \times \bar{r}(s)}. \tag{2}$$

It is essential to note that authors in Stochastic Processes refer to inverse-transforms of equations (2) as the renewal densities.

As an example, suppose $\text{TTF}_i \sim \text{Exp}(0, \lambda)$, i.e., exponential with zero minimum-life and constant hazard rate $h(t) = \lambda$, and $\text{TTR}_i \sim \text{Exp}(0, r)$;

then as has been documented both in Stochastic Processes and Reliability Engineering literature, $\bar{f}(s) = \int_0^\infty \lambda e^{-\lambda t} e^{-st} dt = \lambda/(\lambda + s)$ and $\bar{r}(s) = \int_0^\infty re^{-rt} e^{-st} dt = r/(r + s)$. Such a process is refereed as an "*alternating Poisson Process*" [2], which we acronym as APP. On substituting the above 2 LTs into equation (1a), we obtain the well-known

$$\overline{M}_1(s) = \frac{\lambda r}{s[(\lambda + s)(r + s) - \lambda r]} = \frac{-\lambda r}{s\xi^2} + \frac{\lambda r}{s^2 \xi} + \frac{\lambda r}{\xi^2 (s + \xi)},$$

where $\xi = \lambda + r$, and

$$M_1(t) = \mathcal{L}^{-1}\{\overline{M}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{-\lambda r}{s\xi^2} + \frac{\lambda r}{s^2 \xi} + \frac{\lambda r}{\xi^2 (s + \xi)}\right\}$$

$$= \frac{-\lambda r}{\xi^2} + \frac{\lambda r}{\xi} t + \frac{\lambda r}{\xi^2} e^{-\xi t},$$

which gives the expected number of transitions from a repair-state to an operational-state (or the mean number of cycles). Similarly,

$$\overline{M}_2(s) = \frac{\bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]} = \frac{\lambda^2}{s\xi^2} + \frac{\lambda r}{s^2 \xi} - \frac{\lambda^2}{\xi^2 (s + \xi)},$$

which upon inversion yields $M_2(t) = \frac{\lambda^2}{\xi^2} + \frac{\lambda r}{\xi} t - \frac{\lambda^2}{\xi^2} e^{-\xi t}$, representing the expected number of failures during the interval $(0, t)$. For example, if $\lambda = 0.0005/\text{hour}$ and the constant repair-rate $= r = 0.05$ per hour, then $\xi = \lambda + r = 0.0505$, $M_1(t = 500\,\text{hours}) = 0.237721792$, while

$$M_2(500) = 0.2476227821.$$

Note that the limit of both above RNFs $M_1(t)$ and $M_2(t)$ as repair-rate $r \to \infty$ (i.e., MTTR $\to 0)$ is exactly equal to the exponential RNF $M(t) = \lambda t$, as expected. Further, a comparison of $M_2(t)$ with $M_1(t)$ reveals that

$M_2(t) > M_1(t)$ for all $t > 0,$ which is intuitively meaningful because the expected number of failures must exceed the expected number of cycles for all $t > 0,$ as we are assuming that time zero is when a system starts in the last renewed state *X*.

2. (a) Types 1 and 2 Renewal Functions Using Laplace Transforms

Elsayed [4] obtains the (point) availability function (AVF) for a system comprising of 2 similar components *A* and *B*, using their RNFs $M_A(t)$ and $M_B(t),$ where his $P_A(t)$ represents the Pr that component *A* is in use at time *t*. Using a similar argument, we first obtain $M_2(t)$ and $M_1(t)$ by inverting equations (1), and we later use these 2 functions in order to obtain the availability (AVL) function $A(t).$ Equation (1a) shows that

$$\overline{M}_1(s)[1-\overline{g}(s)] = \frac{\overline{g}(s)}{s} \rightarrow \overline{M}_1(s) = \frac{\overline{g}(s)}{s} + \overline{M}_1(s)\overline{g}(s) \rightarrow$$

$$M_1(t) = G(t) + \int_0^t M_1(t-x)g(x)dx,$$

where $G(t)$ is the cdf of TBCs. Equation (1b) now shows that

$$\overline{M}_2(s)[1-\overline{g}(s)] = \frac{\bar{f}(s)}{s} \rightarrow \overline{M}_2(s) = \frac{\bar{f}(s)}{s} + \overline{M}_2(s)\overline{g}(s) \rightarrow$$

$$M_2(t) = F(t) + \int_0^t M_2(t-x)g(x)dx;$$

thus, in general the well-known expected number of cycles is given by

$$M_1(t) = G(t) + \int_0^t M_1(t-x)g(x)dx. \tag{3a}$$

While the corresponding well-known expected number of failures during $(0,\ t)$ is given by

$$M_2(t) = F(t) + \int_0^t M_2(t-x)g(x)dx. \tag{3b}$$

(b) The Normal TTF and TTR

Suppose time between failures TBFs ~ $N(\mu_x = \text{MTBF}, \sigma_x^2)$ and TTR is also $N(\mu_y, \sigma_y^2)$; then TBCs ~ $N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$. We are making the tacit assumption that both coefficient of variations are sufficiently small (say CV < 15%) such that the normal distribution can qualify as a failure and repair densities; further, the system initially starts in an *X*-state. As a result, equation (3a) shows that $M_1(t) = \sum_{n=1}^{\infty} \Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right)$, where $\mu = \mu_x + \mu_y$, $\sigma = \sqrt{\sigma_x^2 + \sigma_y^2}$, and $M_1(t)$ gives the expected number of cycles. However, because a system is under repair a small (but not negligible) fraction of the interval $(0, t)$, then $M_2(t) \neq \sum_{n=1}^{\infty} \Phi\left(\frac{t - n\mu_x}{\sigma_x\sqrt{n}}\right)$. In order to obtain an approximation for $M_2(t)$ and the resulting availability function (AVF), $A(t)$, defined later, we may argue that the expected duration of time a system is under repair during the interval $(0, t)$ is given by $M_1(t) \times \text{MTTR}$; letting $t_2 \cong t - M_1(t) \times \text{MTTR}$, then equation (3b) shows that the expected number of failures, assuming that the system starts in the *X*-state, is approximately given by $M_2(t) \cong \Phi\left(\frac{t - \mu_x}{\sigma_x}\right) + \sum_{n=2}^{\infty} \Phi\left(\frac{t_2 - n\mu_x}{\sigma_x\sqrt{n}}\right)$.

3. Point Availability

Because we are assuming that a system can be either in an operational-state (X), or under repair, then it has been well-known that the reliability function, $R(t)$, must be replaced by the instantaneous (or point) AVF at time *t*, denoted $A(t)$, which represents the probability (Pr) that a repairable system (or unit) is functioning reliably at time *t*. Thus, if restoration-time is negligible, the AVF is simply $A(t) = R(t)$. However, if a system (or a component of a system) is repairable, then there are two mutually exclusive possibilities [5]:

(1) The system is reliable at *t*, in which case $A_1(t) = R(t)$.

(2) The system fails at time *x*, $0 < x < t$, gets renewed (or restored to almost as-good-as-new) in the interval $(x, x + \Delta x)$ with unconditional Pr element $\rho(x)dx$, $\rho(t)$ being the RNIF of TBCs, and then is reliable from time *x* to time *t*. This second Pr is given by $A_2(t) = \int_0^t \rho(x)dxR(t-x)$; because the above two cases are mutually exclusive, then

$$A(t) = A_1(t) + A_2(t) = R(t) + \int_0^t R(t-x)\rho(x)dx. \tag{4}$$

Taking Laplace transform of the above equation (4) (and observing that the integral is the convolution of $R(t)$ with $\rho(t)$ [6]) yields the very well-known LT of AVF

$$\overline{A}(s) = \overline{R}(s) + \overline{R}(s)\overline{\rho}(s) = \overline{R}(s)[1 + \overline{\rho}(s)]$$

$$= \overline{R}(s)\left[1 + \frac{\bar{f}(s) \times \bar{r}(s)}{1 - \bar{f}(s) \times \bar{r}(s)}\right] = \frac{\overline{R}(s)}{1 - \bar{f}(s) \times \bar{r}(s)}, \tag{5}$$

where $\bar{r}(s)$ is the LT of $r(t)$, the density (or pdf) of repair-time. For the case when the TTF (of a component or system) has a constant failure-rate λ and time to repair (TTR) is also exponential at the rate *r* (i.e., an APP), $\overline{R}(s) = \int_0^\infty e^{-\lambda t}e^{-st}dt = \lambda/(\lambda + s)$, and hence the Laplace transform of AVL from equation (5) is given by

$$\overline{A}(s) = \frac{1/(\lambda + s)}{1 - [\lambda/(\lambda + s)][r(r + s)]} = \frac{r + s}{s[s + (\lambda + r)]}$$

$$= \frac{r}{\xi s} + \frac{\lambda/\xi}{s + \xi} \rightarrow A(t) = \mathcal{L}^{-1}\{\overline{A}(s)\} = \frac{r}{\xi} + \frac{\lambda}{\xi}e^{-\xi t},$$

where $\xi = \lambda + r$, which is provided by many authors in Reliability

Engineering such as [4, 7, 8] and many other notables. For example, given that the failure-rate $= \lambda = 0.0005$ and $r =$ repair-rate $= 0.05$ per hour, then $\xi = \lambda + r = 0.0505$ and the Pr that a network is available (i.e., not under restoration) at $t = 500$ hours is given by $A(500) = \frac{0.05}{0.0505} + \frac{0.0005}{0.0505} e^{-0.0505(500)} = 0.99009901,$ while R (at 500 hours with minimal-repair) $= e^{-0.250} = 0.7788007831 < A(500) = 0.9901$. Thus, restoration has improved AVL by 27.31%. As stated by numerous authors in Stochastic Processes and Reliability Engineering, as $t \to \infty$, $R(t) \to 0$ for all failure densities, while for an APP

$$A(t) \to r/\xi = r/(\lambda + r) = A_{\text{inf}} = \text{MTTF}/(\text{MTTF} + \text{MTTR})$$

$$= 2000/2020 = 0.9901,$$

where restoration includes administrative, logistic and active repair-times.

Note that in the APP case (i.e., both rate-parameters are constants), we can also obtain the AVF, $A(t)$, directly from equation (4) as follows:

$$A(t) = R(t) + \int_0^t R(t-x)\rho_1(x)\,dx = e^{-\lambda t} + \int_0^t e^{-\lambda(t-x)}\rho_1(x)\,dx,$$

where

$$\rho_1(x) = dM_1(x)/dx = \frac{d}{dx}\left(\frac{-\lambda r}{\xi^2} + \frac{\lambda r}{\xi} x + \frac{\lambda r}{\xi^2} e^{-\xi x}\right) = \frac{\lambda r}{\xi} - \frac{\lambda r}{\xi} e^{-\xi x}$$

is the RNIF of the number of cycles. Upon substitution of this RNIF into the expression for $A(t)$, we obtain

$$A(t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-x)} \frac{\lambda r}{\xi}(1 - e^{-\xi x})\,dx = \frac{r}{\xi} + \frac{\lambda}{\xi} e^{-\xi t},$$

as before.

As pointed out by [4], we also observe that

$$\bar{R}(s) = \int_0^\infty e^{-st} R(t)\,dt = \int_0^\infty e^{-st}[(1 - F(t))]\,dt$$

$$= \frac{1}{s} - \int_0^\infty e^{-st} F(t)\,dt = \frac{1}{s} - \bar{F}(s).$$

Hildebrand [6] proves that $\bar{F}(s) = \bar{f}(s)/s$ so that $\bar{R}(s) = \frac{1 - \bar{f}(s)}{s}$; on substitution into equation (5), we obtain

$$\bar{A}(s) = \frac{1 - \bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]} = \frac{1}{s[1 - \bar{f}(s) \times \bar{r}(s)]} - \frac{\bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}$$

$$= \frac{1}{s} + \frac{\bar{f}(s) \times \bar{r}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]} - \frac{\bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}.$$

Inverting these last 3 LTs from equations (1), we obtain

$$A(t) = 1 + M_1(t) - M_2(t) \tag{6}$$

for all underlying failure densities $f(t)$ and TTR-density $r(t)$. Equation (6) is identical to that of Elsayed [4] atop his page 467, which he derived using a system of 2 alternating components. Further, equations (3) imply that $M_2(t) - M_1(t)$ yields the unconditional Pr that a system is under repair at time *t*, and hence equation (6) is intuitively appealing because $A(t) = 1 - [M_2(t) - M_1(t)]$. For the above exponential example with $\lambda = 0.0005$ and repair-rate $r = 0.05$, equation (6) shows that $A(t = 500) = 1 + 0.237721792 - 0.2476227821 = 0.99009901$, as before.

Example 1. Our experience shows that in the case of normal TTF and TTR the approximate value of $t_2 = t - M_1(t) \times \text{MTTR}$ is a bit too small. If $0.005 \le \text{MTTR}/\text{MTTF} \le 0.05$, then $t_2 \cong t - [M_1(t) - 0.475] \times \text{MTTR}$; however, if $0.05 < \text{MTTR}/\text{MTTF} \le 0.10$, then $t_2 \cong t - [M_1(t) - 0.425] \times$ MTTR is a better approximation. These values were obtained such that the limiting AVL, given by $A_{\inf} = \text{MTTF}/(\text{MTTF} + \text{MTTR})$, is approximately equal to $A(t) = 1 + M_1(t) - M_2(t)$ at $t = 120 \times \text{MTTF}$ to 3 decimals. For example, if TTF ~ $N(5000\,\text{hours}, 160000\,\text{hours}^2)$ and also TTR ~ $N(200\,\text{hours}, 576\,\text{hours}^2)$, then

$$M_1(600000) = \sum_{n=1}^{\infty} \Phi\left(\frac{600000 - 5200n}{\sigma\sqrt{n}}\right) = 114.8875850224902$$

expected cycles, where $\sigma = \sqrt{160000 + 576} = 400.7193531637822$, $M_2(t) \cong \Phi\left(\frac{600000 - \mu_x}{\sigma_x}\right) + \sum_{n=2}^{\infty} \Phi\left(\frac{t_2 - n\mu_x}{\sigma_x\sqrt{n}}\right) = 114.926696731$ expected failures, where

$$t_2 \cong t - [M_1(t) - 0.475] \times \text{MTTR} = 577117.4829955020,$$

$$A(600{,}000\,\text{hours}) = 1 + M_1(t) - M_2(t) \cong 0.9608883,$$

which is close to $A_{\inf} = 5000/5200 = 0.96153846154$.

Similar calculations will show if TTR ~ $N(400\,\text{hours}, 2304)$ so that $\text{MTTR}/\text{MTTF} = 0.080$, then $A(t) = 1 + M_1(t) - M_2(t) \cong 0.925806$, $t_2 \cong t - [M_1(t) - 0.425] \times \text{MTTR} = 555924.44217462$, and $A_{\inf} = 5000/5400 = 0.925925926$. It should be noted that if $0 < \text{MTTR}/\text{MTTF} < 0.005$, then from a practical standpoint the renewal process approximately reduces to the minimal-repair case (i.e., a nonhomogeneous Poisson process) for which $M_1(t) \cong M_2(t)$. Further, the normal $A(t)$ generally decreases on the interval $[0, \text{MTTF})$ as t increases, seems to attain its worst value around the MTTF, tends to increase with increasing time beyond MTTF, and then converges toward $A_{\inf}$.

4. Markov Analysis When Only Repair-rate *r* is a Constant

The Markov analysis of AVF, $A(t)$, for case of constant failure- and repair-rates (i.e., the APP) has been reported by nearly all authors in Stochastic Processes and Reliability Engineering. Our objective is to make a slight generalization to when the hazard function (HZF) is time-dependent, i.e., $h(t) \neq \lambda,$ where λ is the CFR (constant failure-rate). We can obtain the AVL of a simple on and off (or up-time and down-time) system from Figure 1, where state "0" represents a system in the reliable-state and "1" represents the same system under repair. The transition-rate in Figure 1 shows that its Kolmogorov equation is given by $dP_0(t)/dt = -h(t)P_0(t) + rP_1(t),$ where $P_0(t) = A(t)$ represents the unconditional Pr of finding the system in the operational state "0" at time *t*, and similarly for $P_1(t)$. Because $P_1(t) = 1 - P_0(t)$ for all *t*, we obtain $dP_0(t)/dt = -h(t)P_0(t) + r(1 - P_0),$ and hence $dP_0(t)/dt + [h(t) + r]P_0(t) = r.$ This last is a simple differential equation with the integrating factor $e^{\int [h(t)+r]\,dt} = e^{H(t)+rt},$ where $H(t) = H$ is the antiderivative of $h(t);$ it should be noted that the antiderivative $\int h(t)dt$ does not seem to match the definition of the cumulative HZF $\int_0^t h(x)dx.$ However, if the cumulative hazard at minimum-life is zero, which is expected, then $H(t)$ is also the cumulative HZF. It is widely known that the general solution of the above differential equation is given by

$$P_0(t) = e^{-(H+rt)} \times \int re^{H(t)+rt}dt + Ce^{-(H+rt)}, \tag{7a}$$

where the constant of integration will be computed as usual from the boundary condition $P_0(t = \delta) = 1,$ δ being the minimum-life. Because $e^{H(t)} = 1/R(t),$ then (7a) is modified to

$$P_0(t) = A(t) = e^{-(H+rt)}\left\{C + \int [re^{rt}/R(t)]dt\right\}$$

$$= e^{-rt}R(t) \times \left\{C + \int [re^{rt}/R(t)]dt\right\}. \tag{7b}$$

Unfortunately, there is no exact solution to (7b) for the general classes of failure distributions, $F(t) = 1 - R(t)$, because the indefinite-integral $I(t) = \int [re^{rt}/R(t)]dt = \int re^{rt}/[1 - F(t)]dt$ has no closed-form antiderivative for all uncountably infinite number of failure distributions. However, we may obtain an exact solution for a few failure distributions $F(t)$, and then have to approximate equation (7b) for others. We start with the simplest case of 2-parameter exponential $F(t) = 1 - e^{-\lambda(t-\delta)}$ and then solve (7b) case by case as listed below, increasing solution difficulty. Further, merely for writing simplicity we let $F = F(t)$, $R = R(t)$, and as stated above the repair-rate stays constant at r.

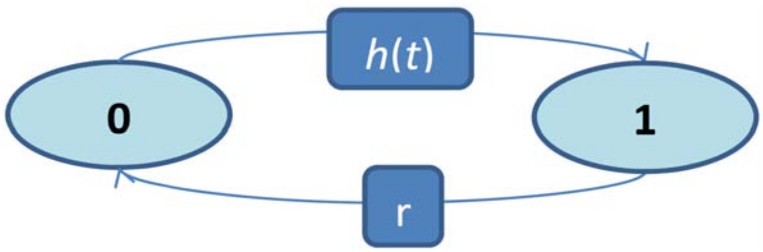

Figure 1. The transition-rate diagram for an on and off system.

Case (a). When the HZF is a CFR but minimum-life δ is not necessarily zero, then the reliability function $R(t) = \begin{cases} 1, & 0 \le t \le \delta \\ e^{-\lambda(t-\delta)}, & \delta \le t < \infty \end{cases}$ and applying the boundary condition $P_0(t = \delta) \equiv 1$, equation (7b) after extensive algebra yields $P_0(t) = A(t) = \frac{r}{\xi} + \frac{\lambda}{\xi} e^{-\xi(t-\delta)}$, $\xi = \lambda + r$, $t \ge \delta$, which at $\delta = 0$ is the same function given in Section 3 for an APP. Clearly, $A(t) = 1$ for $0 \le t \le \delta$. For example, suppose a network's TTF ~ Exp($\delta = 400$, $\lambda = 0.0005$ hours) and constant repair-rate $r = 0.05$ per hour. Then, the characteristic-life now improves to $1/\lambda + \delta = 2400$ hours, which is also equal to MTTF of the Exp(400, 0.0005), and λ is the rate-parameter. As before, $\xi = \lambda + r =$

0.0505 and the Pr that the network is available (i.e., not under repair) at $t = 500$ hours is given by

$$A(500) = \frac{0.05}{0.0505} + \frac{0.0005}{0.0505} e^{-0.0505(100)} = 0.9901624687,$$

while the value of reliability function is R(500, minimal-repair) = 0.9512294245.

Case (b). Secondly, suppose TTF is uniformly distributed over the real-interval $[a, b]$, i.e., $U(a, b)$, where $a \geq 0$ is the minimum-life, $b =$ maximum-life $> a$ and $c = b - a > 0$ is the uniform-density base. Then the cdf $F(t) = (t - a)/c$, $R(t) = (b - t)/c$, $a \leq t < b$, $R(t) \equiv 1$ for all $0 \leq t \leq a$, and $R(t) \equiv 0$ for all $t \geq b$, at which point the system will be transition to the repair-state. For $0 \leq t \leq a$, the substitution of $R(t) \equiv 1$ into (7b) and applying the boundary condition $P_0(t = 0) = 1$ will not yield the value C because the indefinite-integral on the far RHS of (7b) has to be evaluated first. When $t \geq b$, $R(t) \equiv 0$ results in an indeterminate form for the RHS of (7b), $e^{-rt} R(t) \times \int [re^{rt} / R(t)] dt$. Thus, we will have to compute the value of the constant C after obtaining the general solution for $A(t)$. Next, for $a \leq t < b$, $R = (b - t)/c$, $0 < F = (t - a)/c < 1$, $c = b - a > 0$ and substitution into (7b) yields

$$\begin{aligned} P_0(t) = A(t) &= e^{-rt} R(t) \times \left\{ C + \int [re^{rt} / (1 - F)] dt \right\} \\ &= e^{-rt} R(t) \times \left\{ C + \int \left[re^{rt} \sum_{n=0}^{\infty} F^n \right] dt \right\} \\ &= e^{-rt} R(t) \times \{ C + I(t) \}, \end{aligned}$$

where

$$I(t) = \int \left[re^{rt} \sum_{n=0}^{\infty} F^n \right] dt = \sum_{n=0}^{\infty} \int [re^{rt} F^n] dt. \tag{8}$$

Note that the alternative procedure

$$I(t) = \int [re^{rt}/R(t)]dt = \int [re^{rt}/(b-t)/c]dt = \int rce^{rt}/[b(1-t/b)]dt,$$

and using the geometric series for $1/(1-t/b)$, will lead to the same exact result for $A(t)$. We now obtain the antiderivative $I(t)$ of equation (8) as follows:

$$I(t) = \sum_{n=0}^{\infty} \int [re^{rt}F^n]dt = \sum_{n=0}^{\infty} \left[e^{rt}F^n - \int ne^{rt}F^{n-1}(dF/dt) \right] dt$$

$$= \sum_{n=0}^{\infty} \left[e^{rt}F^n - \int ne^{rt}F^{n-1}(1/c) \right] dt, \qquad (9)$$

where $F = (t-a)/c$; the integral under the summation on the far RHS of equation (9) is valid only for the uniform-TTF. Repeated integration by parts, as shown above, will show that at a specific n,

$$\int (re^{rt}F^n)dt = e^{rt} \sum_{k=0}^{n} [(-1)^k ({}_nP_k) \times F^{n-k}/(cr)^k]$$

$$= e^{rt} \sum_{k=0}^{n} [(-1/cr)^k ({}_nP_k) \times F^{n-k}], \qquad (10)$$

where $F = F(t) = (t-a)/c$, ${}_nP_k = n!/(n-k)!$, $0 \le k \le n$, is the permutation of n objects taken k at a time, and $0! = \Gamma(1) = 1$. Substituting equation (10) into (9) yields

$$I(t) = \sum_{n=0}^{\infty} \int [re^{rt}F^n]dt = \sum_{n=0}^{\infty} e^{rt} \sum_{k=0}^{n} [(-1/cr)^k ({}_nP_k) \times F^{n-k}]. \qquad (11)$$

Combining equations (11) and (8) results in

$$P_0(t) = A(t) = e^{-rt}R(t) \times \left\{ C + e^{rt} \sum_{n=0}^{\infty} \sum_{k=0}^{n} [(-1/cr)^k ({}_nP_k) \times F^{n-k}] \right\}. \qquad (12)$$

So far we have argued that $A(t) \equiv 1$ for $0 \le t \le a$, and $A(t)$ is given by equation (12) only for $a \le t < b$; so, what is the AVF for $t \ge b$? Recall that $\text{TBC} = \text{TTF} + \text{TTR}$ so that the support of TBC is $0 \le t < \infty$; this is due to the fact that we are assuming a CFR with exponential repair distribution function $1 - e^{-rt}$, $0 \le t < \infty$. Before providing the overall AVF, we first obtain the value of the constant C in equation (12) by applying the boundary condition $P_0(\text{at } t = a) = P_0(\text{at } F = 0) \equiv 1$. In order to examine and evaluate $P_0(t) = A(t)$ at $t = \delta$, we rewrite the double-sum on the far RHS of equation (12) separating out the constant terms from those whose exponent of F exceeds zero,

$$I(t)/e^{rt} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}[(-1/(cr))^k({}_nP_k) \times F^{n-k}]$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}[(-1/(cr))^k({}_nP_k) \times F^{n-k}] + \sum_{n=0}^{\infty}[(-1/(cr))^n({}_nP_n)], \quad (13)$$

where ${}_nP_n = n!$. Equation (13) clearly shows that

$$\lim_{F \to 0} I(t)/e^{rt} = \lim_{t \to a} I(t)/e^{rt} = 1 - 1/(cr) + 2/(cr)^2 - 6/(cr)^3$$

$$+ 24/(cr)^4 + \cdots = \sum_{n=0}^{\infty}[n!/(-cr)^n],$$

which we denote by A_0. Unfortunately, this last alternating infinite-sum $A_0 = \sum_{n=0}^{\infty}[n!/(-cr)^n]$ does not converge no matter how large cr is; the larger cr is, the more accurate value of C can be obtained. However, equation (12) will provide fairly accurate AVF if the summation over n can be terminated at $n < 171$. It should be highlighted that at $n > 170$ MATLAB will not compute ${}_nP_k = n!/(n-k)!$, $0 \le k \le n$, and hence the infinite double sum in equation (12) has to termite at some reasonable value of n, say $60 \le n \le 100$; this in turn will resolve the divergence problem with A_0 when taken as a sum with the double-sum in equation (12). It should also be noted that the exact average (or expected) hazard rate for the uniform-density $\int_a^b \frac{1}{b-t}\frac{dt}{c}$ does not exist, and the use of approximate value $1/\text{MTTF} = 2/(a+b)$ for $h(t)$ in equation (7a) reduces the process to the case of constant failure-rate during the interval $[a, b]$, which is not realistic. Finally, substituting $t = a$ in equation (12) yields $1 = Ce^{-ra} + A_0$; hence, $C = e^{ra}(1 - A_0)$, and the corresponding AVF is given by

$$A(t) = \begin{cases} 1, \ 0 \le t \le a \\ R(t)\left\{e^{-r(t-a)} + A_0\left(1 - e^{-r(t-a)}\right) \right. \\ \left. \qquad + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\left[\left(\frac{-1}{cr}\right)^k ({}_nP_k) \times F^{n-k}\right]\right\}, \ a \le t < b, \\ 1 - e^{-r(t-b)}, \ b \le t < \infty. \end{cases} \tag{14}$$

In equation (14), $R(t) = (b - t)/c$ and $F = (t - a)/c$. As discussed in Section 3, the widely-known long-term AVL for an APP is $A_{\text{inf}} = \text{MTTF}/(\text{MTTF} + \text{MTTR})$, where the support for TBC is $[0, \infty)$. Because the support for TTF in equation (14) is the finite interval $[a, b]$, taking the limit as $t \to \infty$ is not warranted. Equation (14) shows that at $t = b$, the system fails with certainty and goes under repair and will be AVL with a Pr of $1 - e^{-r(t-b)}$, $b \le t < \infty$, at which point one cycle is completed. However, we can assert with certainty that the glb for average availability is $A_{\text{ave}} = a/(b + 1/r)$, while the lub is $b/(b + 1/r)$, i.e., $a/(b + 1/r) \le A_{\text{ave}} \le b/(b + 1/r)$, where A_{ave} gives the proportion of time that the system is operational. If we examine the AVL for the time intervals $[0, a)$, $[a, b]$, and $(b, b + 1/r]$, then it follows that the system has an AVL $= 1$ with approximate Pr of $a/(b + 1/r)$; it has an AVL $\cong [(a + b)/2]/[(a + b)/2 + 1/r]$ with approximate Pr of $(b - a)/(b + 1/r)$, and an AVL of zero with approximate Pr of $(1/r)/(b+1/r)$. Hence, the weighted-average (or expected) AVL is given by

$$A_{\text{ave}} = 1 \times \frac{a}{b+1/r} + \frac{(a+b)/2}{(a+b)/2+1/r} \times \frac{b-a}{b+1/r} + 0 \times \frac{1/r}{b+1/r}$$

$$= \frac{abr + 2a + rb^2}{(a+b+2/r)(br+1)}. \tag{15}$$

For example, if minimum-life is $a = 200$ hours, maximum-life is $b = 1200$ hours, i.e., TTF ~ $U(200, 1200)$, and repair-rate is 0.02 per hour, then equation (14) gives $A(700\text{ hours}) = 0.9156344$. Further, equation (15) shows that $A_{\text{ave}} = 0.90667$, the lub on AVL is 0.96000, while A_{inf} = $\text{MTTF}/(\text{MTTF} + \text{MTTR}) = 0.93333$.

Case (c). Suppose TTF is distributed like gamma with minimum-life $\delta \geq 0$, shape $\alpha = 2$ and scale $\beta = 1/\lambda$; as before the repair-rate is a constant at *r*. It can easily be verified that

$$R(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq \delta, \\ (1+\lambda x)e^{-\lambda(t-\delta)}, & \delta \leq t < \infty, \end{cases} \quad \text{where } x = t - \delta \geq 0$$

and the HZF is $h(t) = \lambda(\lambda x)/(1+\lambda x)$. Clearly, the AVF for the interval $[0, \delta]$ is equal to 1. In order to obtain the exact expression for $A(t)$ during $[\delta, \infty)$ given in equation (7b), again we have to obtain the antiderivative

$$I(t) = \int [re^{rt}/R(t)]dt = \int re^{rt}/[(1+\lambda x)e^{-\lambda(t-\delta)}]dt$$

$$= re^{r\delta}\int [e^{\xi x}(1+\lambda x)^{-1}]dx, \tag{15a}$$

where we have transformed $t - \delta$ to x so that $dt = dx$ and $\xi = \lambda + r$. Expanding $(1+\lambda x)^{-1}$, $x = t - \delta \geq 0$, geometrically in (15a) we obtain

$$I(t) = re^{r\delta}\int \left[e^{\xi x}\sum_{n=0}^{\infty}(-\lambda x)^n \right]dx = re^{r\delta}\sum_{n=0}^{\infty}\int [e^{\xi x}(-\lambda x)^n]dx. \tag{15b}$$

Bearing in mind that the convergence-radius of $\sum_{n=0}^{\infty}(-\lambda x)^n$ is $0 \le \lambda(t-\delta) < 1$, repeated integration by parts, and letting $\omega = r/\xi$, will reduce equation (15b) to

$$I(t) = \omega e^{r\delta} e^{\xi x} \sum_{n=0}^{\infty} \sum_{k=0}^{n} ({}_nP_k) \times \omega^k (-\lambda x)^{n-k}, \tag{15c}$$

where we remind the reader that $x = t - \delta$. Separating out the constant term from the double-sum on the RHS of (15c) yields

$$I(t) = \omega e^{r\delta} e^{\xi x} \left[\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} ({}_nP_k) \times \omega^k (-\lambda x)^{n-k} + \sum_{n=0}^{\infty} (n! \times \omega^n) \right]. \tag{15d}$$

As a result the AVF for $t \ge \delta$ from equation (7b) is given by

$$A(t) = e^{-rt} R(t) \times \left\{ C + \omega e^{r\delta} e^{\xi x} \left[\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} ({}_nP_k) \times \omega^k (-\lambda x)^{n-k} + C_0 \right] \right\}, \tag{16}$$

where $C_0 = \sum_{n=0}^{\infty} (n! \times \omega^n)$. In order to solve for constant C, we require the initial-condition that $A(x = 0) \equiv 1$; this yields $C = e^{r\delta}(1 - \omega C_0)$, where $0 < \omega = \lambda/\xi \ll 1$. Substituting for C into equation (16) and bearing in mind that $x = t - \delta$, we obtain

$$A(t) = \begin{cases} 1, \ 0 \le t \le \delta, \\ (1 + \lambda x) \times \left\{ (1 - \omega C_0) e^{-\xi x} + \omega C_0 + \right. \\ \left. \qquad \omega \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} ({}_nP_k) \times \omega^k (-\lambda x)^{n-k} \right\}, \ \delta \le t < \infty. \end{cases} \tag{17}$$

Because the gamma mean at shape $\alpha = 2$ is given by $\text{MTTF} = \delta + 2/\lambda = \mu$, it can be argued as in the previous case, where we now divide AVL intervals into $[0, \delta), [\delta, \mu)$ and $[\mu, \mu + 1/r]$, that the expected proportion of time the above system is available is given by

$$A_{\text{ave}} = \frac{\mu^2 + \delta/r}{(\mu + 1/r)^2}. \tag{18}$$

For example, if the TTF ~ gamma ($\delta = 500$ hours, $\alpha = 2$, scale $\beta = 1/\lambda = 12500$ hours) and repair-rate is a constant at $r = 0.02$, then $\mu = \text{MTTF} = 500 + 2/0.00008 = 25{,}500$ hours, and equation (18) gives $A_{\text{ave}} = 0.9961282$, which is close to

$$A_{\text{inf}} = \text{MTTF}/(\text{MTTF} + 1/r) = 0.9980431.$$

For $t = 8500$, $r = 0.02$, and $\lambda = 0.00008$, our MATLAB program using equation (17) gives the point AVL of $A(8500) = 0.99844741$, while at $t = 10000$, $A(10000) = 0.99828087$. Note that equation (17) will not give meaningful answers for $A(t)$ if $\lambda x = \lambda(t - \delta) \geq 1$ because $\sum_{n=0}^{\infty} (-\lambda x)^n$ diverges for $\lambda x \geq 1$.

Case (d). Suppose TTF is distributed like Weibull (W) with minimum-life $\delta \geq 0$, characteristic-life θ, and shape (or slope) $\beta \equiv 2, 3, 4, 5, \ldots$, i.e., an exact positive integer; as before the repair-rate is a constant at r. The Rayleigh failure density is a special case of the $W(\delta = 0, \theta, \beta = 2)$. It has been widely known since the early 1950's that the underlying failure distribution is given by $F(t) = 1 - e^{-(\lambda x)^\beta}$, $R(t) = \begin{cases} 1, & 0 \leq t \leq \delta, \\ e^{-(\lambda x)^\beta}, & \delta \leq t < \infty, \end{cases}$

where $x = t - \delta \geq 0$, the HZF is $h(t) = \beta\lambda(\lambda x)^{\beta - 1}$, the cumulative hazard is $H(t) = (\lambda x)^\beta$, and $\lambda = 1/(\theta - \delta)$. Clearly, the AVF for the interval $[0, \delta]$ is equal to 1. In order to obtain the exact expression for $A(t)$ during $[\delta, \infty)$

given in equation (7b), again we have to obtain the antiderivative

$$I(t) = \int [re^{rt}/R(t)]dt = \int [re^{rt}e^{(\lambda x)^\beta}]dt$$

$$= \int \left[re^{rt} \sum_{n=0}^{\infty} \frac{(\lambda^\beta x^\beta)^n}{n!} \right] dt = \sum_{n=0}^{\infty} \int \left[re^{rt} \frac{(\lambda x)^{\beta n}}{n!} \right] dt. \qquad (19a)$$

Because we are restricting the shape only to positive integers 1, 2, 3, 4, ..., repeated integration by parts will show that at a specific n the value of the indefinite-integral under the summation in equation (19a) is given by

$$\int \left[re^{rt} \frac{(\lambda x)^{\beta n}}{n!} \right] dt = \frac{\lambda^{\beta n} e^{rt}}{n!} \sum_{k=0}^{\beta n} [(-1/r)^k ({}_{\beta n}P_k) x^{\beta n - k}]$$

$$= \frac{(\lambda x)^{\beta n} e^{rt}}{n!} \sum_{k=0}^{\beta n} [({}_{\beta n}P_k) \times (-1/(rx))^k].$$

Substituting this last antiderivative into equation (19a) yields

$$I(t) = \sum_{n=0}^{\infty} \left\{ \frac{(\lambda x)^{\beta n} e^{rt}}{n!} \sum_{k=0}^{\beta n} [({}_{\beta n}P_k) \times (-1/(rx))^k] \right\}. \qquad (19b)$$

As a result the AVF for $t \geq \delta$ from equation (7b) is given by

$$A(t) = e^{-rt}e^{-(\lambda x)^\beta} \times \left\{ C + e^{rt} \sum_{n=0}^{\infty} \frac{\lambda^{\beta n}}{n!} \sum_{k=0}^{\beta n} [({}_{\beta n}P_k) \times (-1/r)^k x^{\beta n - k}] \right\}$$

$$= e^{-rt}e^{-(\lambda x)^\beta} \times \left\{ C + e^{rt} \sum_{n=1}^{\infty} \frac{\lambda^{\beta n}}{n!} \sum_{k=0}^{\beta n - 1} [({}_{\beta n}P_k) \times (-1/r)^k x^{\beta n - k}] \right.$$

$$\left. + e^{rt} \sum_{n=0}^{\infty} \frac{(-\lambda/r)^{\beta n}(\beta n)!}{n!} \right\}. \qquad (20a)$$

In order to solve for the constant C, we require that $A(t = \delta) \equiv 1;$ this yields $C = e^{r\delta}(1 - B_0)$, where $B_0 = \sum_{n=0}^{\infty} \frac{(-\omega)^{\beta n}(\beta n)!}{n!}$, $0 < \omega = \lambda/r \ll 1$. Substituting for C into equation (20a), we obtain

$$A(t) = \begin{cases} 1,\ 0 \le t \le \delta, \\ e^{-(\lambda x)^{\beta}} \times \left\{ e^{-rx} + B_0(1 - e^{-rx}) \right. \\ \left. \qquad + \sum_{n=1}^{\infty} \frac{(\lambda x)^{\beta n}}{n!} \sum_{k=0}^{\beta n - 1} [({}_{\beta n}P_k) \times (-1/(rx))^k] \right\},\ \delta \le t < \infty. \end{cases} \tag{20b}$$

For example, if the TTF ~ $W(200,\ 2200,\ 2)$ and repair-rate is a constant at $r = 0.04$ per hour, then $\mu = \delta + (\theta - \delta)\Gamma(1 + 1/\beta) = \text{MTTF} = 200 + 2000\Gamma(1 + 1/2) = 1972.453851$ hours, and equation (18) gives $A_{\text{ave}} = 0.976378$, which is close to $A_{\text{inf}} = \text{MTTF}/(\text{MTTF} + 1/r) = 0.9874841$. For $t = 1500$, $r = 0.04$, and $\lambda = 0.0005$, our MATLAB program using equation (20b) gives the point AVL of $A(1500) = 0.98430786$, while at $t = 3000$, $A(3000) = 0.96646568$.

It must be highlighted that there are computational problems with equation (20b) for larger values of t and slope β, as MATLAB will not do computations for factorials beyond $n = 170$; thus we were unable to verify that for very large values of t that equation (20b) gives results that are close to $\text{MTTF}/(\text{MTTF} + 1/r)$. However, we have proven that at shape $\beta = 1$, equation (20b) identically reduces to $P_0(t) = A(t) = \frac{r}{\xi} + \frac{\lambda}{\xi} e^{-\xi(t-\delta)}$, $\xi = \lambda + r$; MATLAB also verifies this claim computationally.

Case (e). Suppose TTF is distributed like Weibull (W) with $\delta \ge 0$ minimum-life, characteristic-life θ, and shape (or slope) β that is not an exact integer; as before the repair-rate is a constant at r. For example, suppose TTF ~ $W(\delta,\ \theta,\ \beta = 1.5)$; then from equation (15a), $I(t) = \sum_{n=0}^{\infty} \int \left[re^{rt} \frac{(\lambda x)^{1.5n}}{n!} \right] dt$. It should be clear that at $n = 1$, the $\int [re^{rt} (\lambda x)^{1.5}] dt$ has no closed-form antiderivative and hence no exact solution for $A(t)$ can be obtained. Further, approximating this last antiderivative by expanding re^{rt} in a Maclaurin series and using only the first 10 terms of the infinite series will not lead to an adequate approximation. Work will be in progress to develop a method to approximate $A(t)$ for the general class of failure distributions.

5. The Renewal and Availability Functions when TTF is Gamma and TTR is Exponential

It is well known that the LT of an underlying gamma failure density with shape α and scale $\beta = 1/\lambda$ is given by $\bar{f}(s) = \lambda^{\alpha}/(\lambda + s)^{\alpha}$; note that only when α is a positive integer this last closed-form is valid. When α is not an exact positive integer, there is no closed-form solution for the LT of a gamma density because the integration-by-parts never terminates. Thus, in the case of shape being an exact positive integer, i.e., Erlang underlying failure density, we have the well-known LT of AVL:

$$\overline{A}(s) = \frac{1 - \bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]} = \frac{1 - \dfrac{\lambda^{\alpha}}{(\lambda + s)^{\alpha}}}{s\left[1 - \dfrac{\lambda^{\alpha}}{(\lambda + s)^{\alpha}} \times \dfrac{r}{r + s}\right]}$$

$$= \frac{(\lambda + s)^{\alpha}(s + r) - \lambda^{\alpha}(s + r)}{s[(\lambda + s)^{\alpha}(r + s) - \lambda^{\alpha} r]}. \tag{21}$$

At $\alpha = 2$, equation (21) reduces to

$$\overline{A}(s) = \frac{s^2 + (2\lambda + r)s + 2\lambda r}{s[s^2 + (2\lambda + r)s + \lambda^2 + 2\lambda r]} = \frac{c_1}{s} + \frac{c_2}{s - r_1} + \frac{c_3}{s - r_2},$$

where r_1 and r_2 are the roots of the polynomial $s^2 + (2\lambda + r)s + \lambda^2 + 2\lambda r = 0$. Thus, $r_1 = -(\lambda + r/2) - \sqrt{(r/2)^2 - \lambda r}$, $r_2 = -(\lambda + r/2) + \sqrt{(r/2)^2 - \lambda r}$, $c_1 = 2r/(2r + \lambda)$, $c_2 = \dfrac{-\lambda(2\lambda + r + r_1)}{(\lambda + 2r)\sqrt{r^2 - 4\lambda r}}$, and $c_3 = \dfrac{\lambda(2\lambda + r + r_2)}{(\lambda + 2r)\sqrt{r^2 - 4\lambda r}}$. Inverting back to the t-space, we obtain $A(t) = 2r/(2r + \lambda) + c_2 e^{r_1 t} + c_3 e^{r_2 t}$. This last AVF clearly shows that as $t \to \infty$,

$$A(t) \to 2r/(2r + \lambda) = r/(r + \lambda/2) = \text{MTTF}/(\text{MTTF} + \text{MTTR}),$$

and further, $A(0) \equiv 1$, as expected.

Example 2. Suppose a system has an underlying gamma failure distribution with shape $\alpha = 2$, scale $\beta = 1/\lambda = 1000$ hours and TTR has a constant repair-rate $r = 0.05$, then the availability at 500 hours is given by $A(500) = 0.99385551$; while the same system with minimal-repair has an $A(t = 500) = R(500) = \int_t^\infty \lambda(\lambda x)e^{-\lambda x}dx = e^{-\lambda t}(1 + \lambda t) = 0.90979599$. That is, repair will improve availability by 9.24%. We also used our equation (17) with n terminated at 130 and obtained $A(500) =$ 0.99355668 for the same system of $\delta = 0$, $\alpha = 2$, scale $\beta = 1/\lambda = 1000$ hours and $r = 0.05$. The steady-state (or long-term) AVL of such a system as discussed by many other authors is $A = 0.05/(0.05 + 0.0005) =$ 0.99009901.

At $\alpha = 2$, the LT of expected number of cycles reduces to

$$\overline{M}_1(s) = \frac{r\lambda^2}{s^2[s^2 + (2\lambda + r)s + \lambda^2 + 2\lambda r]} = \frac{c_4}{s} + \frac{c_5}{s^2} + \frac{c_6}{s - r_1} + \frac{c_7}{s - r_2},$$

where r_1 and r_2 are the same roots, $c_4 = \dfrac{-r(2\lambda + r)}{(\lambda + 2r)^2}$, $c_5 = \lambda r/(\lambda + 2r)$, $c_6 = \dfrac{c_5 - c_4 r_2}{\sqrt{r^2 - 4\lambda r}}$, and $c_7 = \dfrac{c_4 r_1 - c_5}{\sqrt{r^2 - 4\lambda r}}$. Upon inversion, we obtain $M_1(t) = c_4 + c_5 t + c_6 e^{r_1 t} + c_7 e^{r_2 t}$. For the same parameters as the above example, we obtain $M_1(t = 10{,}000\text{ hours}) =$ 4.69561808 expected cycles. Similarly, it can be shown that the LT of the expected number of failures is given by

$$\overline{M}_2(s) = \frac{\lambda^2(r+s)}{s^2[s^2 + (2\lambda + r)s + \lambda^2 + 2\lambda r]} = \frac{c_8}{s} + \frac{c_9}{s^2} + \frac{c_{10}}{s - r_1} + \frac{c_{11}}{s - r_2},$$

where $c_8 = \dfrac{(\lambda^2 - r^2)}{(\lambda + 2r)^2}$, $c_9 = \lambda r/(\lambda + 2r)$, $c_{10} = \dfrac{(2\lambda + r + r_1)c_8 + c_9}{\sqrt{r^2 - 4\lambda r}}$, and $c_{11} = \dfrac{-(2\lambda + r + r_2)c_8 - c_9}{\sqrt{r^2 - 4\lambda r}}$. Upon inversion to the t-space, we obtain $M_2(t) = c_8 + c_9 t + c_{10}e^{r_1 t} + c_{11}e^{r_2 t}$. The value of expected number of failures during a mission of length 10,000 hours is $M_2(t = 10000) =$ 4.70551907, which exceeds $M_1(10000) =$ 4.69561808, as expected. Further, $M_1(10000) - M_2(10000) + 1 =$ 0.99009901, which is identical to the value of AVF obtained from $A(t) = 2r/(2r + \lambda) + c_2 e^{r_1 t} + c_3 e^{r_2 t}$ at $t = 10000$.

Unfortunately, when TTF is Erlang at $\alpha = 3, 4, 5, 6$ and 7 and a constant repair-rate r, the corresponding LT denominators $D(s) = s[1 - \bar{f}(s) \times \bar{r}(s)]$ have at least 2 complex roots, which have been well known to be complex conjugate pairs. Yet, after partial-fractioning, the LT's can be inverted to yield real-valued $M_1(t)$ and $M_2(t)$, as demonstrated below.

At $\alpha = 3$,

$$\overline{M}_1(s) = \frac{\bar{f}(s) \times \bar{r}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]} = \frac{\lambda^3}{(\lambda + s)^3} \times \frac{r}{r+s} \Bigg/ \left\{ s\left[1 - \frac{\lambda^3}{(\lambda+s)^3} \times \frac{r}{r+s}\right]\right\}$$

$$= \frac{\lambda^3 r}{s[(\lambda + s)^3(r + s) - r\lambda^3]}$$

$$= \frac{\lambda^3 r}{s^2[s^3 + (3\lambda + s)s^2 + (3\lambda r + 3\lambda^2)s + \lambda^3 + 3r\lambda^2]}$$

$$= \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s - r_1} + \frac{c_4}{s - r_2} + \frac{c_5}{s - r_3},$$

where the root r_1 will be real, while r_2 and r_3 will be complex conjugates, i.e., both $r_2 + r_3$ and $r_2 \times r_3$ will for certain be real numbers. In order to maintain equality in the above partial fraction, it can be shown that

$$c_2 = \frac{-\lambda^3 r}{r_1 r_2 r_3}, \qquad c_1 = c_2 \times \frac{r_1 r_2 + r_1 r_3 + r_2 r_3}{r_1 r_2 r_3};$$

further, letting the constants $a_1 = c_2(r_1 + r_2 + r_3) - c_1(r_1 r_2 + r_1 r_3 + r_2 r_3)$, $a_2 = c_2 - c_1(r_1 + r_2 + r_3)$, then c_3, c_4, and c_5 are the unique solutions given by $C = [c_3\ c_4\ c_5]' = A^{-1} \times b$, where C is the 3×1 solution vector, b is a 3×1 vector $b = \begin{bmatrix} a_1 \\ a_2 \\ -c_1 \end{bmatrix}$ and the 3×3 matrix

$$A = \begin{bmatrix} r_2 r_3 & r_1 r_3 & r_1 r_2 \\ r_2 + r_3 & r_1 + r_3 & r_1 + r_2 \\ 1 & 1 & 1 \end{bmatrix}.$$

A MATLAB program was devised to obtain the expected number of cycles $M_1(t)$ as outlined above. The program also uses similar procedure to compute $M_2(t)$ and the resulting $A(t)$. The MATLAB program has the capability to compute the 3 renewal measures $M_1(t)$, $M_2(t)$, and $A(t)$ for shapes $\alpha = 2, 3, 4, 5, 6$ and 7.

6. Conclusions

This article provided the exact RNFs and AVF for the case of normal TTF and TTR. Then Markov analysis was used to obtain the AVL functions for the cases of 2-parameter exponential TTF, the uniform, the gamma at shape 2, and Weibull with positive integer shape TTF, while the repair-rate was held constant at *r*. Further, LTs were used to obtain both RNFs and the AVF of the gamma at shapes $2, 3, 4, \ldots, 7$ at constant repair-rate. The gamma AVL obtained in Sections 4 and 5 were the same up to 4 decimal accuracies.

References

[1] S. Maghsoodloo and D. Helvaci, Renewal and renewal-intensity functions with minimal repair, J. Quality and Reliability Engineering 2014 (2014), Article ID 857437, 10 pp.

[2] U. N. Bhat, Elements of Applied Stochastic Processes, 2nd ed., John Wiley & Sons Inc., 1984.

[3] D. R. Cox and H. D. Miller, The Theory of Stochastic Processes, Wiley, New York, 1968.

[4] E. A. Elsayed, Reliability Engineering, 2nd ed., Wiley, Hoboken, N.J., 2012.

[5] K. S. Trivedi, Probability and Statistics with Reliability, Queuing and Computer Science Applications, Prentice Hall, New York, 1982.

[6] F. Hildebrand, Advanced Calculus for Applications (1962 Edition), Prentice-Hall, 1962.

[7] L. M. Leemis, Reliability: Probabilistic Models and Statistical Methods, 2nd ed., Ascended Ideas, 2009.

[8] C. E. Ebeling, An Introduction to Reliability and Maintainability Engineering, 2nd ed., Waveland Press, Long Grove, Ill, 2010.