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Strengths and Limitations of Taguchi's Contributions to Quality, Manufacturing, and Process Engineering

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Abstract

This paper reviews Genichi Taguchi's contributions to the field of quality and manufacturing engineering from both a statistical and an engineering viewpoint. His major contributions are first listed and then described in a systematic and analytical manner. The concepts underlying Taguchi's univariate quality loss functions (QLFs), his orthogonal arrays (OAs), robust designs, signal-to-noise (S/N) ratios, and their corresponding applications to quality and process engineering are examined and described in great detail. Some of Taguchi's OAs are related to the classical (fractional) factorial designs (a field that was started by Sir Ronald A. Fisher in the early 1920s). The applications of Taguchi's robust (parameter and tolerance) designs to manufacturing engineering are illustrated through designed experiments.

Keywords: *Taguchi Methods, Quality Loss Functions, Orthogonal Arrays, Parameter Design, Tolerance Design, Signal-to-Noise Ratios*

1. Introduction

Dr. Genichi Taguchi is a Japanese engineer whose contributions to the field of quality engineering were not publicized in the Western Hemisphere until the early 1980s. There have been at least 100 articles written on Taguchi methods (a term that was coined by the American Supplier Institute (ASI), Inc., and one that Taguchi did not greatly admire) in western journals in the past 20 years, some very supportive of his methodology and others critical of his quality engineering methods. On the other hand, some articles have fairly and justly evaluated his significant contributions without any bias. Further, there have been many papers on the extension of his methodology and its applications. The objective in this paper is to present an analytical review of his contributions and present examples to illustrate the applications of Dr. Taguchi's methodology to product and process engineering.

2. Taguchi's Contributions to Quality Engineering and Design of Experiments

1. Taguchi quantified the definition of Quality using Karl Gauss's quadratic loss function.
2. He introduced orthogonal arrays (OAs), although almost half of them are the classical fractional (or factorial) designs developed by Sir Ronald A. Fisher, G.E.P. Box and J.S. Hunter, F. Yates, O. Kempthorne, S.R. Searle, N.R. Draper, R.L. Plackett and J.P. Burman, J.W. Tukey, H. Scheffé, and countless others. Taguchi's OAs, however, are already in developed format so that the engineer does not have to design the experiment from scratch, even though the engineer should have some knowledge of their development to make proper use of Taguchi's OAs. In other words, the contribution Dr. Taguchi has made in this area is simply making it easier for an engineer to use design of experiments (DOE).
3. Taguchi introduced robust (i.e., parameter and tolerance) designs.
4. He defined a set of measures called signal-to-noise (S/N) ratios that combine the mean and standard deviation into one measure in analyzing data from a robust design.

The following sections will discuss these topics in the same order as they were introduced above and highlight a few strengths and minor limitations of Taguchi's contributions that have been debated by the statistical and engineering community. The emphasis will be to illustrate to the process engineer how to apply Taguchi methods and, in general, DOE. Further, for the readers' convenience, a comprehensive list of symbols and acronyms is provided under Nomenclature before the References section.

3. Taguchi's Definition of Quality

According to Taguchi, quality is the amount of losses a product imparts to society from the moment of shipment. Let X and Y be measurable, static, continuous quality characteristics (QCHs); then a QCH can be of two types: (1) magnitude or (2) nominal. If a QCH is of magnitude type, then Y may be smaller-the-better (STB), or the QCH Y may be larger-the-better (LTB), in which case X = 1/Y must be an STB-type QCH. Thus, when the QCH (such as Y = yield, efficiency, or breaking strength) is LTB, a simple transformation, X = 1/Y, is made so that X is now STB.

Examples of directly STB QCHs are lateral force harmonic or eccentricity of a tire, loudness of compressors or engines, warpage, and braking distance of an automobile. All STB static QCHs have two aspects in common: (1) their ideal target is zero, (2) they all have only a single consumer's upper specification limit (USL), denoted by y_u. In real-life engineering applications, when y is an STB QCH its values can never be negative (for example, braking distance cannot be negative). Similarly, if the response y is LTB, then the ideal target is ∞, y > 0, and y has only a single lower specification limit, LSL = y_L.

Several notable quality gurus (such as W. Edwards Deming) have alluded to the fact that when a product barely meets specification, its quality level cannot much differ from one that does not barely meet specifications. For example, if the design tolerance for a resistor is 5 ± 0.10 ohm, then there can hardly be any difference between the quality levels of two resistors with resistances of 5.099 and 5.101 ohms. However, to the best of our knowledge, Genichi Taguchi was the first quality engineer who recommended the use of this very concept (through Gauss's quadratic loss function) to quantify quality. Accordingly, the quality loss of an item according to Taguchi is defined as in Eq. (1).

$$L(y)=\begin{cases} k\,y^2, & \text{y is an STB type QCH} \\ k/y^2 = k\,x^2, & \text{y is an LTB type QCH, and x is STB} \\ k(y-m)^2, & \text{y is a nominal dimension} \end{cases} \tag{1}$$

When y is a nominal dimension, the design (or consumer's) specifications are given by m ± Δ, where m stands for the midpoint of tolerance range starting from LSL = m – Δ to USL = m + Δ, and the constant k is always positive. Some authors use τ and a few others use T for the ideal target, but m was used to represent midpoint by Taguchi. Further, Taguchi refers to Δ as the allowance on either side of the midpoint, m, of tolerance range, while in manufacturing engineering, Δ is usually referred to as the tolerance (or perhaps the semi-tolerance). Note that we are considering only the case of symmetric tolerances, but asymmetric tolerances often occur in manufacturing processes and are treated by several authors (see Taguchi, Elsayed, and Hsiang 1989, pp. 30-39; Maghsoodloo and Li 2000; Li 2000). The quality loss functions (QLFs) in Eq. (1) measure how far the dimension of a unit is from its ideal target value m. The farther y departs from its ideal target, the larger the QLF becomes exponentially (in powers of 2). The constant k can be computed immediately once the amount of quality loss (QL) at a specification limit is known. There is generally more information about quality losses at a specification limit than at any other point in the y-space (due to customer complaints). For example, if the 5-ohm resistor with specs 5 ± 0.10 imparts a monetary loss of \$0.30 at the LSL = 4.90, or USL = 5.10 ohms, then $0.30 = k(5 \pm 0.10 - 5)^2 \rightarrow k = 30.00$ \$/ohm^2. Thus, the QLF for one resistor takes the form L(y) = \$30$(y - 5)^2$, and the constant k has converted the units of $(y - 5)^2$, which is ohms^2, to dollars. Taguchi's QLF can take into account not only the QL due to deviation from the ideal target but also all losses to society (such as repair cost, time loss to the consumer, damage to the environment, warranty cost, and all other side effects from the use of the product). Accordingly, depending on what kind of societal losses L(y) represents, the positive constant k should be computed in such a manner that all pertinent societal losses at a specification limit are taken into account.

Before illustrating an application of the quadratic loss function, suppose that we take a random sample of size n from a process or from a supplier's lot. The loss due to the i-th unit in the sample for a nominal dimension will equal $L_i = k(y_i - m)^2$. If the measurable QCH is STB, then $L_i = k\,y_i^2$, and if y is an LTB type QCH, then $L_i = k/y_i^2 = k\,x_i^2$, where $x_i = 1/y_i$. It seems that without loss of generality the total QLs (quality losses) from the n items in the sample is $\sum_{i=1}^{n} L_i = \sum_{i=1}^{n} k(y_i - m)^2$, where m is generally different from zero for a nominal dimension, m = 0 when y is STB, and for an LTB type QCH, $\sum_{i=1}^{n} L_i = \sum_{i=1}^{n} k\,x_i^2$, where again the ideal target for x = 1/y is also zero. Thus, the average QL per unit for a sample of size n is given by $\bar{L} = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{1}{n}\sum_{i=1}^{n} k(y_i - m)^2 = \frac{k}{n}\sum_{i=1}^{n}(y_i - m)^2 = k \times (\text{MSD})$, where the statistic $\frac{1}{n}\sum_{i=1}^{n}(y_i - m)^2$ was named the mean squared deviation (MSD) by Dr. Taguchi, i.e., MSD = $\frac{1}{n}\sum_{i=1}^{n}(y_i - m)^2$ so that $\bar{L} = k \times (\text{MSD})$.

Note that sample MSD measures variation around the ideal target m, while sample variance measures variation from the sample mean $\bar{y}$. It can be shown (see Maghsoodloo 1992, p. 19) that the expectation of sample average QL is given by

$$E(\bar{L}) = k\left[\sigma^2 + (\mu - m)^2\right] \tag{2}$$

where μ is the process mean and σ^2 is the process variance. It now follows that

$$\bar{L} = \frac{k}{n}\sum_{i=1}^{n}(y_i - m)^2 = \frac{k}{n}\sum_{i=1}^{n}\left[(y_i - \bar{y}) + (\bar{y} - m)\right]^2 =$$

$$k\left\{\frac{1}{n}\sum_{i=1}^{n}\left[(y_i - \bar{y}) + (\bar{y} - m)\right]^2\right\} =$$

$$k\left\{\frac{1}{n}\sum_{i=1}^{n}\left[(y_i - \bar{y})^2 + 2(\bar{y} - m)(y_i - \bar{y}) + (\bar{y} - m)^2\right]\right\} =$$

$$k\left\{\frac{1}{n}\left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + 2(\bar{y} - m)\sum_{i=1}^{n}(y_i - \bar{y}) + \sum_{i=1}^{n}(\bar{y} - m)^2\right]\right\} =$$

$$k\left\{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 + 2(\bar{y} - m)\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y}) + \frac{1}{n}\sum_{i=1}^{n}(\bar{y} - m)^2\right\} =$$

$$k\left\{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 + 2(\bar{y} - m)\frac{1}{n}(0) + (\bar{y} - m)^2\right\} =$$

$$k\left\{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 + (\bar{y} - m)^2\right\} \rightarrow$$

$$\bar{L} = \frac{1}{n}\sum_{i=1}^{n} L_i = \frac{k}{n}\sum_{i=1}^{n}(y_i - m)^2 = k\left[\hat{\sigma}_n^2 + (\bar{y} - m)^2\right] \tag{3}$$

where $\hat{\sigma}_n^2 = (n-1)S^2/n$ is the sample variance and $S^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2 / (n-1)$ is the unbiased estimator of process variance σ^2.

The above developments leading to Eq. (3) make use of the fact that the sum of deviations of n data points from their own sample mean $\bar{y}$ is identically zero, i.e., $\sum_{i=1}^{n}(y_i - \bar{y}) \equiv 0$ and the fact that $\sum_{i=1}^{n} c = n \times c$ for any constant c relative to the index i, i.e., $\sum_{i=1}^{n}(\bar{y} - m)^2 = n(\bar{y} - m)^2$. Equation (3) reveals a remarkable story, and that is, if we wish to reduce quality losses, then we must reduce process variance σ^2 estimated by the biased sample variance $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$ and then get the process mean on target. That is to say, at the second step, we must reduce $(\mu - m)^2$, which is estimated by $(\bar{y} - m)^2$. Note that the estimator $(\bar{y} - m)^2$ of the off-center parameter $(\mu - m)^2$ is also biased because $E\left[(\bar{y} - m)^2\right] = E\left[(\bar{y} - \mu) + (\mu - m)\right]^2 = (\mu - m)^2 + E\left[(\bar{y} - \mu)^2\right] = (\mu - m)^2 + V(\bar{y}) = (\mu - m)^2 + \frac{\sigma_y^2}{n}$, which implies that the amount of bias in the estimator $(\bar{y} - m)^2$ is $B\left[(\bar{y} - m)^2\right] = \frac{\sigma_y^2}{n}$. In this last development, E stands for the expected value, which is a linear operator, and V stands for the variance operator, which is nonlinear.

One major application of Taguchi's QLF is the comparison of two (or more) suppliers. Suppose we wish to compare the breaking strength of cables from two suppliers for which the consumer's LSL is 1400 psi = 1.40 ksi. Although direct measurement of breaking strength may be very difficult because it would involve destructive testing, as in example 3.3 of Taguchi, Elsayed, and Hsiang (1989), we may be able to use the fact that the cable's breaking strength is directly proportional to its cross-sectional area. Because this example is developed just to illustrate the application of the quadratic loss to quantify quality in dollars, we assume that we will be able to determine breaking strength directly and that the manufacturing cost of the two cables is almost the same. Further, we assume that field failure is very expensive and results in a societal loss of \$5000.00 per unit. That is, when y = 1.40 ksi for a single cable, then L(y) = \$5000.00. Because $L(y) = k/y^2$, this yields $5000 = k/(1.4^2) \rightarrow k = 9800\ \$(\text{ksi})^2 \rightarrow L(y) = \$9800/y^2 = \$9800x^2$ per cable. Suppose we randomly measure n_1 = 10 cables' strength from suppl[illegible]1 and $n_2 = 13$ from supplier 2. In practice, one should allocate more observations to the process that has larger variability. For example, if we have total resources of $N = 42$ items to be sampled and we know that $\sigma_2 = 2\sigma_1$, then process 2 must be allocated a sample of size $n_2 = \left(\frac{\sigma_2}{\sigma_1+\sigma_2}\right)N = \frac{2}{3} \times 42 = 28$, while $n_1 = 14$. Of course, the experimenter may not have any knowledge of the variability of the two processes, in which case the sampling must be done in (at least) two stages to assess variation at the first stage, followed by completing the sample allocations at the second stage. The two data sets are given below.

y_{1j} : 1.5, 1.4, 1.7, 1.5, 1.6, 1.5, 1.8, 1.8, 1.7, 1.6
y_{2j} : 1.9, 1.9, 2.2, 2.5, 1.6, 2.1, 2.0, 1.8, 1.7, 2.5, 2.1, 1.8, 1.5 ksi

It seems that from a traditional view of quality there are no differences in the quality of the two suppliers because neither sample contains a nonconforming unit (that is, all y values $\geq$ 1.4 ksi). However, based on the modern view of quality using Taguchi's QLF, there is much difference between the quality levels of the two types of cables. Converting to the variable $x = 1/y$, we obtain

x_{1j} : 0.6667, 0.7143, 0.5882, 0.6667, 0.6250, 0.6667, 0.5556, 0.5556, 0.5882, 0.6250
x_{2j} : 0.5263, 0.5263, 0.4545, 0.4000, 0.6250, 0.4762, 0.5000, 0.5556, 0.5882, 0.4000, 0.4762, 0.5556, 0.6667

Note that x is an STB type QCH with ideal target $m = 0$.

Then, $\text{MSD}_1 = \frac{1}{10}\sum_{i=1}^{10} x_i^2 = \hat{\sigma}_x^2 + (\bar{x})^2 = 0.002553 + (0.62518674)^2 = 0.3934113$, where the sample variance $\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. Then the average quality loss of supplier 1 is $\bar{L}_1 = 9800 \times 0.3934113 = \$3855.431$ per cable. Similarly, for the supplier 2, $\bar{x}_2 = 0.5193$ and $\hat{\sigma}_2^2 = 0.005932 \rightarrow \text{MSD}_2 = 0.005932 + (0.5193)^2 = 0.27558 \rightarrow \bar{L}_2 = 9800 \times 0.27558 = \$2700.66673$ per cable, and hence the quality difference of supplier 2 over supplier 1 is $\text{QD}_{21} = \$1154.7639$ per cable. This is in complete contrast to zero QD between the two suppliers from the traditional (or conventional) quality viewpoint.

The quality engineer should observe that when y is a magnitude type QCH then—unless the coefficient of variation $\widehat{cv} = \hat{\sigma}_x / \bar{x}$ (or variation coefficient, which is the reciprocal of S/N ratio and is equal to 8.0817% for the sample of supplier 1, and 14.83% for supplier 2) exceeds 30% for an STB QCH and 17% for an LTB QCH—the mean (or the signal) will play a much more important role in improving quality than reducing variability (Maghsoodloo 1990). Further, if the QCH values are far above the LSL (for LTB) and far below USL (for an STB type QCH), then we can withstand quite a bit of variation compared with when the output values are close to a specification limit. For the above example on breaking strength, given that the breaking strength's LSL = 1.40 ksi, the sample 3.1, 5.6, 1.9, 2.8, 7.9 ksi (with variance $\hat{\sigma}_1^2 = 4.81840$) implies far superior quality over the sample 1.6, 1.7, 1.5, 1.6, 1.7 ksi (with variance $\hat{\sigma}_2^2 = 0.00560$). In fact, the QD of the former sample over the latter is equal to $\text{QD}_{12} = 3758.7624952 - 1090.795726 = \$2667.96677$ per cable, but the y-variance of the former sample is 860.4286 times the y-variance of the latter sample, and the x-variance of the former sample is 23.5551 times larger than the latter. (Note that for $x > 1$ the transformation $x = 1/y$ is of the variance-reduction type, causing the x-variance of the first sample to reduce substantially more than that of the second sample.)

However, the above assertion cannot be made for a nominal dimension because the process mean μ may be below the ideal target m (in which case the signal must be increased), or μ may be above the midpoint of tolerances, m, in which case the amount of off-center $(\mu - m)$ exceeds 0, and the signal must be reduced to improve quality. Further, the process variance σ^2 will almost always play an important role for a nominal dimension in quality improvement (QI) because the standardized amount of off-center $(\mu - m)/\sigma$ is a truer measure of lack of quality than is the off-center distance $|\mu - m|$.

Relationship Between Natural Tolerances and Taguchi's Expected Quality Loss for a Nominal Dimension

By definition, if a process is six-sigma capable (that is, process capability ratio $\text{PCR} = C_p = \frac{\text{USL}-\text{LSL}}{6\sigma} = 1$), then in the symmetric case the distance between USL and LSL is exactly six sigma, that is, six-sigma process capability implies that USL – LSL = 6σ, which in turn implies that $2\Delta = 6\sigma$, or $\sigma = \Delta/3$. Note that six-sigma capability is different from Motorola's definition of Six-Sigma Quality, where both the LSL and USL are at six sigma below and above the process mean μ, respectively.

However, as Montgomery (2001b, p. 24) points out, and we concur, there is a slight inconsistency in this definition when the process becomes off-centered or is out of control only with respect to its mean. In fact, we think that there is a slight misconception with this definition when the process is out of control only with respect to its mean. The problem arises due to the fact that the values of LSL = m − Δ and USL = m + Δ are fixed and set by the manufacturing designer or consumer, and consequently, a machining process is either capable of meeting specifications (i.e., $p < \alpha$) or not capable (i.e., $p > \alpha$), where p is the process fraction nonconforming (FNC) and α is the company-wide tolerable FNC set in such a manner that the company will prosper in global competition. If the process is centered such that μ = m, the point that we are raising is moot; however, when μ shifts, say by one sigma to the right of m, then it will be impossible for both LSL and USL to be six sigma away from μ because LSL and USL are fixed and do not change as μ shifts. In fact, with an upward shift in μ of one sigma to the right of m, the LSL will be seven sigma to the left of μ, while the USL will be five sigma to the right of μ.

It would be more prudent to incorporate Taguchi's view of quality with Motorola's definition of six-sigma quality and slightly modify Motorola's six-sigma quality as LSL and USL at six sigma from the ideal target m (not μ). Notwithstanding, this modification will not alter the amount of FNC that a process produces when only the mean is out of control. For example, when a process is Gaussian and centered (i.e., μ is in control at m) and operating at Motorola's six-sigma quality, then its FNC is 0.00197317540085 parts per million (ppm), or roughly 2 parts per billion. However,

if the mean of a Gaussian process is out of control by $\frac{1}{2}$ sigma (i.e., $\mu = m \pm \sigma/2$), its FNC is increased to 0.0190297223534586 ppm;
if the mean of a Gaussian process is out of control by 1 sigma (i.e., $\mu = m \pm \sigma$), its FNC increases to 0.28665285167762 ppm;
if the process is off-centered by $1\frac{1}{2}$ sigma (i.e., $\mu = m \pm 1.5\sigma$), then the Gaussian process FNC increases to 3.3976731564911 ppm;
if $\mu = m \pm 2\sigma$, the FNC increases to 31.671241833786 ppm;
if $\mu = m \pm 2.5\sigma$, the FNC increases to 232.62907903554 ppm;
if $\mu = m \pm 3\sigma$, the FNC increases to 1349.898031630096 ppm;
if $\mu = m \pm 3.5\sigma$, the FNC increases to 6209.66532578 ppm;
if $\mu = m \pm 4\sigma$, the process FNC increases to 22750.13194818 ppm;
if $\mu = m \pm 4.5\sigma$, the process FNC increases to 66807.20126886 ppm;
and finally, when the mean is way out of control by 5σ, the Gaussian FNC increases to 158655.253931457 ppm (or roughly 0.158655254).

In general, if $\mu = m \pm r\,\sigma$ ($r \geq 0$), the Motorola Gaussian FNC is given by $p_M = 2 - \Phi(6 - r) - \Phi(6 + r)$, where Φ represents the cdf (cumulative distribution function) of a unit normal distribution.

In practice, the values of process mean and standard deviation, μ and σ, are generally unknown and have to be estimated by the sample statistics $\bar{y}$ and S, respectively. To ascertain whether a process (producing a nominal dimension y) is operating at Motorola's six-sigma quality, one must compute the tolerance interval $\bar{y} \pm K\,S$ and determine if this tolerance interval is contained in the tolerance range (LSL, USL). Unfortunately, the values of the tolerance factor, K, for a tolerable FNC of α = 2 parts per billion at any confidence probability of γ has not been reported, as far as we know, in statistical literature. Ms. Hsin-Cheng Chiu (1995) developed a software that computes the values of K for nearly any situation, but her program does not allow an α value less than 0.000001 (= 1 ppm, which pertains to Motorola's 4.892-sigma quality not 6). For example, for a random sample of size n = 20 from a $N(\mu, \sigma^2)$, α = 0.000001, γ = 0.99, and a nominal dimension y, her program reports K = 7.88, while for n = 30 her program gives K = 7.01. Further, the Tolerance Limits software reports that a random sample of size at least n = 83 is needed to obtain a tolerance factor of $K \leq 6$ at α = 0.000001 and γ = 0.99. Since obtaining the exact tolerance factor to attain Motorola's six-sigma quality is not an easy task, it is best to estimate the sample FNC assuming that the process is Gaussian. Thus, the recommendation is first to compute $\hat{Z}_L = (LSL - \bar{y})/S$, $\hat{p}_L = \Phi(\hat{Z}_L)$, $\hat{Z}_U = (USL - \bar{y})/S$, $\hat{p}_U = \Phi(-\hat{Z}_U)$, and $\hat{p} = \hat{p}_L + \hat{p}_U$. If $\hat{p} = \hat{p}_L + \hat{p}_U$ is less than two parts per billion, then conclude that the machining process is practically operating at six-sigma quality; otherwise, the process is not operating at six-sigma quality. Further, the experimenter must be cognizant that the statistic $\hat{p}_L + \hat{p}_U$ is subject to sampling error and/or fluctuations and the fact that $\hat{p}$ is less than two parts per billion does not guarantee that the machining process is operating at Motorola's six-sigma quality.

To relate the natural tolerances of a normal process to Taguchi's QLF, the four most common possibilities (out of infinite) are considered, as outlined below:

(i) $\mu = m$ and a Six-Sigma Process Capability Ratio ($C_p = 1$) $\rightarrow$ USL – LSL = 6σ

$E(QL) = k\ \sigma^2 = \frac{A_c}{\Delta^2}\ [(USL - LSL) / 6]^2 = \frac{A_c}{\Delta^2}(2\Delta/6)^2 = A_c/9$; note that in this case $\sigma = \Delta/3$, where A_c is the amount of QL (quality loss) at either the LSL, or USL. This is in contrast to the evaluation of quality from a conventional viewpoint because if a process is six-sigma capable and is also normally distributed with $\mu = m$, then the amount of traditional QL (based on either meeting or not meeting specs) is simply $0.0027 \times A_c$ because the traditional QL function is given by $QL_{Trad}(y) = \begin{cases} 0, & \text{if } LSL \le y \le USL \\ A_c, & \text{otherwise} \end{cases}$. This implies that the conventional (or traditional) method of quality evaluation underestimates quality losses by 97.57%.

(ii) $\mu = m$ and an Eight-Sigma Process Capability $\rightarrow$ USL – LSL = 8σ

$\rightarrow$ PCR = $1.3333\overline{3}$ and $E(QL) = \frac{A_c}{\Delta^2}\ [(USL - LSL) / 8]^2 = A_c/16$, but $E(QL_{Trad}) = 0.0000633425\ A_c$. In this case, the conventional method of quality evaluation underestimates quality losses by 99.8986520%.

(iii) $\mu = m + 0.50\sigma$ and a Six-Sigma Process Capability $\rightarrow$ USL – LSL = 6σ

$E(QL) = k\ [\sigma^2 + (\mu - m)^2] = \frac{A_c}{\Delta^2}\ [(\Delta/3)^2 + 0.25\ (\Delta/3)^2] = A_c\ (1/9 + 0.25/9) = 1.25\ A_c/9$, while $E(QL_{Trad}) = 0.0064423\ A_c$, underestimating QLs by 95.361544%.

(iv) $\mu = m + 0.50\sigma$ and an Eight-Sigma Process Capability $\rightarrow$ USL – LSL = 8σ

PCR = $1.3333\overline{3} \rightarrow \sigma = \Delta/4 \rightarrow E(QL) = \frac{A_c}{\Delta^2}\ [(\Delta/4)^2 + 0.25\ (\Delta/4)^2] = 1.25\ A_c/16$, and $E(QL_{Trad}) = 0.00023603\ A_c$, underestimating QLs by 99.6978816%.

In general, if a process is off-centered such that $\mu - m = r\ \sigma$ ($r \neq 0$) and PCR stands at $t \times \sigma$, that is, $t \times \sigma = 2\Delta$, then it can be verified that the $E(QL) = 4(1 + r^2)A_c/t^2$. Further, if a process is Gaussian (that is, normal) and off-centered by $r\ \sigma$ and operating at a PCR $= t\ \sigma$, then the amount of traditional QL is equal to $E(QL_{Trad}) = A_c[\Phi(-r - t/2) + \Phi(r - t/2)]$, where $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}du$ is the cumulative distribution of the standardized (or unit) normal random variable.

As an example, suppose that QI (quality improvement) efforts on a machine have improved the process mean from the off-target value of $m + 0.75\sigma$ to $\mu = m$ (i.e., the process has been centered) and the existing six-sigma process capability (that is, PCR = 1) has been improved to seven-sigma process capability, that is, PCR = 7/6. We wish to compute the percent reduction in Taguchi's expected societal QLs and also the amount of conventional QI. Before QI, the amount of expected Taguchi's QL is given by $E(QL_b) = 4(1 + 0.75^2)A_c/6^2 = 0.173611\ A_c$; after QI, $E(QL_a) = 4A_c/7^2 = 0.081633 \times A_c$, so that the amount of QI is given by $0.09198\ A_c$. The conventional expected QL before improvement is given by $A_c[\Phi(-0.75 - 6/2) + \Phi(0.75 - 6/2)] = A_c[\Phi(-3.75) + \Phi(-2.25)] = A_c[0.00008842 + 0.0122245] = 0.012313A_c$. After QI, the traditional QL is given by $A_c[\Phi(-0 - 7/2) + \Phi(0 - 7/2)] = 2A_c\Phi(-3.5) = 2A_c \times 0.000233 = 0.0004653A_c$. This yields a conventional QI equal to $0.01185A_c$, which underestimates the expected QI by 87.12%.

We have not found articles that are critical of Dr. Taguchi's use of Karl Gauss's quadratic loss function to quantify product quality. In fact, Pignatiello and Ramberg (1991) list this contribution as number four among Taguchi's top-10 triumphs. The quadratic loss function discovered by Karl Gauss has been in existence well over 200 years, but Dr. Taguchi was the first who formalized its use to quantify quality, and hence he fully deserves credit for this particular application of Gauss's quadratic loss function.

4. Literature Review

Taguchi methods have been discussed extensively in different platforms, such as panel discussions, books, articles, and so on, especially since the early 1980s, when applications to different industries began in the Western Hemisphere. Below is a brief (yet incomplete) summary of these discussions. We will not discuss every article published in this area in the past 24 years, but will provide numerous references for the interested reader in the Bibliography section.

Taguchi's two most important contributions to quality engineering are the use of Gauss's quadratic loss function to quantify quality and the development of robust designs (parameter and tolerance design). Taguchi's robust designs have widespread applications upstream in manufacturing to fine-tune a process in such a manner that the output is insensitive to noise factors. Nearly half of this article deals with Taguchi's parameter and tolerance designs.

Several papers about Taguchi methods originated from the Center for Quality and Productivity Improvement (CQPI) at the University of Wisconsin. A number of reports evaluated Taguchi methods from a statistical standpoint. The primary ones are by Box and Fung (1986); Box, Bisgaard, and Fung (1988); Box and Jones (1992); Bisgaard (1990, 1991, 1992); Czitrom (1990); Bisgaard and Diamond (1990); Bisgaard and Ankenman (1993); and Steinberg and Burnsztyn (1993). In these reports, the parameter design received the most attention. These authors confirm that Dr. Taguchi has made important contributions to quality engineering; however, it may not be easy to apply his techniques to real-life problems without some statistical knowledge. Specifically, the use of signal-to-noise ratios in identifying the nearly best factor levels in order to minimize quality losses may not be efficient.

Three important discussions on Taguchi methods have been published in *Technometrics* by Leon, Shoemaker, and Kackar (1987); Box (1988); and Nair (1992). Some other performance measures are given and discussed as alternatives to signal-to-noise ratios by Leon, Shoemaker, and Kackar (1987) and Box (1988). Taguchi's parameter design is discussed extensively by a group of scientists in a discussion panel chaired by Nair (1992). The scientists' major point is that Taguchi methods do not have a statistical basis and signal-to-noise ratios pose some computational problems.

Shoemaker and Tsui (1991) studied Taguchi's parameter design from the standpoint of cost. They claimed that putting controllable and uncontrollable factors in two separate arrays, inner and outer, will result in more experimental runs. Montgomery (1997, pp. 622-641) highlights the same difficulty in a Taguchi parameter design. We tend to agree with these authors that more cost may be involved in a Taguchi crossed-array design than in a combined single-array classical design as long as the output is either an STB or LTB type QCH. When the output is of magnitude type (i.e., QI requires either decreasing the signal or increasing the signal), we illustrated above that unless the CV is larger than 17%, the traditional classical DOE will identify factors that significantly impact the mean of the output, and this will in turn pave the way to improve quality. However, when the output is of nominal dimension, it is best to invest the extra capital to identify the *controls* (these are the factors that control process variation) and *signals* (these are factors that impact the mean but have negligible effect on variability) and go through the Taguchi two-step procedure of first reducing variability followed by getting the mean on target. Thus, our recommendation to any engineer is to use DOE by all means as an upstream QI tool. If the engineer does not have sufficient recourses and the QCH is either STB or LTB and the CV < 20%, then use a single-array classical FFD maximizing design resolution (defined later). If the QCH is of nominal-the-best type, then by all means use Taguchi's parameter design even if more experimentation is required. Further, if the noise factors are environmental variables, it is generally best to place such variables in an outer array and treat them as uncontrollable. Box and Jones (1992) discuss an alternative to a Taguchi crossed-array design when the noise factors are environmental.

Tsui (1996) reviews and gives probable problems with Taguchi methods. He compares Taguchi methods with other alternative approaches in the literature. According to Kim and Cho (2000), it is expensive to arrive at a process having on-target mean and minimum variance with Taguchi methods. They suggest an alternative model based on an asymmetric quality loss to obtain the most economical process mean. Robinson, Borror, and Myers (2004) in a recent article gather previous arguments and alternative approaches to Taguchi methods. Alternative performance measures are discussed and are compared with signal-to-noise ratios. Also Taguchi's parameter design is reviewed from different perspectives.

It is nearly impossible to discuss all of the works related to Taguchi methods. We have tried to mention the main articles that discuss the pros and cons of Taguchi's contributions. There are several other papers that are listed in the Bibliography but are not specifically discussed here.

5. Factorial Designs and Taguchi's Orthogonal Arrays

Because roughly half of Taguchi's orthogonal arrays (OAs) are related to classical (fractional) factorial designs, this section is a short summary of factorial designs that were developed mainly by Sir Ronald A. Fisher (1966), Kempthorne (1952), Yates (1937), Graybill (1961), Tukey (1949), Cochran and Cox (1957), Scheffé (1953, 1956, 1959), Searle (1971a), and other notables. The factorial b^k means that the design matrix, X, contains $k \geq 2$ different factors (or process parameters, or k inputs) each at b levels (b = 2, 3, 4, …), contains b^k factor level combinations (FLCs, or treatment combinations), and possesses exactly k arbitrary columns. The factorial design b^k, $k \geq 2$, is complete (or a full factorial) only if at least one response is obtained at each of the b^k FLCs. Further, if the number of responses at each FLC is the same, namely n, then the design matrix X is also said to be balanced and orthogonal. A matrix X is orthogonal if and only if $(X^TX) = (X'X)$ is diagonal or can be diagonalized through a linear transformation, where T and "′" stand for transpose. The number "b" is called the design base, and the total number of columns of a design matrix will be given in the following subsection. Before relating Taguchi's OA to classical factorial (or fractional factorial) designs, general rules will be listed that will apply to all balanced orthogonal fractional factorial designs (FFDs) to facilitate the understanding of OAs and how to put them to use in practice.

Review of Fractional Factorial Designs

Fractional (or incomplete) factorial designs, or fractional replicates, were developed mainly by Box and Hunter (1961a,b), John (1961, 1962, 1964), Fries and Hunter (1980), Kempthorne (1952), Montgomery and Runger (1996), and other notables. We now summarize the rules that will apply to all balanced and orthogonal FFDs.

1. The notation b^{k-p} means that the design matrix, X, contains $k \geq 3$ different factors (or process parameters, or k inputs) each at b levels (b = 2, 3, 5, etc.), but only the $\frac{1}{b^p}$th fraction of all possible b^k FLCs (or treatment combinations) are experimentally tested. For example, the FFD 2^{5-2} implies that our design matrix has five factors (A, B, C, D, and E) each at two levels, but only $N_{flc} = 2^{5-2} = 2^3 = 8$ distinct FLCs out of the possible $2^5 = 32$ are studied. (Note that N_{flc} stands for the number of distinct FLCs that comprise the design matrix X, and just to simplify notation, we let $N_f = N_{flc}$.) Further, in any design of experiments the grand total number of observations in the entire experiment, assuming a balanced and orthogonal design, can be written as $N = n \times N_f$. Similarly, a 3^{4-1} is a 3^{-1} = 1/3rd fraction of a full 3^4 factorial design, and hence its design matrix will have $N_f = 3^{4-1} = 3^3 = 27$ distinct FLCs (or experimental runs). By a 3^{6-2} FFD we mean a 3^{-2} = 1/9th fraction of a complete 3^6 factorial, and only $N_f = 81 = 3^{6-2} = 3^4$ FLCs out of the possible $729 = 3^6$ FLCs are tested experimentally. The reader should note that for the case of FFDs the values of the design base "b" have been intentionally restricted to prime numbers 2, 3, 5, 7, etc., because of the fact that direct meaningful fractionalization in nonprime bases (such as 4 and 6) is impossible, at least to the knowledge of the authors. Indirect fractionalization in nonprime bases will require the use of pseudo-factors and hence is much more laborious to carry out. Further, the FFD b^{k-p} will have exactly $(b^{k-p} - 1)/(b - 1)$ columns, only $k - p$ of which can be written arbitrarily and will have exactly $N_f = b^{k-p}$ distinct number of rows (or FLCs, or treatments). For example, the design matrix of a 2^{6-2} FFD has $(2^{6-2} - 1)/(2 - 1) = 15$ distinct columns, four of which are arbitrary, and $2^{6-2} = 2^4 = 16$ distinct rows (or distinct FLCs). While a 3^{6-2} FFD has $(3^{6-2} - 1)/(3 - 1) = 40$ columns, only 4 (= 6 – 2) of which are arbitrary, and its design matrix X has $N_f = 3^{6-2} = 3^4 = 81$ rows, or distinct FLCs. The complete factorial b^k can be fractionalized into b^p, ($p < k - 1$, $k \geq 3$), distinct blocks each containing b^{k-p} runs (or distinct FLCs) for b = 2, 3, and 5.
2. The elements (or factor levels) for a base-b design are simply 0, 1, 2, 3, …, b – 1. Taguchi adds 1 to the elements 0, 1, 2, …, b – 1 to ob-

tain his factor levels as 1, 2, 3, …, b. Thus, in the classical FFD notation the elements of base 2 are 0, 1; the levels of base-3 designs are 0, 1, 2; and the elements of base-5 designs are 0, 1, 2, 3, 4, while in Taguchi's designs the levels are (1, 2), (1, 2, 3), and (1, 2, 3, 4, 5), respectively. In base 2 algebra, 2 will equal to 0 (modulus 2) and 3 will be referred to as 1 (mod 2). Similarly, in base 3 algebra, 3 = 0 (modulus 3, or mod 3), 4 = 1 (mod 3), and 5 = 2 (mod 3), etc. For example, a 3^3 factorial design has 27 = 3^3 distinct FLCs starting with 000 (all three factors at their low levels), 001 (factors A and B at their low levels while factor C at its medium level), 002 (factors A and B at their low levels while factor C at its high level), 010, 011, …, 221, and ending with 222 (where all three factors are at their high levels).

3. The degree(s) of freedom of a column (or an effect) in a b^k complete factorial or b^{k-p} FFD is simply b – 1.
4. A generator of a FFD is a high-order effect whose contrast function values (defined below) are the same for all the FLCs in the same fraction (or block) so that the generator's effect is sacrificed (or lost) and thus cannot be studied. The FFD b^{k-p} has exactly p independent generators, which divide the b^k distinct FLCs into b^p different fractions (or blocks). Each block has b^{k-p} distinct FLCs. The principal block (PB) is the one for which all the design generators have the value of 0 for all their contrast functions. Only the PB has the group property that can more easily generate the other $b^p - 1$ blocks.
5. An effect is defined in a b^k factorial design in such a manner that it can occupy only one column and hence it will carry exactly (b – 1) degrees of freedom (df). As an example, in a 2^3 factorial design, there are seven effects, A, B, C, AB, AC, BC, and ABC, each carrying 1 df and each occupying exactly one column, while in a 3^2 full factorial design has four effects, A, B, AB, and AB^2, each with 2 df, and hence a 3^2 factorial design must have $(b^k - 1)/(b - 1) = 4$ distinct columns that are occupied by the orthogonal effects A, B, AB, and AB^2. Note that AB and AB^2 represent the two orthogonal components of the first-order interaction A × B, which carries 2 × 2 = 4 df. Some authors conveniently use AB to denote interaction in any base b, but it should be clear by now that only in base-2 designs can the notation AB be used to denote the 1-df interaction A × B. Using AB to denote the interaction A × B in base 3 is somewhat misleading because the AB effect has 2 df, while the A × B interaction has 4 df. In general, the b^k factorial design has $\sum_{j=0}^{k-1} b^j = (b^k - 1)/(b - 1)$ orthogonal effects (or columns), each with (b – 1) df. For example, a 2^5 factorial has $(2^5 - 1)/(2 - 1) = 31$ effects, A, B, …, E, AB, …, DE, ABC, …, CDE, ABCD, …, BCDE, and ABCDE, each with 1 df. The 3^3 factorial design has $(3^3 - 1)/(3 - 1) = 13$ orthogonal effects, which are A, B, C, AB, AB^2, AC, AC^2, BC, BC^2, ABC, AB^2C, ABC^2, and AB^2C^2, each with b – 1 = 2 df. Thus, the 3^3 complete factorial will have 27 = 3^3 distinct FLCs but only $(3^3 - 1)/(3 - 1)$ = 13 distinct effects (or orthogonal columns). Similarly, the 5^2 factorial has $(5^2 - 1)/(5 - 1)$ = 6 orthogonal effects, A, B, AB, AB^2, AB^3, and AB^4, each with 4 df. The design 5^2 has 25 distinct FLCs, 00, 01, 02, 03, 04, 10, 11, … 34, 40, …, and 44, and six orthogonal columns, which are occupied by the 4-df orthogonal effects A, B, AB, AB^2, AB^3, and AB^4. Again the interaction A × B in base 5 has 4 × 4 = 16 df and can be orthogonally decomposed into the 4-df components AB, AB^2, AB^3, and AB^4 because 5 is a prime number. We have emphasized that to fractionalize directly in a b^k factorial design, the design base, b, must be a prime number because it can be shown that in a base-4 design the orthogonal decomposition of the 9-df interaction A × B into the 3-df effects AB, AB^2, and AB^3 (or AB, AB^3, and A^2B^3) is impossible. In other words, the effects AB, AB^2, and AB^3 (or AB, AB^3, and A^2B^3) are meaningless in base 4 because they do not form an orthogonal (i.e., additive) decomposition of A × B (with 9 df).
6. The contrast function in base 2 for the effect AB is $\xi(AB) = x_1 + x_2$, where x_1 represents the levels of factor A (0 for low or 1 for high) and x_2 represents the levels of factor B (0 or 1); the contrast function for the effect AB^2C in base-3 design is $\xi(AB^2C) = x_1 + 2x_2 + x_3$, where x_3 represents the levels of factor C (0 for low, 1 for medium, and 2 for the high level); the con-

trast function for the effect AB^3 in a base-5 design is $\xi(AB^3) = x_1 + 3x_2$ (where x_1 and $x_2 = 0$, 1, 2, 3, or 4). Note that a contrast function in base 2 can take on only the values of 0, or 1; a contrast function in base 3 can have only the values 0, 1, or 2, while in base 5 a contrast function can have only the values 0, 1, 2, 3, or 4.

7. The FFD b^{k-p} has p independent generators and a total of $(b^p - 1)/(b - 1)$ generators, which comprise its defining identity I, and each of the $(b^p - 1)/(b - 1)$ generators in I is called a "word." For example, the 2^{8-3} FFD has p = 3 independent generators and a total of $(2^3 - 1)/(2 - 1) = 7$ generators, each of which produces one alias for each effect. The 3^{5-2} FFD has two independent generators and a total of $(3^2 - 1)/(3 - 1) = 4$ generators, each of which produces $3 - 1 = 2$ aliases for each effect so that each effect has $3^2 - 1 = 8$ aliases. While, the 5^{4-2} FFD has a total of $(5^2 - 1)/(5 - 1) = 6$ generators, each of which produces $b - 1 = 4$ aliases for each effect so that each effect has $5^2 - 1 = 24$ aliases. Note the precise pattern that for each block that is not studied in a b^{k-p} FFD, exactly one alias is generated for each effect. This pattern of $(b^p - 1)$ aliases for each effect will prevail for all orthogonal FFDs in the universe (b = 2, 3, 5) because $b^p - 1$ blocks are left out of the experimentation. Again, for each block of FLCs that is not studied, exactly one alias is generated for each effect.
8. The resolution of a b^{k-p} FFD, as defined by Box and Hunter (1961a,b), is the length of the shortest word in the defining relation I. For example, the 2^{5-2} FFD with generators $g_1 = ABC$, $g_2 = CDE$, and $g_3 = ABC^2DE = ABC^0DE = ABDE$ has a resolution R = III because the length of shortest words ABC and CDE is three letters. While, the FFD 3^{6-2} with generators $g_1 = ABC^2D$, $g_2 = CDE^2F^2$, $g_3 = g_1 \times g_2 = ABD^2E^2F^2$, and $g_4 = g_1 \times g_2^2 = ABCEF$ has a resolution R = IV.
9. Statistical literature dictates that in designing a fractional (or incomplete) factorial, the experimenter must always strive to maximize design resolution. A resolution of R < III practically renders the design useless because at least two main factors will be aliased (i.e., at least two main factors will occupy the same column of an OA). However, to attain a resolution R = III, the design matrix must have $k \geq 3$ factors and sufficient number of columns to assign all main factors separately from each other to different columns. To attain a resolution R = IV (for $k \geq 4$ factors), the design matrix must have sufficient number of columns to assign all main factors (i.e., first-order effects) separate from all components of two-way interactions (or second-order effects). To attain a resolution V for $k \geq 5$ factors, the design matrix X must have sufficient number of columns to assign all main factors and all second-order effects (or components of two-way interactions) to separate columns. This begs the question "Given k > 2 , b, and p, how many columns or rows (or distinct experimental FLCs) are needed or sufficient to attain at least a resolution III, IV, V, or VI design?" This question has been treated extensively in statistical literature by Webb (1968) and Margolin (1969), although resolution VI designs are not as important and as practical as those with R = IV and V because resolution VI designs require substantially more runs. In a resolution III design, main factors are separate from each other, and thus the b^{k-p} FFD needs at least k orthogonal columns, and because each column has (b – 1) df, the necessary and sufficient required minimum number of distinct FLCs (or rows), N_{min}, to attain R = III is given by $1 + (b - 1)k$, where the one extra run is needed for the estimation of process mean μ. The experimenter should be aware of the fact that for base-2 FFDs, it is unwise to generate a resolution III design for the 2^{4-1} FFD because an R = IV is always possible (by simply selecting g = ABCD as the design generator). Similarly, while it is possible to generate a resolution R = III design with k = 8 factors, it is again unwise to do so because the minimal FFD 2^{8-4} has a resolution R = IV with the choice of p = 4 independent generators, ABCE, ABDF, BCDG, and ACDH (the 2^{8-5} FFD is a resolution R = II design because the eight main factors require 8 df but the design matrix provides only 7 df for studying effects, and thus two main factors will have to be aliased, or inseparable from each other occupying the same column). In summary, for base-b designs a resolution III is guaran-

Table 1a
All Minimal Resolution V Designs in Base 2 Through k = 20 Factors

Design	N_f	$(k^2 + k + 2)/2$	A Set of p Independent Generators
2_V^{5-1}	16	16	ABCDE
2_V^{8-2}	64	37	ABCDG, ABEFH
2_V^{10-3}	128	56	ABCEF, BCDGH, ABDIJ
2_V^{11-4}	128	67	ABCEH, ACDFI, BCDGJ, ABCDEFGK
2_V^{13-5}	256	92	ABCDEI, ABCFGJ, ABDFHK, BCEGHL, ACDFM
2_V^{14-6}	256	106	ABCDHI, BCEHJ, BDFHK, ACEFHL, CDGHM, ADEGHN
2_V^{15-7}	256	121	ABCDHI, BCEHJ, BDFHK, ACEFHL, CDGHM, ADEGHN, AFGHO
2_V^{16-8}	256	137	ABCDHI, BCEHJ, BDFHK, ACEFHL, CDGHM, ADEGHN, AFGHO, BEFGHP
2_V^{17-9}	256	154	ABCDHI, BCEHJ, BDFHK, ACEFHL, CDGHM, ADEGHN, ABCFGHO, CEFGP, ABCDEFGQ
2_V^{19-10}	512	191	ABCDEJ, ABCFGK, ABCHIL, ABDFHM, ABDGN, ABEGIO, ABEHP, ABFIQ, ACDIR, ACEFS
2_V^{20-11}	512	211	ABCDEK, ABCFGL, ABCHIM, ABDFJN, ABEHJO, ABGIJP, ACDIJQ, ACEGJR, ACFHJS, ADGHJT, ABCEFIJ

teed (assuming that a resolution III design exists) if $b^{k-p} \geq 1 + (b - 1)k$. For some values of k such as k = 8 and k = 16, no minimal resolution III design exists in base 2.

In a resolution V design ($k \geq 5$), both the main factors and two-way interactions are completely separate from each other (i.e., they occupy separate columns of an OA). That is to say, main factors and two-way interactions can be estimated unbiased from each other, assuming effects of order three or higher are negligible. The main factors need $(b - 1)k$ df, and because each two-way interaction in a base-b design has $(b - 1)^2$ df, the necessary minimum number of distinct FLCs (or rows) for the design matrix is $N_{min}(V) \geq 1 + (b-1)k + (b-1)^2 \times \binom{k}{2} = 1 + (b-1)[k + (b-1) \times {}_kC_2]$, where ${}_kC_2 = \binom{k}{2} = \frac{k(k-1)}{2}$ represents the combinations of k items taken two at a time. Note that for base-2 designs, this last requirement, $1 + (b - 1)[k + (b - 1) \times {}_kC_2]$, reduces to $\frac{1}{2}(k^2 + k + 2)$, which is precisely the minimum distinct run requirements given by Margolin (1969, p. 435). However, Margolin is not emphatic in his statement because this last necessary value of $N_{min}(V)$ is not always sufficient to generate a resolution V design. Further, if the number of distinct runs $N_f < 1 + (b - 1)[k + (b - 1) \times {}_kC_2]$, then a resolution V design cannot be generated. *Table 1a* shows that for certain values of k (= 6, 7, 9, 12, 18) a minimal resolution V design is probably impossible to generate. After an exhaustive search, the 11 minimal resolution V designs listed in *Table 1a* were all that could be found through k = 20 factors (except for the fact that for each 2_V^{k-p} FFD, k > 5, there are many sets of p independent generators). For example, *Table 1a* shows that for k = 14 if N_f = 128 runs > $(k^2 + k + 2)/2$, the FFD 2_V^{14-7} cannot be generated and that 256 runs are needed to generate the FFD 2_V^{14-6}. Thus, the $(k^2 + k + 2)/2$ runs are needed but are not generally sufficient to attain the resolution R = V in base 2. This is similar to Webb's (1968) necessary condition of at least 2k runs for a resolution IV design but the 2k runs are not generally sufficient to yield a resolution IV design in base 2. The resolution V designs in *Table 1a* are consistent with the maximum number of factors that can be accommodated with a 256-run design reported by Draper and Mitchell (1970) to be k = 17.

The minimum required number of runs for a resolution IV design, $k \geq 4$ factors, is not as easily obtained because in such a design some of the second-order effects are aliased with each other and the rest are free from effects through the second order. (A first-order effect is a main factor, and a second-order effect is either a two-way interaction or a component of a first-order interaction.) Clearly, from the requirements for resolution III and V designs, it can be inferred that to attain a resolution R = IV, the necessary number of distinct experimental runs must satisfy $1 + (b-1)k < N_{min}(IV) < 1 + (b-1) \times [k + (b-1) \times {}_kC_2]$. Margolin (1969, p. 437) gives the minimum run requirements for the 2^n3^m resolution IV design as $3(n + 2m - 1)$ for $n \geq 0$ and $m > 0$, where n and m are the number of two-level and three-level factors, respectively. Further, Webb (1968) proved that a resolution IV 2^{k-p} FFD needs at least 2k distinct experimental runs. Note that the value 2k does lie within the interval $1 + (b-1)k < N_{min}(IV) < 1 + (b-1)[k + (b-1) \times {}_kC_2]$ for b = 2 and $k \geq 4$. Further, Margolin's Theorem 2 requires that m > 0 so that the minimum run requirements for base-2 resolution IV designs cannot be ascertained from his Theorem 2. From the above discussions, for base-2 designs, the sufficient number of runs to attain a resolution IV design, if it is possible to generate R = IV, is $N_{min}(IV) > 1 + k + \frac{1}{2}({}_kC_2) = 1 + k + [k(k-1)/4] = 1 + \frac{k(k+3)}{4}$. The multiplier of $\frac{1}{2}$ in front of ${}_kC_2$ originates from the fact that in a resolution IV design at least two second-order effects (or components of two-way interactions) can occupy the same column of an OA. Because in larger resolution IV FFDs, more than two second-order effects can occupy the same column of an OA, the $\frac{1}{2}$ multiplier leads to very conservative sufficient run requirements for k > 10 factors. Thus, if k is such that a resolution IV design can be generated and the value of $N_f = 2^{k-p} > 1 + \frac{k(k+3)}{4}$, then the FFD 2_R^{k-p} always has a resolution R = IV for (k = 4, 6, 7, 8, …). But the converse of this stringent requirement is not always true, that is, if $N_f < 1 + \frac{k(k+3)}{4}$, it is sometimes possible to generate a resolution IV design if the value of 2^{k-p} is close to $1 + \frac{k(k+3)}{4}$, such as the case of the FFD 2_{IV}^{7-3} for which $2^{k-p} = 16$ and $1 + \frac{k(k+3)}{4} = 18.5$. This is due to the fact that in a base-2 resolution IV design, as k (> 4) and $p \leq k - 3$ increase, the number of two-way interactions that occupy the same column of an OA increases (i.e., the number of two-way interactions that are aliased together increases). *Table 1b* gives a summary of minimal resolution IV designs for base 2.

In summary, if an R = IV is attainable in base 2, the requirement $N_f > 1 + \frac{k(k+3)}{4} \geq 2k$ will guarantee an R = IV (for $k \geq 4$), but this is not a necessity like Webb's needed 2k runs, but merely a sufficient condition to attain a minimal resolution IV in base-2 designs. Note that a minimal resolution IV design does not exist for k = 5 factors; further, *Table 1b* shows that no minimal resolution III design exists for k = 8 and 16 (what is meant by nonexistence is the fact that the design 2^{16-12} with 16 runs has a resolution II and it would be inefficient to select the independent generators in such a manner as to obtain the FFD 2_{III}^{16-11} because, as *Table 1b* shows, a resolution R = IV exists with 2^{16-11} = 32 runs).

In base 2, a minimal resolution VI design is guaranteed if $N_f > 1 + k + {}_kC_2 + ({}_kC_3)/2 = (k^3 + 3k^2 + 8k + 12)/12$ and $k \geq 9$ because the main factors need k columns, the two-way interactions need ${}_kC_2$ separate columns, and the three-way interactions need roughly $({}_kC_3)/2$ columns because in a resolution VI design the main factors and all two-way interactions can be estimated unbiased from each other, but three-way interactions are aliased with other three-way interactions. *Table 1c* gives the minimum resolution VI base-2 designs through k = 20 factors, where $(k^3 + 3k^2 + 8k + 12)/12$ is sufficient but not the necessary number of runs to attain R = VI. Draper and Mitchell (1970) also report that the maximum number of factors that can be accommodated with a 512-run design of resolution VI is k = 18.

Table 1b
Minimal Resolution IV Designs in Base 2 Through k = 20 Factors

Design	N_f	$1+\frac{k(k+3)}{4}$	A Set of p Independent Generators
2_{IV}^{6-2}	16	14.5	ABCE, BCDF
2_{IV}^{7-3}	16	18.5	ABCE, BCDF, ACDG
2_{IV}^{8-4}	16	23	ABCE, ABDF, BCDG, ACDH
2_{IV}^{9-4}	32	28	ABCF, BCDG, CDEH, ABDI
2_{IV}^{10-5}	32	33.5	ABCF, ABDG, ABEH, BCDI, BCEJ
2_{IV}^{11-6}	32	39.5	ABCF, ABDG, ABEH, BCDI, BCEJ, CDEK
2_{IV}^{12-7}	32	46	ABCF, ABDG, ABEH, BCDI, BCEJ, CDEK, ABCDEL
2_{IV}^{13-8}	32	53	ABCF, ABDG, ABEH, ACDI, ACEJ, ADEK, BCDL, BCEM
2_{IV}^{14-9}	32	60.5	ABCF, ABDG, ABEH, ACDI, ACEJ, ADEK, BCDL, BCEM, CDEN
2_{IV}^{15-10}	32	68.5	ABCF, ABDG, ABEH, ACDI, ACEJ, ADEK, BCDL, BCEM, BDEN, CDEO
2_{IV}^{16-11}	32	77	ABCF, ABDG, ABEH, ACDI, ACEJ, ADEK, BCDL, BCEM, BDEN, CDEO, ABCDEP
2_{IV}^{17-11}	64	86	ABCFG, ABDFH, ABEFI, ACDFJ, ACEFK, ADEFL, BCDFM, BCEFN, BDEFO, CDEP, ABCDEQ
2_{IV}^{18-12}	64	95.5	ABFG, ACFH, ADFI, AEFJ, BCFK, BDFL, BEFM, CDFN, CEFO, DEFP, ABCQ, CDER
2_{IV}^{19-13}	64	105.5	ABFG, ACFH, ADFI, AEFJ, BCFK, BDFL, BEFM, CDFN, CDEO, DEFP, ABCQ, ABDR, ABES
2_{IV}^{20-14}	64	116	ABHT, ABGS, ABFR, ABEQ, ABDP, ABCO, BFLM, CJKN, DJLN, EKLN, FJKL, GJMN, HKMN, ILMN

For base-3 designs, the sufficient number of runs for a resolution III design is $N_{min} = 1 + 2k$, but the 1 + 2k distinct rows are generally smaller than $N_f = 3^{k-p}$ rows of an orthogonal array and hence 1 + 2k must be rounded up to the next higher integer in powers of 3 (although the experimenter can leave some columns of a 3^{k-p} OA empty and use them as residuals or simply unused). The minimum required number of runs for a resolution V design in base 3 is given by $N_{min} = 1 + 2k + 4({}_kC_2) = 1 + 2k^2$ (only for $k \geq 5$), and again the value of $1 + 2k^2$ (if not already in powers of 3) has to be rounded up to the next higher integer that can be expressed in powers of 3. *Table 2a* summarizes minimal resolution V designs through k = 12 factors in base 3. Note that Margolin (1969) does not provide sufficient run requirements for a base-3 resolution $R = \text{V}$ designs, but Conner and Zelen (1959) provide the independent generators for a few of the 3_R^{k-p} FFDs through k = 10, which are 3_V^{7-2}, 3_V^{9-4}, 3_V^{10-5}, 3_{IV}^{6-2}, 3_{IV}^{7-3}, 3_{IV}^{8-4}, 3_{VI}^{6-1}, 3_{VI}^{9-3}, and 3_{IV}^{9-5}.

For base-3 designs, the sufficient required number of runs for a minimal resolution IV design is N_f

Table 1c
Minimal Resolution VI Designs in Base 2 Through k = 20 Factors

Design	N_f	$(k^3 + 3k^2 + 8k + 12)/12$	A Set of p Independent Generators
2^{9-2}_{VI}	128	88	ABCDFH, BCEFGI
2^{10-2}_{VI}	256	116	ABCFGH, CDEHIJ
2^{11-3}_{VI}	256	149.5	ABCDFI, BCDEGJ, ACDEHK
2^{12-4}_{VI}	256	189	ABCDEI, ABCFGJ, ABDFHK, ACEGHL
2^{14-5}_{VI}	512	288	ABCDEJ, ABCFGK, ABCHIL, ABDFHM, ABEGIN
2^{15-6}_{VI}	512	348.5	ABCDEJ, ABCFGK, ABCHIL, ABDFHM, ABEGIN, ADEFIO
2^{16-7}_{VI}	512	417	ABCDEJ, ABCFGK, ABDFHL, ABEGIM, ADEFIN, AEFGHO, ABCEFHIP
2^{17-8}_{VI}	512	494	ABCDEJ, ABCFGK, ABCHIL, ABDFHM, ABEGIN, ADEFIO, BDGHIP, ACDEFGHQ
2^{18-9}_{VI}	512	580	ABCDEJ, ABCFGK, ABCHIL, ABDFHM, ABEGIN, ADEFIO, BDGHIP, ACDEFGHQ, BCEFGHIR
2^{19-9}_{VI}	1024	675.5	ABCGHK, ABDGIL, ABEGJM, BEFHIN, ACDHJO, ACEIJP, CFGHIQ, DEGHJR, DFHIJS
2^{20-10}_{VI}	1024	781	ABCDEK, ABCFGL, ABDFHM, ACDFIN, AEGHIO, ABEFJP, BCGIJQ, BCDFGHJR, BEFGHS, CDEGIT

$> 1 + 2k + \frac{1}{2}\,[4 \times {}_kC_2] = 1 + k + k^2$ for $k \geq 4$. *Table 2b* summarizes resolution IV designs for base 3. The reader should observe that our sufficient run requirements for base-3 resolution IV designs ($k \geq 4$) are consistent with that of Margolin's (1969) Theorem 2 requirement for minimum needed number of runs, which he lists as $3(2k - 1)$ because $k^2 + k + 1 > 3(2k - 1)$ for all $k > 4$. In a similar manner, the minimum and sufficient number of runs for a resolution VI design in base 3 is $1 + 2k + 4({}_kC_2) + \frac{1}{2}\,[4({}_kC_3)] = (k^3 + 3k^2 + 2k + 3)/3$. *Table 2c* summarizes resolution VI designs for base 3.

The above summary completes our review of classical FFDs. Before relating some of Taguchi's OAs to the classical FFDs, an example will be presented to illustrate how to compute the sum of squares (SS) of any effect for balanced orthogonal factorial designs in base b (most commonly b = 2, 3, and 5). The concept of orthogonality and balance will become clearer at the end of the example.

Consider a 3^3 factorial with n = 2 observations per FLC, where all the $N = n \times N_f = 2 \times 27 = 54$ observations are taken in a completely random order. The qualitative factor A represents a type of drum (type 0, 1, 2), factor B represents the speed differential between the conveyer belts for the liner/ply and the drum ($B_0 = 5\%$, $B_1 = 7.5\%$, and $B_2 = 10\%$), and factor C represents the concentricity of the bead with respect to the drum (0 = not concentric, 1 = fairly concentric, and 2 = very concentric). The response variable, y, is the radial force harmonic (RFH) of a passenger tire coded by subtracting 20 pounds from all 54 observations. The coding was done just to simplify computations and will not affect any of the corrected SS in the analysis of variance (ANOVA) table. *Table 3* lists the data.

The arbitrary columns of the OA for *Table 3* are obtained by writing three (the exponent of b) columns of 0's, 1's, and 2's arbitrarily, i.e., the first column will be nine 0's, nine 1's, followed by nine 2's.

Table 2a
Minimal Resolution V Designs in Base 3 Through k = 12 Factors

Design	N_f	$N_{min} = 1 + 2k^2$	A Set of p Independent Generators
3_V^{5-1}	81	51	ABCDE
3_V^{7-2}	243	99	ABCDE, $CD^2EF^2G^2$
3_V^{8-3}	243	129	BCDFG, CDE^2F^2H, ABD^2E^2F
3_V^{9-4}	243	163	BCDEFG, CDE^2F^2H, BD^2E^2FI, ABC^2EF^2
3_V^{10-5}	243	201	BCDEFG, $ACDE^2F^2H$, ABD^2E^2FI, ABC^2EF^2J, AB^2C^2DF
3_V^{11-6}	243	243	BCDEFG, $ACDE^2F^2H$, ABD^2E^2FI, ABC^2EF^2J, AB^2C^2DFK, BE^2FG^2J

Table 2b
Minimal Resolution IV Designs in Base 3

Design	N_f	$N_{min} = 1 + k + k^2$	A Set of p Independent Generators
3_{IV}^{4-1}	27	21	AB^2CD
3_{IV}^{6-2}	81	43	ACDE, BC^2DE^2F
3_{IV}^{7-3}	81	57	CDEF, $BE^2F^2G^2$, ACE^2G^2
3_{IV}^{8-4}	81	73	ADEG, BC^2EF, $ABCG^2$, BC^2D^2H
3_{IV}^{9-5}	81	91	BCDEFG, $ACDE^2F^2H$, ABD^2E^2FI, ABC^2EF^2, AB^2C^2DF
3_{IV}^{10-6}	81	111	ABC^2D, $CDEF^2$, ACF^2G, AEFH, ADFI, AD^2H^2J
3_{IV}^{12-7}	243	157	$BCDE^2F$, ACE^2GH, AB^2E^2FI, AC^2EF^2J, BCD^2F^2K, BD^2FGL, CDF^2GH^2

Table 2c
Minimal Resolution VI Designs in Base 3

Design	N_f	$(k^3 + 3k^2 + 2k + 3)/3$	A Set of p Independent Generators
3_{VI}^{6-1}	243	113	AB^2CDE^2F
3_{VI}^{8-2}	729	241	$ABCDEH^2$, $CD^2EF^2G^2H$
3_{VI}^{9-3}	729	331	BCDEFG, $ACDE^2F^2H$, ABD^2E^2FI
3_{VI}^{10-4}	729	441	BCDEFG, $ACDE^2F^2H$, ABD^2E^2FI, ABC^2EF^2J
3_{VI}^{11-5}	729	573	BCDEFG, $ACDE^2F^2H$, ABD^2E^2FI, ABC^2EF^2J, AB^2C^2DFK
3_{VI}^{12-6}	729	729	BCDEFG, $ACDE^2F^2H^2$, ABD^2E^2FI, ABC^2EF^2J, AB^2C^2DFK, $BC^2D^2FJ^2L$

Table 3
Data for the 3^3 Complete Factorial Example

Factor B	B at level "0" → B at 5%			B_1 → B at 7.5%			B_2 → B at 10%			
C / A	C_0	C_1	C_2	C_0	C_1	C_2	C_0	C_1	C_2	$y_{i...}$
Type 0	4.8	1.0	–9.1	2.2	–1.1	–3.4	10.3	6.8	3.5	34.8
	6.9	–2.1	–6.8	4.7	–5.6	2.1	9.2	4.2	7.2	
Type 1	3.2	1.3	1.5	2.7	–2.1	–10.1	8.3	3.4	1.3	25.0
	5.7	0.0	–3.2	6.9	–3.5	–7.7	9.2	5.2	2.9	
Type 2	8.6	3.2	–2.1	8.6	4.1	–6.8	11.2	7.6	6.6	79.4
	7.7	5.5	3.1	5.8	2.3	–4.2	10.7	1.7	5.8	
$y_{.jk.}$	36.9	8.9	–16.6	30.9	–5.9	–30.1	58.9	28.9	27.3	$y_{....} = 139.2$

The second column will consist of three 0's, three 1's, three 2's, and repeated twice more. The third column will consist of 0, 1, and 2, but repeated eight times more to yield 27 rows. The total SS is obtained by summing the square of all 54 observations (which is called the uncorrected SS denoted by USS) and then subtracting the correction factor CF = $(y_{....})^2/54$, where $y_{....} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{r=1}^{2} y_{ijkr}$, and Taguchi uses the uncommon notation Sm for the correction factor; note that the index i extends over the factor A levels, j refers to factor B levels, k extends over factor C levels, and r = 1, 2 implies that there are two repeat observations (or replications) within each cell. For example, the cell (or FLC) 201 contains the responses $y_{3121} = 3.2$ and $y_{3122} = 5.5$ so that $y_{312.} = 8.7$, etc. The uncorrected SS for *Table 3* data is given by USS = $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{r=1}^{2} y_{ijkr}^2 = 4.8^2 + 6.9^2 + \ldots + 5.8^2 = 1850.40$. The reader should be cognizant of the fact that in developing an ANOVA, each time that a real number is squared one degree of freedom is generated. Further, because degrees of freedom are additive (the origin of these concepts lies in the assumption of normality for y_{ijkr} and the resulting noncentral chi-square distribution), then the USS for *Table 3* will have $N = 2 \times 3^3 = 54$ df because there are n = 54 normally distributed random numbers that are being squared and added. The CF = $\frac{139.2^2}{54} = 358.8266\overline{6}$, which has only one degree of freedom because only one Gaussian number is being squared. Thus, the corrected SS is given by CSS = SS(Total) = USS – CF = 1850.40 – 358.82667 = $1491.5733\overline{3}$, which has 54 – 1 = 53 df.

Another reason that the CSS = $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{r=1}^{2}\left(y_{ijkr} - \bar{y}_{....}\right)^2$ has 53 df (instead of 54) is the fact that the 54 squared terms in this last CSS have one constraint among them, namely $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{r=1}^{2}\left(y_{ijkr} - \bar{y}_{....}\right) \equiv 0$. In all factorial (or FF) designs, the Total SS decomposes into two orthogonal components, namely Model SS and Residual SS, i.e., SS(Total) = SS(Model) + SS(Residuals). For the data of *Table 3*, the source of Residuals is from pure experimental error that is generated from the variation within each of the 27 cells. If repeat observations are not made in at least one FLC, then pure experimental error cannot be retrieved, and unless the design provides leftover degrees of freedom for residuals, no exact statistical test of significance can be made. For example, the FLC 000 has two responses, 4.8 and 6.9, which contribute $4.8^2 + 6.9^2 - \frac{(4.8+6.9)^2}{2} = 4.8^2 + 6.9^2 - 11.7^2/2 = 2.2050$ to the overall pure error SS [denoted as SS_{PE} or SS(PE)], and recalling the premise that every real number squared generates one degree of freedom, then the SS_{PE} from the cell 000 carries a net of 1 + 1 – 1 = 1 degree of freedom.

Similarly, the cell 202 has two responses, –2.1 and 3.1, which contribute $(-2.1)^2 + 3.1^2 - \frac{(-2.1+3.1)^2}{2}$ = 13.52 (with one degree of freedom) to the overall SS_{PE}. Because the above factorial design has 3 × 3 × 3 = 27 distinct FLCs and each cell contributes one

Table 4
Depicting Variability Among (or Between) the 27 Cells

Factor B	B at level "0" → B at 5%			B at 1 → 7.5%			B at 2 → 10%			
C / A	C_0	1	2	0	C at 1	2	0	1	C at 2	$y_{i...}$
Type 0	11.7	–1.1	–15.9	6.9	–6.7	–1.3	19.5	11.0	10.7	34.8
Type 1	8.9	1.3	–1.7	9.6	–5.6	–17.8	17.5	8.6	4.2	25.0
Type 2	16.3	8.7	1.0	14.4	6.4	–11.0	21.9	9.3	12.4	79.4
$y_{.jk.}$	36.9	8.9	–16.6	30.9	–5.9	–30.1	58.9	28.9	27.3	$y_{....}$ = 139.2

degree of freedom to the overall pure error, SS_{PE} must have 27 df. Adding the 27 one-degree-of-freedom SS yields SS(PE) = 2.2050 + 4.8050 + 2.6450 + 3.1250 + 10.1250 + 15.1250 + 0.6050 + 3.3800 + 6.8450 + 3.1250 + 0.8450 + 11.0450 + 8.8200 + 0.9800 + 2.8800 + 0.4050 + 1.6200 + 1.2800 + 0.4050 + 2.6450 + 13.5200 + 3.9200 + 1.6200 + 3.3800 + 0.1250 + 17.4050 + 0.3200 = 123.2000.

Because the above factorial design has a total of 53 df, due to orthogonality the Model SS must have 53 – 27 = 26 df. Clearly, one way to obtain the SS(Model) is by subtracting SS(PE) from SS(Total), i.e., SS(Model) = SS(Total, with 53 df) – SS(PE, with 27 df) = $1491.5733\overline{3}$ – 123.20 = $1368.3733\overline{3}$ (with 26 df). The above method of computing the SS(PE) and SS(Model) is at best time-consuming and cumbersome. Because pure experimental error originates from the internal variation within the same cell, variation due to the model must originate from the fact that cell averages are different. In short, the Model SS must come from variability among distinct FLCs. Further, because n = 2 for all 27 cells, then we may as well compare cell subtotals directly (instead of their averages) to assess the contribution of model terms to the SS(Total). If we remove the internal variation within all cells from *Table 3*, the resulting *Table 4* will depict the variations among (or between) the 27 cells.

Another pattern that will prevail in computing any SS in all orthogonally balanced factorial designs is the fact that every squared term has a specific divisor. The divisor is always the number of individual observations that comprise the squared term. The formula for the correction factor is CF = (grand sum of all observations)^2/divisor. To determine what the value of the divisor is, the question to ask is how many individual observations have to be added to obtain $y_{....}$ = "grand sum of all observations." The answer is N = 54, and hence this divisor has to be 54, i.e., CF = $(y_{....})^2/54$. As yet another example, if we wish to square the subtotal for level zero of A, denoted by A_0, then the required divisor for $A_0^2 = (34.8)^2$ (see *Table 3*) has to be 18 because 18 individual observations have to be added to obtain A_0 = 34.8.

Having established some rules for SS computations, we are now in a position to compute the overall Model SS as follows. Recall that the model describes the variation among different distinct FLCs, of which there are 27. Thus, we have to square the 27 terms in *Table 4* but divide by 2 because every term in *Table 4* was obtained from adding two individual responses. However, such an USS will have 27 df (this is due to the fact that a single Gaussian term squared generates exactly 1 df, the origin of which lies in the noncentral chi-squared (χ^2) distribution) and we have already argued that the model for the above experiment must have 26 df because the 27 squared terms in SS(Model) = $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} 2\left(\bar{y}_{ijk.} - \bar{y}_{....}\right)^2$ have one constraint $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{r=1}^{2}\left(\bar{y}_{ijk.} - \bar{y}_{....}\right) \equiv 0$ among them; thus, we have to correct by subtracting the CF, i.e.,

$$\mathrm{SS}(\mathrm{Model}) = \frac{11.7^2 + (-1.1)^2 + (-15.9)^2 + \ldots + 9.3^2 + 12.4^2}{2} - \mathrm{CF} = \frac{345.4}{2} - \frac{139.20^2}{54} = 1368.3733\overline{3}\,(\text{with 26 df})$$

which is in complete agreement with the previously computed value from SS(Model) = SS(Total) – SS(PE). The reader should be cognizant of the fact that the breakdown of SS(Model) that is to follow is possible only if (1) the design is orthogonal and balanced, and (2) the quantitative factor levels are equally spaced (or at least some transformation, such as logarithm, of the factor levels is equally spaced.) Clearly, in our example, the levels of factor B have equal spacing of 2.5%, and we have to assume that factor C levels are also equally spaced. Further, because we are in base 3, each effect will have 2 df and thus there must be 13 orthogonal effects, i.e., the Model SS should break down into thirteen 2-df orthogonal (first, second, and third-order) effects listed below:

$$A, B, C, AB, AB^2, AC, AC^2, BC, BC^2, ABC, ABC^2, AB^2C, AB^2C^2$$

Recall that we have defined an effect in base b as one that has (b – 1) df and occupies one column of an OA. We will now compute the SS of the above 13 effects term by term, which is essential in developing Taguchi's L_{27} OA, and will also provide each contrast function ξ.

Factor A:

$\xi(A) = x_1$; $A_0 = 34.8$, $A_1 = 25.0$, $A_2 = 79.4 \rightarrow$ SS(A) $= \dfrac{34.8^2+25^2+79.4^2}{18} - \dfrac{139.20^2}{54} = 93.4177\overline{7}$ (with 2 df)

Factor B:

$\xi(B) = x_2$; $B_0 = 29.2$, $B_1 = -5.1$, $B_2 = 115.1 \rightarrow$ $\mathrm{SS(B)} = \dfrac{29.2^2+(-5.1)^2+111.5^2}{18} - \dfrac{139.20^2}{54} = 425.9877\overline{7}$

Factor C:

$\xi(C) = x_3$; $C_0 = 126.7$, $C_1 = 31.9$, $C_2 = -19.4 \rightarrow$

$$\mathrm{SS(C)} = \frac{126.7^2+31.9^2+(-19.4)^2}{18} - \frac{139.20^2}{54} = \frac{126.7^2+31.9^2+(-19.4)^2}{18} - \mathrm{CF} = 610.4433\overline{3}$$

The experimenter should note that it can be ascertained from the above three SS's that factor C has the strongest impact on the mean of the response variable y = RFH.

The Effect AB:

$\xi(AB) = x_1 + x_2$; to compute the value of $(AB)_0$, we have to add all cell subtotals whose $\xi(AB) = 0$ (mod 3). The FLCs that make $\xi(AB) = 0$ (mod 3) are 000, 001, 002, 120, 121, 122, 210, 211, 212 $\rightarrow (AB)_0 = 11.7 - 1.1 - 15.9 + 17.5 + 8.6 + 4.2 + 14.4 + 6.4 - 11.0 = 34.80$ (see *Table 4*). To compute the value of $(AB)_1$, all cell subtotals whose $\xi(AB) = 1$ (mod 3) are added. The FLCs that make $\xi(AB) = 1$ (mod 3) are 010, 011, 012, 100, 101, 102, 220, 221, 222 $\rightarrow (AB)_1 = 6.9 - 6.7 - 1.3 + 8.9 + 1.3 - 1.7 + 21.9 + 9.3 + 12.4 = 51.0$. To compute $(AB)_2$, all cell subtotals whose $\xi(AB) = 2$ (mod 3) are added. The FLCs that make $x_1 + x_2 = 2$ (mod 3) are 020, 021, 022, 110, 111, 112, 200, 201, 202 $\rightarrow (AB)_2 = 19.5 + 11.0 + 10.7 + 9.6 - 5.6 - 17.8 + 16.3 + 8.7 + 1.0 = 53.40$ $\rightarrow \mathrm{SS(AB)} = \dfrac{34.8^2+51.0^2+53.4^2}{18} - \dfrac{139.20^2}{54} = 11.3733\overline{3}$ (with 2 df).

The Effect AB²

$\xi(AB^2) = x_1 + 2x_2$; to compute the value of $(AB^2)_0$, all cell subtotals whose $\xi(AB^2) = 0$ (mod 3) are added. The FLCs that make $\xi(AB^2) = 0$ (mod 3) are 000, 001, 002, 110, 111, 112, 220, 221, 222 $\rightarrow (AB^2)_0 = 11.7 - 1.1 - 15.9 + 9.6 - 5.6 - 17.8 + 21.9 + 9.3 + 12.4 = 24.50$. The FLCs that make $\xi(AB^2) = x_1 + 2x_2 = 1$ (mod 3) are 020, 021, 022, 100, 101, 102, 210, 211, 212 $\rightarrow (AB^2)_1 = 19.5 + 11.0 + 10.7 + 8.9 + 1.3 - 1.7 + 14.4 + 6.4 - 11.0 = 59.50$. The FLCs that make $\xi(AB^2) = x_1 + 2x_2 = 2$ (mod 3) are 010, 011, 012, 120, 121, 122, 200, 201, 202 $\rightarrow (AB^2)_2 = 6.9 - 6.7 - 1.3 + 17.5 + 8.6 + 4.2 + 16.3 + 8.7 + 1.0 = 55.20 \rightarrow \mathrm{SS}(AB^2) = \dfrac{24.5^2+59.5^2+55.2^2}{18} - \dfrac{139.20^2}{54} = 40.4811\overline{1}$ (with 2 df).

To illustrate the meaning of the above two sum of squares, SS(AB) and SS(AB²), we will compute the SS(A×B), having 4 df, by crossing factors A and B, as depicted in *Table 5*. The A×B interaction in *Table 5* has 3 × 3 = 9 cells (or FLCs), and thus *Table 5* has 9 – 1 = 8 df, two of which are absorbed by factor A, two by factor B, and the remaining 4 df belong to the A×B interaction, i.e., SS(A) + SS(B) + SS(A×B) =

$$\frac{(-5.3)^2+(-1.1)^2+41.2^2+8.5^2+(-13.8)^2+30.3^2+26.0^2+9.8^2+43.6^2}{6} - \frac{139.20^2}{54} =$$

$571.2600 \rightarrow$ SS(A×B) = 571.2600 – SS(A) – SS(B) $\rightarrow$ SS(A×B) $= 571.2600 - 93.4177\overline{7} - 425.9877\overline{7} = 51.8544\overline{4}$ (with 4 df).

Recall that SS(AB) $= 11.3733\overline{3}$ (with 2 df) and SS(AB²) $= 40.4811\overline{1}$ (with 2 df), and hence SS(AB)

Table 5
A×B Interaction Table

	B_0	B_1	B_2	$y_{i...}$
A_0	–5.30	–1.10	41.20	34.8
A_1	8.500	–13.80	$30.30 = y_{23..}$	25.0
A_2	26	9.80	43.60	79.4
$y_{.j..}$	29.2	–5.1	115.1	$y_{....} = 139.2$

+ SS(AB^2) = $11.3733\overline{3}$ + $40.4811\overline{1}$ = $51.8544\overline{4}$ = SS(A×B). Thus, we have established that SS(A×B) with 4 df decomposes into two orthogonal (i.e., additive) components, SS(AB) and SS(AB^2), each with 2 df because the design base is a prime number. The above algebraic procedure of decomposing SS(A×B) into SS(AB) and SS(AB^2) is easier and less confusing than the tabular procedure used in most DOE books [such as Davies et al. (1967), Hicks and Turner (1999), and Montgomery (2001a)]. Yates (1937) referred to AB^2 as the I(AB) component of A×B and to AB as the J(AB) component of A×B. Such classifications in statistical literature seem somewhat arbitrary because from I(AB) one cannot discern what its contrast function is, while using AB^2 identifies the contrast function $\xi(AB^2) = x_1 + 2x_2$ immediately. The reader may wonder if a component such as A^2B exists. The answer is yes, but $\xi(A^2B) = 2x_1 + x_2 \rightarrow 2\xi(A^2B) = 2(2x_1 + x_2) = 4x_1 + 2x_2 = x_1 + 2x_2$ (modulus 3) $= \xi(AB^2) \rightarrow \xi(A^2B) = 2\ \xi(AB^2)$ mod 3. Thus, the two components AB^2 and A^2B are identical. For this reason, when working in bases 3 and 5, we can always, without loss of generality, keep the exponent of the first letter of any effect as 1. Further, because A×B has 4 df and each effect in base 3 has 2 df, A×B cannot have more than two orthogonal 2-df components. Thus, we have established that a two-way interaction A×B in base-3 designs with 4 df decomposes into two additive (or orthogonal) components, each with 2 df, and as a result, to compute SS(AC) and SS(AC^2), it will be helpful to cross factors A and C, just like *Table 5* for factors A and B, and then use the resulting A×C table (not shown herein) to compute the nine subtotals needed to compute SS(AC) and SS(AC^2).

The Effect AC:

$\xi(AC) = x_1 + x_3 \rightarrow (AC)_0 = 38.1 - 15.3 + 24.4 = 47.20$; $(AC)_1 = 3.2 + 36.0 + 2.4 = 41.60$; $(AC)_2 = -6.5 + 4.3 + 52.6 = 50.40 \rightarrow SS(AC) = \dfrac{47.2^2 + 41.6^2 + 50.4^2}{18} - \dfrac{139.20^2}{54} = 2.204444$ (with 2 df).

The Effect AC^2:

$\xi(AC^2) = x_1 + 2x_3 \rightarrow (AC^2)_0 = 38.1 + 4.3 + 2.40 = 44.80$; $(AC^2)_1 = -6.5 + 36.0 + 24.4 = 53.90$; $(AC^2)_2 = 3.2 - 15.3 + 52.6 = 40.50 \rightarrow SS(AC^2) = 5.201111$ (with 2 df) $\rightarrow SS(A\times C) = 7.40555\overline{5}$. One can easily verify from the A×C table that $SS(A\times C) = 711.26666\overline{6} - SS(A) - SS(C) = 7.405555\overline{5} = SS(AC) + SS(AC^2)$.

The Effect BC:

$\xi(BC) = x_2 + x_3 \rightarrow (BC)_0 = 36.9 - 30.1 + 28.9 = 35.70$; $(BC)_1 = 8.9 + 30.9 + 27.3 = 67.10$; $(BC)_2 = -16.6 - 5.9 + 58.9 = 36.40 \rightarrow SS(BC) = 35.7211\overline{1}$ (with 2 df).

The Effect BC^2:

$\xi(BC^2) = x_2 + 2x_3 \rightarrow (BC^2)_0 = 36.9 - 5.9 + 27.3 = 58.300$; $(BC^2)_1 = -16.6 + 30.9 + 28.9 = 43.20$; $(BC^2)_2 = 8.9 - 30.1 + 58.9 = 37.70 \rightarrow SS(BC^2) = \dfrac{58.3^2 + 43.2^2 + 37.7^2}{18} - \dfrac{139.20^2}{54} = 12.6411\overline{1}$ (with 2 df) $\rightarrow SS(B\times C) = 48.3622\overline{2}$. One can easily verify from the B×C interaction table that $SS(B\times C) = \dfrac{36.9^2 + 8.9^2 + (-16.6)^2 + 30.9^2 + (-5.9)^2 + (-30.1)^2 + 58.9^2 + 28.9^2 + 27.3^2}{6} - \dfrac{139.20^2}{54} - SS(B) - SS(C) = 1084.7933\overline{3} - 425.9877\overline{7} - 610.4433\overline{3} = 48.3622\overline{2} \rightarrow (B\times C) = 48.3622\overline{2}$ (with 4 df) $= 35.7211\overline{1} + 12.6411\overline{1} = SS(BC) + SS(BC^2)$.

The Effect ABC:

$\xi(ABC) = x_1 + x_2 + x_3 \rightarrow$ From *Table 4*, the value of $(ABC)_0$ is computed using the nine FLCs (000, 012, 021, 102, 120, 111, 201, 210, 222) that make the contrast function $\xi(ABC) = x_1 + x_2 + x_3 = 0$ (mod 3). Thus, $(ABC)_0 = 11.7 - 1.3 + 11.0 - 1.7 + 17.5 - 5.6 + 8.7 + 14.4 + 12.4 = 67.10$; $(ABC)_1 = -1.1 + 6.9 + 10.7 + 8.9 - 17.8 + 8.6 + 1.0 + 6.4 + 21.9 = 45.50$; $(ABC)_2 = -15.9 - 6.7 + 19.5 + 1.3 + 9.6 + 4.2 + 16.3 - 11.0 + 9.3 = 26.60 \rightarrow SS(ABC) = \frac{67.1^2 + 45.5^2 + 26.6^2}{18} - \frac{139.20^2}{54} = 45.630$ (with 2 df).

The Effect ABC²:

$\xi(ABC^2) = x_1 + x_2 + 2x_3 \rightarrow$ From *Table 4*, the value of $(ABC^2)_0$ is computed using the nine FLCs that make the contrast function $\xi(ABC^2) = x_1 + x_2 + 2x_3 = 0$ (mod 3). The nine FLCs that make the contrast function $\xi(ABC^2) = x_1 + x_2 + 2x_3 = 1$ are 002, 021, 010, 100, 111, 122, 220, 212, 201. Thus, $(ABC^2)_0 = 11.7 - 6.7 + 10.7 + 1.3 - 17.8 + 17.5 + 1.0 + 14.4 + 9.3 = 41.40$; $(ABC^2)_1 = -15.9 + 11.0 + 6.9 + 8.9 - 5.6 + 4.2 + 21.9 - 11 + 8.7 = 29.10$; similarly, $(ABC^2)_2 = -1.1 - 1.3 + 19.5 - 1.7 + 9.6 + 8.6 + 16.3 + 6.4 + 12.4 = 68.70 \rightarrow SS(ABC^2) = \frac{41.4^2 + 29.1^2 + 68.7^2}{18} - \frac{139.20^2}{54} = 45.6433\overline{3}$ (with 2 df).

The Effect AB²C:

$\xi(AB^2C) = x_1 + 2x_2 + x_3 \rightarrow$ From *Table 4*, the value of $(AB^2C)_0$ is computed using the nine FLCs that make the contrast function $\xi(AB^2C) = x_1 + 2x_2 + x_3 = 0$ (mod 3). Thus, $(AB^2C)_0 = 11.7 - 6.7 + 10.7 - 1.7 + 9.6 + 8.6 + 8.7 - 11.0 + 21.9 = 51.80$; $(AB^2C)_1 = -1.1 - 1.3 + 19.5 + 8.9 - 5.6 + 4.2 + 1.0 + 14.4 + 9.3 = 49.30$; $(AB^2C)_2 = -15.9 + 6.9 + 11.0 + 1.3 - 17.8 + 17.5 + 16.3 + 6.4 + 12.4 = 38.10 \rightarrow SS(AB^2C) = \frac{51.8^2 + 49.3^2 + 38.1^2}{18} - \frac{139.20^2}{54} = 5.9144\overline{4}$ (with 2 df).

The Effect AB²C²:

$\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3 \rightarrow$ From *Table 4*, the value of $(AB^2C^2)_0$ is computed using the nine FLCs that make the contrast function $\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3 = 0$ (mod 3). Thus, $(AB^2C^2)_0 = 11.7 - 1.3 + 11.0 + 1.3 + 9.6 + 4.2 + 1.0 + 6.4 + 21.9 = 65.80$; $(AB^2C^2)_1 = -15.9 - 6.7 + 19.5 + 8.9 - 17.8 + 8.6 + 8.7 + 14.4 + 12.4 = 32.10$; $(AB^2C^2)_2 = -1.1 + 6.9 + 10.7 - 1.7 - 5.6 + 17.5 + 16.3 - 11.0 + 9.3 = 41.30 \rightarrow SS(AB^2C^2) = \frac{65.8^2 + 32.1^2 + 41.3^2}{18} - \frac{139.20^2}{54} = 33.7144\overline{4}$ (with 2 df).

The above base-3 algebraic procedure using the contrast function to compute the four orthogonal components of the second-order (or three-way) interaction A×B×C (with 8 df) is straightforward and leaves no room for confusion and/or error. However, the tabular procedure that has been used in statistical literature in the past 50 years can lead to confusion and misclassification of the four orthogonal components $SS(ABC)$, $SS(ABC^2)$, $SS(AB^2C)$, and $SS(AB^2C^2)$ of SS(A×B×C). For example, due to the use of tabular procedure the two components $SS(ABC^2)$ and SS(ABC) atop p. 372 of Montgomery (2001a) are reversed (albeit a minor misclassification) because the value of $SS(ABC^2)$ is 584.11 (not 18.77 as reported) and the value of SS(ABC) is 18.77. Similarly, the two components SS(DOC) and $SS(DOC^2)$ near the bottom of p. 286 of Hicks and Turner (1999) are also reversed. The value of SS(DOC) to two decimals is 0.30 (not 0.19 as reported) and the correct value of $SS(DOC^2)$ is 0.19.

The last 13 SS computations verify the fact that SS(Model) = $1368.3733\overline{3}$ (with 26 df) decomposes into 13 orthogonal components, A, B, C, AB, AB^2, AC, AC^2, BC, BC^2, ABC, ABC^2, AB^2C, and AB^2C^2. By orthogonal breakdown of SS(Model) is meant that SS(Model) = SS(A) + SS(B) + SS(C) + SS(AB) + $SS(AB^2)$ + SS(AC) + $SS(AC^2)$ + SS(BC) + $SS(BC^2)$ + SS(ABC) + $SS(ABC^2)$ + $SS(AB^2C)$ + $SS(AB^2C^2)$ = $1368.3733\overline{3}$.

It is paramount to note that if a complete factorial design of any base b is orthogonal and balanced, then the SS(Model) always decomposes into the SS of main factors and the SS of interactions of all possible orders. For example, a complete orthogonal 4^3 factorial design with the same number of observations, $n \geq 1$, per FLC possesses the orthogonal decomposition of SS(Model) = SS(A) + SS(B) + SS(C) + SS(A×B) + SS(A×C) + SS(B×C) + SS(A×B×C). However, because the design base b = 4 is not a prime number, the orthogonal decomposition of SS(A×B) with 9 df into SS(AB), $SS(AB^2)$, and $SS(AB^3)$ [or into $SS(A^2B^3)$, $SS(A^3B^2)$, and $SS(A^2B^2)$] each with 3 df does not exist. We have verified that the three components SS(AB), $SS(AB^2)$, and $SS(A^3B^2)$ are orthogonal in some 4^2 designs, but these three components are not orthogonal to factors A and B and hence useless for confounding in blocks or direct fractionalization in base 4. Thus, the Taguchi $L_{16}(4^5)$ (Taguchi and Konishi 1987, p. 59) is an orthogonal array but its columns (3), (4), and (5) cannot be generated from its columns (1) and (2) using mod 4 algebra because 4 is not a prime number and the orthogonal decomposition of A×B into Taguchi components AB, A^2B, and A^3B is impossible. The same cannot be said for the Taguchi $L_{25}(5^6)$ OA because 5 is a prime number and the columns (3), (4), (5), and (6) of $L_{25}(5^6)$ can for certain be generated from its columns (1) and (2) by first converting to base-5 elements 0, 1, 2, 3, and 4 and then generating column (3) from (1) + (2) (mod 5), column (4) from 2 × (1) + (2) (mod 5), column (5) from 3 × (1) + (2) (mod 5), and generating column (6) from 4 × (1) + (2) (mod 5) (Taguchi and Konishi 1987, p. 64). When the design base is a prime number, we need the decomposition of A×B×C (or higher order interactions if the design had more than three factors) into its orthogonal components, each with (b – 1) df, only for fractionalizing (and obtaining the corresponding alias structure) and confounding in blocks. When the design is a full (or complete) factorial with no blocking, then the orthogonal decompositions of A×B×C… into (b – 1) df effects is not needed and perhaps quite useless. The complete ANOVA table from Minitab for the above balanced orthogonal design is provided below.

ANOVA: y versus A, B, C

Factor	Type	Levels	Values
A	fixed	3	0 1 2
B	fixed	3	0 1 2
C	fixed	3	0 1 2

Analysis of Variance for y

Source	DF	SS	MS	F	P
A	2	93.418	46.709	10.24	0.000
B	2	425.988	212.994	46.68	0.000
C	2	610.443	305.222	66.89	0.000
A*B	4	51.854	12.964	2.84	0.044
A*C	4	7.406	1.851	0.41	0.803
B*C	4	48.362	12.091	2.65	0.055
A*B*C	8	130.902	16.363	3.59	0.006
Error	27	123.200	4.563		
Total	53	1491.573			

The last column of the above Minitab output shows that factors A, B, C, and interactions A×B and A×B×C are all statistically significant at the 5% level because their P-values (or probability levels) are all less than 0.05. The interaction B×C is significant at the 5.5% level because its exact P-value (using Matlab) is given by $\hat{\alpha} = P(F_{4,27} \geq 2.64971591) = 0.054991546802 < 0.055$. The smaller the P-value is, the more significant the impact of the corresponding effect is on the mean of response y.

Now that we have defined a foundation for balanced orthogonal factorial designs, the next objective is to relate some of Taguchi's OAs (orthogonal arrays) to FFDs. The best source that lists nearly all of Taguchi's OAs is *Taguchi Methods, Orthogonal Arrays and Linear Graphs – Tools for Quality Engineering*, by G. Taguchi and S. Konishi (1987). This source lists 18 OAs and two arrays $L'_9(2^{21})$ and $L'_{27}(3^{22})$ that are reported to be partially orthogonal. Further, on p. iii of the introduction section Y. Wu and S. Taguchi report that the most frequently used arrays are: L_{16}, L_{18}, L_{27}, and L_{12}. Because it will be nearly impossible to discuss all of Taguchi's arrays, we will relate his $L_8(N_f = 8)$, L_{16}, and $L_{27}(N_f = 27)$ to classical FFDs, discuss his L_{12} and L_{18}, and by then the reader should have a good grasp of orthogonality and how to actually design a FF experiment. The reader is also referred to an excellent exposition by Box, Bisgaard, and Fung (1988) that traces the origin of some of Taguchi's OAs.

Note that out of the 18 OAs that are listed in Taguchi and Konishi (1987), the $L_4(2^3)$, $L_8(2^7)$, $L_{16}(2^{15})$, $L_{32}(2^{31})$, $L_{64}(2^{63})$, $L_9(3^4)$, $L_{27}(3^{13})$, $L_{81}(3^{40})$, and $L_{25}(5^6)$ are classical FFDs (or complete factorials). The $L_{16}(4^5)$ OA can be used as a full factorial but when used as a FFD its alias structure cannot be determined because modulus 4 algebra cannot be used to generate columns (3), (4), and (5) of the $L_{16}(4^5)$ from its columns (1) and (2) (see Taguchi and Konishi 1987, p. 59). The $L_{64}(4^{21})$ is an OA; however, as a FFD, its alias structure is unknown to us because 4 is not a prime number.

In addition to L_{12}, $L'_9(2^{21})$, and $L'_{27}(3^{22})$, the remaining Taguchi's OAs listed in Taguchi and S. Konishi (1987) are mixed-level designs and, as such, their alias structures are complicated and not known to the authors. The partially OAs $L'_9(2^{21})$ and $L'_{27}(3^{22})$ are difficult to analyze because in the case of $L'_9(2^{21})$ the design matrix, X, provides only 8 df for studying effects yet it has 21 separate columns, each with 1 df. It is highly improbable that one could study 21 distinct 1-df effects separately from one another with nine runs that yield only 8 df. We ran Taguchi's $L'_9(2^{21})$ on Minitab's GLM and Minitab reported rank deficiency and positive SS's only for eight effects, as expected. The remaining 13 SS's were reported to have zero SS with zero df. Similarly, the $L'_{27}(3^{22})$ partially orthogonal design matrix provides 26 df for studying different 2-df effects, but it has 22 columns that need 44 df. Note that with a 26-df design matrix in base 3 one can study a maximum of thirteen 2-df effects without aliasing them. We would have to recommend against the use of these last two Taguchi's partially orthogonal arrays.

6. Some Commonly Used Taguchi's OAs (Taguchi $L_8(2^7)$ OA)

The subscript 8 in $L_8(2^7)$ implies that the design matrix has $N_f = 8$ distinct rows (or distinct FLCs) and thus provides 7 (= 8 – 1) df for studying seven orthogonal effects, implying that the design matrix can have a maximum of seven orthogonal columns. Further, because $8 = 2^3$, exactly three arbitrary columns can be written, but the remaining four columns must be obtained from the three arbitrary columns.

We use a variation of the procedure first introduced by Kackar and Tsui (1990) by first displaying the three arbitrary columns in *Table 6* (using the base-2 elements 0 for the low level, 1 for the high level, and later on converting to Taguchi's notation of 1 and 2), and then embedding a 2^3 full factorial into the L_8 OA. The reader should bear in mind that for Taguchi's OAs, we are using the notation that the numbers inside the parentheses ( ) generally imply columns, that is, (1) means column 1, (2) means column 2, and so on. To generate column 3, we simply add columns 1 and 2 (modulus 2), that is, (3) = (1) + (2), mod 2. To generate (5), we add (1) and (4) mod 2, that is, (5) = (1) + (4), mod 2; similarly, (6) = (2) + (4), and (7) = (1) + (2) + (4) mod 2. The entire design matrix using this procedure is provided in *Table 7a*. Note that in *Table 7a* it will be totally impossible to generate another column (i.e., an eighth distinct column) that is orthogonal to the above seven columns because the matrix has only 7 df and each column (because of two levels) carries exactly 1 df. To convert *Table 7a* to Taguchi's format, we simply place the third arbitrary column in column 4 and the interaction (1) + (2) column in column 3 as shown in *Table 7b*.

Table 7b now shows that the interaction of columns 1 and 2 is column 3, that is, if factor A is assigned to (1) and B is assigned to column 2, then their interaction A×B must be assigned to (3) because the contrast function of A×B interaction is $\xi(AB) = x_1 + x_2 = (1) + (2)$, mod 2. Similarly, if factor C is assigned to (4), then AC interaction must be assigned to column (5) (if the experimenter desires to study AC interaction). Further, the BC interaction must be assigned to (6) because (2) + (4) = (6), mod 2, and ABC interaction must be assigned to (7) because (1) + (2) + (4) = (7), mod 2.

Table 6
Three Arbitrary Columns of the Taguchi L_8 OA in Base-2 Notation

(1)	(2)	(4)
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

We next convert to Taguchi's notation by transforming $0 \rightarrow 1$ and $1 \rightarrow 2$ as displayed in *Table 7c*. *Table 7c* is identical to the Taguchi's OA on p. 1 of Taguchi and Konishi (1987). So far, we have discussed how to construct the Taguchi L_8 OA for a full 2^3 factorial. The next step is to construct the 2^{4-1} FFD using Taguchi's L_8 OA. Here there are four factors, A, B, C, and D, that will occupy four out of the seven columns. Although not necessary, it is usually best to assign the main factors to the three arbitrary columns, which are (1), (2), and (4). Because the FFD 2^{4-1} has only p = 1 generator, it is best to maximize resolution by selecting g = ABCD as the design generator. This means that we should assign our factors to the columns of Taguchi's L_8 OA in such a manner as to attain the alias structure A = BCD, B = ACD, C = ABD, D = ABC, AB = CD, AC = BD, and AD = BC. If our design matrix shows that D = ABC, then a resolution R = IV is guaranteed. One way to attain this maximum resolution is to assign A to (1), B to (2), and C to (4), and because (1) + (2) + (4) = (7), mod 2, we must assign factor D to column (7); this assignment will ensure a resolution IV design because both effects ABC and D will occupy

Table 7a
The Seven Orthogonal Columns of an L_8 OA

(1)	(2)	(4)	(1)+(2)	(1)+(4)	(2)+(4)	(1)+(2)+(4)
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	1	0	0	0	1

Table 7b
The Seven Orthogonal Columns of Taguchi's L_8 OA in Base-2 Notation

(1)	(2)	(3)	(4)	(1)+(4)	(2)+(4)	(1)+(2)+(4)
0	0	0	0	0	0	0
0	0	0	1	1	1	1
0	1	1	0	0	1	1
0	1	1	1	1	0	0
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1

Table 7c
The Seven Columns of Taguchi's L_8 OA in His Notation

(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1	1	1	1	1	1
1	1	1	2	2	2	2
1	2	2	1	1	2	2
1	2	2	2	2	1	1
2	1	2	1	2	1	2
2	1	2	2	1	2	1
2	2	1	1	2	2	1
2	2	1	2	1	1	2

Table 7d
The Inferior Assignment of Four Factors to a Taguchi L_8 OA

A = (1)	B = (2)	AB = (3)	C = (4)	AC = (5)	D = (6)	AD = (7)
1	1	1	1	1	1	1
1	1	1	2	2	2	2
1	2	2	1	1	2	2
1	2	2	2	2	1	1
2	1	2	1	2	1	2
2	1	2	2	1	2	1
2	2	1	1	2	2	1
2	2	1	2	1	1	2

column (7) and hence will be aliased. A minor deficiency of using Taguchi's L_8 OA is to haphazardly assign the four factors in the 2^{4-1} FFD to any column of his L_8 array because the experimenter may disregard Taguchi's two linear graphs (see Taguchi and Konishi 1987, p. 1) and indeed end up with the inferior resolution III design. If the experimenter follows the column assignments of Taguchi's linear graphs, he or she is assured of a resolution IV design.

A poor choice of column assignments is depicted in *Table 7d* as yielding a resolution III design but does not comply with the guidelines set forth by

Table 7e
The L_8 OA in the Actual Base-2 Notation with *R* = III

A = (1)	B = (2)	AB = (3)	C = (4)	AC = (5)	D = (6)	AD = (7)	**BCD**
0	0	0	0	0	0	0	**0**
0	0	0	1	1	1	1	**0**
0	1	1	0	0	1	1	**0**
0	1	1	1	1	0	0	**0**
1	0	1	0	1	0	1	**0**
1	0	1	1	0	1	0	**0**
1	1	0	0	1	1	0	**0**
1	1	0	1	0	0	1	**0**

Table 7f
The Taguchi L_8 OA in the Base-2 Notation with *R* = IV

A = (1)	B = (2)	AB = (3)	C = (4)	AC = (5)	AD = (6)	D = (7)	ABCD
0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0
0	1	1	0	0	1	1	0
0	1	1	1	1	0	0	0
1	0	1	0	1	0	1	0
1	0	1	1	0	1	0	0
1	1	0	0	1	1	0	0
1	1	0	1	0	0	1	0

Taguchi's two linear graphs. To illustrate to the reader that the FFD in *Table 7d* is indeed inferior, we revert back to the actual base-2 notation, where 0 represents low and +1 represents the high level of a factor, as shown in *Table 7e*. *Table 7e* clearly shows that the BCD effect cannot be assessed (or has been sacrificed) to generate the seven orthogonal columns of the L_8 array, i.e., the generator of the design in *Table 7e* is g = BCD and hence a resolution *R* = III because g = BCD consists of three letters. Further, the contrast function for g = BCD is $\xi(BCD) = x_2 + x_3 + x_4$, which shows that all eight FLCs inside the brackets [0000 = (1), 0011 = cd, 0101 = bd, 0110 = bc, 1000 = a, 1011 = acd, 1101 = abd, and 1110 = abc] make the contrast function $\xi(BCD) = x_2 + x_3 + x_4$ equal to zero (mod 2). Hence, the design matrix in *Table 7e*, which does not match either of Taguchi's two linear graphs, is the PB of the 2^{4-1}_{III} FFD with the generator g = BCD.

To attain a resolution IV design using Taguchi's L_8 involving four factors, we must make the column assignments depicted in *Table 7f*, which does follow the column assignments permitted under either of his two linear graphs. The FFD in *Table 7f* has a resolution *R* = IV because the design generator g = ABCD has four letters, i.e., the design matrix in *Table 7f* is the PB of a 2^{4-1}_{IV} FFD.

From the above discussions, it is concluded that when designing a 2^{4-1} FFD using Taguchi's L_8 OA, the experimenter should follow the column assignment guidelines set forth by either of the two linear graphs given at the bottom of p. 1 of Taguchi and Konishi (1987). Otherwise, he or she may attain a resolution III for the constructed 2^{4-1} design matrix. Before we discuss Taguchi's L_{16} OA, it is critical to mention that the classical notation for base-2 designs is –1 for the low level and +1 for the high level of a factor. The use of –1 and +1 is appropriate because when a factor is at two levels, only its linear effect (or impact) on the mean of response variable y can be assessed, and the contrast coefficients for any two-level factor, say A, are simply –1 and +1. Thus,

Table 7g
The Taguchi L_8 OA in the Classical FFD Notation with *R* = IV

A = (1)	B = (2)	AB = (3)	C = (4)	AC = (5)	AD = (6)	D = (7)	**I = ABCD**	**FLC**
–1	–1	+1	–1	+1	+1	–1	**+1**	**(1)**
–1	–1	+1	+1	–1	–1	+1	**+1**	**cd**
–1	+1	–1	–1	+1	–1	+1	**+1**	**bd**
–1	+1	–1	+1	–1	+1	–1	**+1**	**bc**
+1	–1	–1	–1	–1	+1	+1	**+1**	**ad**
+1	–1	–1	+1	+1	–1	–1	**+1**	**ac**
+1	+1	+1	–1	–1	–1	–1	**+1**	**ab**
+1	+1	+1	+1	+1	+1	+1	**+1**	**abcd**

contrast(A) = $-1 \times A_0 + (+1) \times A_1$, where A_0 is the grand subtotal of all responses for which factor A is at its low level, and where A_1 is the grand total of all responses for which factor A is at its high level. In classical FFD notation, the columns A, B, C, and D of *Table 7f* will take the format presented in *Table 7g*. The signs under the generator g = ABCD are obtained by simply multiplying the signs under A, B, C, and D. Because all the signs under the ABCD column are +1, the ABCD effect is also called the identity, I, of the design matrix in *Table 7g*, and as a result, D = +ABC. Note that the equality D = +ABC can be multiplied through by D to obtain D^2 = ABCD, but *Table 7g* shows that if column D is squared, all its eight entries will equal +1, and hence D^2 = I. By the identity element it is meant that any of the columns (1) through (7) of *Table 7g* can be multiplied by column I = ABCD without changing the column signs of (1) through (7). The identity element for the other $\frac{1}{2}$ fraction (with eight FLCs) of the 2_{IV}^{4-1} FFD in *Table 7g* is simply I = –ABCD so that the signs under D will be obtained from –ABC. In other words, to determine the eight FLCs in the other $\frac{1}{2}$ fraction, simply multiply column (7) signs of *Table 7g* by –1 to obtain the FLCs [d, c, b, bcd, a, acd, abd, abc]. Further, the orthogonality of Taguchi's $L_8(2^7)$ design matrix can be verified by the fact that the dot product of any two columns, (1) through (7), of *Table 7g* is zero because each column is simply an 8×1 vector.

The L_8 OA is also very useful for designing a 2_{III}^{5-2} FF, which is a 1/4th fraction of a 2^5 factorial. Further, it is impossible to generate a resolution IV design with k = 5 factors (similarly, it will be impossible to generate a resolution V design with k = 6 or 7 factors in base 2; see *Table 1a*). For example, if we use g_1 = ABCD and g_2 = BCDE as independent design generators, then the third generator will be (ABCD) × BCDE = AE, which is a resolution II design so that factors A and E will be aliased. To attain an *R* = III, one possible assignment is A on column 1 [A → (1) , B → (2), C → (3), D → (4), E → (5)]; there are now two columns left, and the experimenter may not arbitrarily select two desired first-order interactions to study. The only two-way interactions that can be studied are BD = CE, both on (6), and CD = BE, both on (7). These lead to one of the three generators, g_1 = BCDE. Because both factor C and the AB interaction occupy (3), C = AB implies that g_2 = ABC, and hence g_3 = ADE. Note that the two linear graphs provided by Taguchi for the L_8 OA can be used for designing the 2_{III}^{5-2} FFD, but the experimenter is limited to study only two related interactions, such as (AB, AC), (AB, BC), (AC, AD), (AC, CD), … , or (CD, DE), and no others. In other words, it will be impossible to embed the seven effects, A, B, C, D, E, AB, and CD, into an L_8 OA without aliasing at least two of these seven effects because the two interactions AB and CD have no common letters.

The Taguchi L_8 array can also be used as the 2_{III}^{6-3} and 2_{III}^{7-4} FFDs. In the case of 2_{III}^{6-3}, we have six factors that will occupy six columns of an L_8 array, and the one interaction that can be studied should be determined from Taguchi's two linear graphs. The 2_{III}^{7-4} FFD matrix is saturated because every column will be occupied by a separate main factor and thus no

Table 8
Taguchi's L_{16} OA Generated Using the Base-2 Notation of 0 and 1

	A	B	AB	C	AC	BC	ABC	D	AD	BD	ABD	CD	ACD	BCD	ABCD
Run No.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
3	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
4	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
5	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
6	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
7	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
8	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
9	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
10	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
11	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
12	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
13	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
14	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1
15	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1
16	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

two-way interaction can be studied separately from the main factors (hence a resolution R = III design). In summary, a Taguchi L_8 OA can be used to accommodate a 2^3 complete factorial and any of the four FFDs, 2_{IV}^{4-1}, 2_{III}^{5-2}, 2_{III}^{6-3}, and 2_{III}^{7-4}.

The Taguchi $L_{16}(2^{15})$ OA

Because this OA has $N_f = 16$ distinct rows and $16 = 2^4$, the exponent 4 shows that the design matrix will have exactly four arbitrary columns (see pp. 3-6 of Taguchi and Konishi 1987 for the design matrix and the associated linear graphs) and provides 15 df for studying distinct effects. Hence the matrix can have up to and including 15 orthogonal columns. The L_{16} on p. 3 of Taguchi and Konishi (1987) clearly shows that columns (1), (2), (4), and (8) have been written completely arbitrarily. As in the L_8 array, the L_{16} can easily be generated by first embedding a complete 2^4 factorial in its 15 columns and using the modulus 2 notation of 0 for the low and 1 for the high level of a factor. It is paramount that the four factors of the full 2^4 factorial be assigned to the four arbitrary columns (1), (2), (4), and (8). Without loss of generality, we assign factor A to (1), B to (2), C to (4), and D to (8), as depicted in *Table 8*. Then column (3) of *Table 8* is generated by adding columns (1) and (2) mod 2; column (11) is generated by adding columns (1), (2), and (8) mod 2 because the contrast function for the ABD effect is given by $\xi(\text{ABD}) = x_1 + x_2 + x_4$. In *Table 8*, if we replace all 0's by 1's and all 1's by 2's, we will obtain the Taguchi L_{16} OA in his own notation. In addition to a 2^4 full factorial, a Taguchi L_{16} array can be used to accommodate the FFDs 2_{V}^{5-1}, 2_{IV}^{6-2}, 2_{IV}^{7-3}, 2_{IV}^{8-4}, 2_{III}^{9-5}, 2_{III}^{10-6}, 2_{III}^{11-7}, 2_{III}^{12-8}, 2_{III}^{13-9}, 2_{III}^{14-10}, and 2_{III}^{15-11}. This last FFD, 2_{III}^{15-11}, is saturated because every column of the L_{16} OA is occupied by a main factor, and consequently, each effect has $2^{11} - 1 = 2047$ aliases (this is due to the fact that there are 2047 blocks each with 16 FLCs that are not studied). The three linear graphs (1a, b, and c) on p. 4 of Taguchi and Konishi (1987) pertain to a 2_{V}^{5-1} FFD, and *Table 8* shows that resolution V is obtained by simply assigning the fifth factor, E, to column (15) so that E = ABCD $\rightarrow$ g = ABCDE. In this case, the experimenter may also assign the five main factors and the 10 two-way interactions according to the linear graphs 1b and c to attain a resolution V design. The linear graphs 2a, b, and c on p. 4 of Taguchi and Konishi (1987) will generate a 2^{7-3}_{III} FFD, thus the linear graphs 2a, b, and c are deficient in this case because they yield only a resolution III design. Because the L_{16} meets Webb's minimum required number of runs of 2k = 14, it may be possible to generate a resolution IV design by selecting the independent generators g_1 = ABCE, g_2 = BCDF, and g_3 = ABFG. Note that this set of generators does yield a resolution IV design because the other four generators are g_4 = ADEF, g_5 = CEFG, g_6 = ACDG, and g_7 = BDEG, each of which has four letters. However, the experimenter will have to follow the guidelines set forth by Bulington, Hool, and Maghsoodloo (1990) to attain a resolution IV design using Taguchi's L_{16} OA, but must assign a three-way interaction to column (1). Similarly, a 2^{6-2}_{IV} FFD can be embedded into a Taguchi L_{16} OA, but the experimenter must assign two three-way interactions to two of the columns. The linear graphs 3, 4, and 6a, b, and c on pp. 5-6 of Taguchi and Konishi (1987) produce a 2^{8-4}_{III} FFD, but it is possible to generate a resolution IV design using the independent generators g_1 = ABEF, g_2 = ACEG, g_3 = ADEH, and g_4 = ACFH and assigning A to column (1) of an L_{16}, B to (3), C to (5), D to (7), E to (9), F to (11), G to (13), and assigning H to column (15) of an L_{16} Taguchi OA. The linear graphs 5a, b, and c produce a 2^{10-6}_{III} FFD, and because 2k = 20 > 16, it is impossible to generate a resolution IV design with 16 runs involving 10 factors, i.e., the five remaining columns of an L_{16} do not provide sufficient df (or room) for the ${}_{10}C_2$ = 45 interactions to be placed two at a time (up to six at a time) on the remaining five columns.

Y. Wu and S. Taguchi (p. iii of the introduction to Taguchi and Konishi 1987) state that the $L_{12}(2^{11})$, $L_{16}(2^{15})$, $L_{18}(2\times3^7)$, and $L_{27}(3^{13})$ are the most commonly used of Taguchi's OAs. We have discussed how to generate an $L_{16}(2^{15})$, and the $L_{12}(2^{11})$ Taguchi OA is a modification of the Plackett-Burman design (for more information see Montgomery (2001a), pp. 343-345), where every pair of columns is orthogonal in the sense that the pairs (1, 1), (1, 2), (2, 1), and (2, 2) appear exactly three times together in any two columns of the L_{12}. We defer the discussion of $L_{18}(2\times3^7)$ OA before we discuss Taguchi's parameter design. Thus, we next discuss Taguchi's L_{27} OA.

The Taguchi $L_{27}(3^{13})$ OA

The $L_{27}(3^{13})$ is an OA of 27 distinct FLCs in base 3; because the N_f = 27 distinct rows provide 26 df for studying different effects and each column of an L_{27} has 2 df (because of three levels), this design matrix can be used to examine a maximum of 26/2 = 13 two-df effects. Thus, the L_{27} can be used to accommodate a full 3^3 factorial and the FFDs 3^{4-1}_{IV}, 3^{5-2}_{III}, 3^{6-3}_{III}, 3^{7-4}_{III}, 3^{8-5}_{III}, 3^{9-6}_{III}, 3^{10-7}_{III}, 3^{11-8}_{III}, 3^{12-9}_{III}, and 3^{13-10}_{III}. The last FFD, 3^{13-10}_{III}, is saturated in the sense that every 2-df column of the L_{27} array is occupied by a 2-df main factor and each effect will have $3^{10} - 1 = 59048$ aliases. As in the case of L_{16}, the simplest way of generating the L_{27} is to embed a 3^3 complete factorial design into its 13 columns. The exponent 3 (in $3^3 = 27$) implies that the levels of the three factors, A, B, and C, can be written arbitrarily in three columns. The three arbitrary columns of the Taguchi L_{27} are columns (1), (2), and (5) (see Taguchi and Konishi 1987, pp. 37-38). Column (1) is arbitrary because it consists of nine 1's (low level of factor A), followed by nine 2's (the middle level of factor A), and then nine 3's (the high level of factor A). Similarly, column (2) was arbitrarily written as three 1's, three 2's, followed by three 3's, and this pattern is repeated twice more, and column (5) is written arbitrarily as levels 1, 2, and 3 of factor C and repeated eight more times. Because two-way interactions, such as A×B, have 4 df, then each two-way interaction can be embedded in two 2-df columns of an L_{27} OA. For example, the A×B effect will occupy columns (3) and (4) of the L_{27}, assuming that A is on column (1) and B is on column (2). As was illustrated in section 5, the A×B interaction decomposes into two orthogonal components, AB and AB^2, but Taguchi replaces the component AB^2 by the statistically unconventional component A^2B. Converting to base-3 notation of 0 for low, 1 for middle, and 2 for the high level of a factor, it is noted that the contrast function for A^2B is $\xi(A^2B) = 2x_1 + x_2 = 2\xi(AB^2)$ because in base-3 algebra, 4 = 1 modulus (3), and thus the two components A^2B and AB^2 represent the same effect. Column (3) of L_{27} is occupied by the AB component of A×B, and because the contrast function of AB is $\xi(AB) = x_1 + x_2$, column (3) is generated by adding columns (1) and (2) modulus 3. Similarly, column (4) is generated by adding 2 × (1) + (2) (mod 3) because $\xi(A^2B) = 2x_1 + x_2$. The

Table 9
Generating Taguchi's L_{27} Using Microsoft Excel Mod 3 Function

	A	B	AB	A^2B	C	AC	A^2C	BC	ABC	A^2BC	B^2C	AB^2C	**A^2B^2C**
Run No.	(1)	(2)	(3)	(4)	(5)	(6)	('7)	(8)	(9)	(10)	(11)	(12)	**(13)**
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	1	1	1	1	1	1	1	1	1
3	0	0	0	0	2	2	2	2	2	2	2	2	2
4	0	1	1	1	0	0	0	1	1	1	2	2	2
5	0	1	1	1	1	1	1	2	2	2	0	0	0
6	0	1	1	1	2	2	2	0	0	0	1	1	1
7	0	2	2	2	0	0	0	2	2	2	1	1	1
8	0	2	2	2	1	1	1	0	0	0	2	2	2
9	**0**	**2**	**2**	**2**	**2**	**2**	**2**	**1**	**1**	**1**	**0**	**0**	**0**
10	1	0	1	2	0	1	2	0	1	2	0	1	2
11	1	0	1	2	1	2	0	1	2	0	1	2	0
12	1	0	1	2	2	0	1	2	0	1	2	0	1
13	1	1	2	0	0	1	2	1	2	0	2	0	1
14	1	1	2	0	1	2	0	2	0	1	0	1	2
15	1	1	2	0	2	0	1	0	1	2	1	2	0
16	1	2	0	1	0	1	2	2	0	1	1	2	0
17	1	2	0	1	1	2	0	0	1	2	2	0	1
18	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**1**	**2**	**0**	**0**	**1**	**2**
19	2	0	2	1	0	2	1	0	2	1	0	2	1
20	2	0	2	1	1	0	2	1	0	2	1	0	2
21	2	0	2	1	2	1	0	2	1	0	2	1	0
22	2	1	0	2	0	2	1	1	0	2	2	1	0
23	2	1	0	2	1	0	2	2	1	0	0	2	1
24	2	1	0	2	2	1	0	0	2	1	1	0	2
25	2	2	1	0	0	2	1	2	1	0	1	0	2
26	2	2	1	0	1	0	2	0	2	1	2	1	0
27	2	2	1	0	2	1	0	1	0	2	0	2	1

Microsoft Excel Mod function was used to generate the entire L_{27} OA given in *Table 9*. If 0 is replaced by 1, 1 by 2, and 2 by 3 in *Table 9*, Taguchi's L_{27} OA in his own notation is obtained, as shown on p. 37 of Taguchi and Konishi (1987). The table of interactions between two columns (TOIBTC) on p. 38 of Taguchi and Konishi (1987) assisted in determining which second-order effect would occupy which column of the L_{27} OA. The L_{27} OA can provide a resolution IV design in only one instance, namely for the FFD 3_{IV}^{4-1}, and to ensure that an R = IV is attained, all the experimenter has to do is assign the fourth factor D to one of the columns (9), (10), (12), or (13), while factors A, B, and C must be embedded in the three arbitrary columns (1), (2), and (5).

As an example, suppose we wish to study six factors, A, B, C, D, E, and F, each at three levels and the two-way interactions C×D, C×E, and D×E. These six factors and three two-way interactions will need $2 \times 6 + 3 \times 4 = 24$ df and will occupy $24/2 = 12$ of the 13 columns of an L_{27} OA, and the remaining column can be used to study one more effect that can be determined after column assignments are completed. Suppose factor C is assigned to column (1), D to (2), then CD → (3), C^2D will have to be assigned to column (4). Assigning E to (5), then *Table 9* shows that CE will have to be embedded onto (6), C^2E onto column(7), DE onto (8), and D^2E to column (11). These assignments leave columns (9), (10), (12), and (13) available. Without loss of generality, factor A may be assigned to column (9), factor B to (10), and factor F to (12), which leaves only column (13) empty. The TOIBTC on p. 38 of Taguchi and Konishi (1987) now shows that (1) × (12) = (11) & (13), and hence column (13) may be used to study one component of C×F, namely, CF or C^2F. Microsoft Excel is again used to verify that (1) + (12) = (13) mod 3, and hence the CF component of C×F can also be studied. The question that now arises is, "What is the alias structure of the above 3^{6-3}_{III} FFD?" Because three independent generators are needed out of the total of $(3^3 - 1)/(3 - 1) = 13$ generators and the Taguchi L_{27} always yields the PB of a FFD, the TOIBTC on p. 38 of Taguchi and Konishi (1987) is used to assist in identifying the alias structure. This table shows that (13) = (2) × (10) and thus CF = BD, or $C^3F^3 = BC^2DF^2$, which shows that one of the design generators is $g_1 = BC^2DF^2$ because $C^3F^3 = C^0F^0 = I$. Next, the TOIBTC on p. 38 of Taguchi and Konishi (1987) shows that (2) × (6) = (9), or (12). Thus, CDE = A, or $g_2 = AC^2D^2E^2$. Lastly, the same TOIBTC of Taguchi and Konishi (1987) and our *Table 9* show that 2(9) × (10) = (1), which yields $A^2B = C$, or $g_3 = AB^2C$. Thus, the other 10 generators are

$$g_4 = g_1 \times g_2 = BC^2DF^2 \times AC^2D^2E^2 = ABCE^2F^2,$$
$$g_5 = g_1^2 \times g_2 = B^2CD^2F \times AC^2D^2E^2 = AB^2DE^2F,$$
$$g_6 = g_1 \times g_3 = BC^2DF^2 \times AB^2C = ADF^2,$$
$$g_7 = g_1^2 \times g_3 = B^2CD^2F \times AB^2C = ABC^2D^2F,$$
$$g_8 = g_2 \times g_3 = AC^2D^2E^2 \times AB^2C = ABDE,$$
$$g_9 = g_2 \times g_3^2 = AC^2D^2E^2 \times A^2BC^2 = BCD^2E^2,$$
$$g_{10} = g_1 \times g_2 \times g_3 = BC^2DF^2 \times AC^2D^2E^2 \times AB^2C = ACEF,$$
$$g_{11} = g_1^2 \times g_2 \times g_3 = B^2CD^2F \times AC^2D^2E^2 \times AB^2C = AB^2C^2D^2EF^2,$$
$$g_{12} = g_1 \times g_2^2 \times g_3 = BC^2DF^2 \times A^2CDE \times AB^2C = CD^2EF^2, \text{ and}$$
$$g_{13} = g_1^2 \times g_2^2 \times g_3 = B^2CD^2F \times A^2CDE \times AB^2C = BEF$$

Note that the minimum length word among the above 13 generators in the defining relation I is three and hence a Resolution III design. Further, because the FFD 3^{6-3}_{III} is a 1/27th fraction and only one block out of the 27 blocks is studied and 26 blocks are left out of experimentation, each effect must have $3^3 - 1 = 26$ aliases. For example, to obtain the 26 aliases of factor A, we either multiply A by the 13 generators and also multiply A by the 13 generators squared modulus 3. Or, we may multiply A and A^2 by the 13 generators modulus 3 using the statistical convention that the first letter must have an exponent of 1. Following this procedure, the 26 aliases of factor A are

$$A = ABC^2DF^2 = AB^2CD^2F = ACDE = CDE = ABC^2 = BC^2 = AB^2C^2EF = BCE^2F^2 = ABD^2EF^2 = BD^2EF^2 = AD^2F = DF^2 = AB^2CDF^2 = BC^2D^2F = AB^2D^2E^2 = BDE = ABCD^2E^2 = AB^2C^2DE = AC^2E^2F^2 = CEF = ABCDE^2F = BCDE^2F = ACD^2EF^2 = AC^2DE^2F = ABEF = AB^2E^2F^2.$$

The Taguchi $L_{18}(2 \times 3^7)$ OA

The L_{18} is the most commonly used mixed-level Taguchi's OA and can accommodate one two-level factor and a maximum of seven three-level factors. Because the total number of distinct runs $N_f = 18 = 2^1 \times 3^2$, summing the exponents $1 + 2 = 3$ implies that exactly three columns are written completely arbitrarily [namely columns (1), (2), and (3)]. The design matrix provides $18 - 1 = 17$ df for studying effects, where the two-level factor A on column (1) will absorb 1 df, and the seven three-level factors, B, C, D, E, F, G, and H, each absorb 2 df. Thus, the eight factors altogether will absorb 15 out the possible 17 df that the design matrix provides. This leaves two unused df that can be used to study the interaction only between columns (1) and (2). Assuming, without loss of generality, that A is the two-level factor, the experimenter must embed the three-level factor whose interaction with factor A he or she would like to examine in column (2). Later it will be shown in a parameter design example that the only possible interaction that can be studied is (1)×(2), but this interaction cannot be embedded into the design matrix. The question that now arises is how Dr. Taguchi developed the other five columns (4, 5, 6, 7, and 8) of the L_{18} so that the design matrix is orthogonal. As stated earlier, the first three columns are written completely arbitrarily. Then, we need to address the construction of columns (4) through (8) of the L_{18}. The reader must be informed that we are not sure how G. Taguchi developed his L_{18} OA, and what follows is our explanation. To this end, let us define a group of three 3×1 vectors, $G_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}'$, $G_2' = \begin{bmatrix}2 & 3 & 1\end{bmatrix}$, and $G_3' = \begin{bmatrix}3 & 1 & 2\end{bmatrix}$, where prime is used to denote matrix transpose. Note that column (3) of L_{18} is arbitrarily written as $\begin{bmatrix}G_1' & G_1' & G_1' & G_1' & G_1' & G_1'\end{bmatrix}'$. Next, the above three vectors are translated by subtracting 2 from each element so that G_1 transforms to $G_4 = \begin{bmatrix}-1 & 0 & 1\end{bmatrix}'$, G_2 transforms to $G_5 = \begin{bmatrix}0 & 1 & -1\end{bmatrix}'$, and G_3 translates to $G_6' = \begin{bmatrix}1 & -1 & 0\end{bmatrix}$. (Note that G_4 is the linear contrast in base 3 for a quantitative factor.) It is well known that two vectors are orthogonal if and only if their dot product is zero. Clearly, $G_4' \times G_5 = \begin{bmatrix}-1 & 0 & 1\end{bmatrix} \times \begin{bmatrix}0 & 1 & -1\end{bmatrix}' = -1 = G_4' \times G_6 = G_5' \times G_6$, which implies that the vectors G_1 and G_2 are not orthogonal, G_1 and G_3 are not orthogonal, and neither are G_2 and G_3. Further, $G_i' \times G_i = +2$ for all i = 4, 5, and 6. A close examination of the fourth column of L_{18} on p. 36 of Taguchi and Konishi (1987) reveals that column (4) can be written as (4) = $\begin{bmatrix}G_1' & G_1' & G_2' & G_3' & G_2' & G_3'\end{bmatrix}'$ and column (5) of L_{18} is simply (5) = $\begin{bmatrix}G_1' & G_2' & G_1' & G_3' & G_3' & G_2'\end{bmatrix}'$. If we now take the dot product of column (4) with (5) after the translation, we obtain $\begin{bmatrix}G_4' & G_4' & G_5' & G_6' & G_5' & G_6'\end{bmatrix} \times \begin{bmatrix}G_4\\G_5\\G_4\\G_6\\G_6\\G_5\end{bmatrix} = 2 - 1 - 1 + 2 - 1 - 1 = 0$, which shows that columns (4) and (5) of L_{18} are orthogonal. Another pattern that is obvious in the L_{18} is that columns (3) through (8) have their first three rows as the 3×1 vector G_1. To generate column (6), we have to find another permutation of $\begin{bmatrix}G_1' & G_1' & G_2' & G_2' & G_3' & G_3'\end{bmatrix}'$, keeping G_1 in the first position, which is orthogonal to both columns (4) and (5). One such permutation is $\begin{bmatrix}G_1' & G_2' & G_3' & G_2' & G_1' & G_3'\end{bmatrix}'$, which comprises column (6) of Taguchi's L_{18}. Similarly, two other permutations of $\begin{bmatrix}G_1' & G_1' & G_2' & G_2' & G_3' & G_3'\end{bmatrix}'$ that are orthogonal to columns (3), (4), (5), and (6) are columns (7) = $\begin{bmatrix}G_1' & G_3' & G_2' & G_2' & G_3' & G_1'\end{bmatrix}'$ and (8) = $\begin{bmatrix}G_1' & G_3' & G_3' & G_1' & G_2' & G_2'\end{bmatrix}'$ of Taguchi's L_{18} OA. The last five columns of the L_{18} Taguchi OA in terms of the 3×1 vectors G_1, G_2, and G_3 are given below and denoted as the matrix **G**.

$$\mathbf{G} = \begin{array}{c} \begin{matrix}(4) & (5) & (6) & (7) & (8)\end{matrix} \\ \begin{bmatrix} G_1 & G_1 & G_1 & G_1 & G_1 \\ G_1 & G_2 & G_2 & G_3 & G_3 \\ G_2 & G_1 & G_3 & G_2 & G_3 \\ G_3 & G_3 & G_2 & G_2 & G_1 \\ G_2 & G_3 & G_1 & G_3 & G_2 \\ G_3 & G_2 & G_3 & G_1 & G_2 \end{bmatrix} \end{array}$$

The above developments indicate that the L_{18} is not unique in the sense that there are other permutations of the last five rows of the matrix **G** that yield an L_{18} orthogonal array. We have identified all other **G** matrices, of which there are 11, that are somewhat distinct from the above Taguchi's layout but will also yield an L_{18} OA. The other 11 are listed atop the next page. Note that the orthogonality of the 11 alternatives to Taguchi's L_{18} was verified by first replacing column (1) by nine –1's followed by nine +1's, then subtracting 2 from every element of the remaining seven columns so that all eight columns summed to zero, and finally computing the resulting $X' \times X$ matrix. In all the 11 alternative cases, the resulting 8×8 matrix $X' \times X$ was diagonal, and Minitab's GLM also verified that all 11 design matrices were orthogonal.

7. Taguchi's Parameter Design

A Taguchi parameter design experiment consists of two orthogonal arrays. The inner array accom-

$$
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_2&G_2&G_1&G_3&G_3\\G_2&G_3&G_3&G_2&G_1\\G_3&G_1&G_2&G_2&G_3\\G_1&G_3&G_2&G_3&G_2\\G_3&G_2&G_3&G_1&G_2\end{bmatrix}\end{array},
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_1&G_2&G_2&G_3&G_3\\G_2&G_3&G_1&G_3&G_2\\G_2&G_1&G_3&G_2&G_3\\G_3&G_2&G_3&G_1&G_2\\G_3&G_3&G_2&G_2&G_1\end{bmatrix}\end{array},
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_1&G_2&G_2&G_3&G_3\\G_3&G_3&G_2&G_2&G_1\\G_3&G_2&G_3&G_1&G_2\\G_2&G_1&G_3&G_2&G_3\\G_2&G_3&G_1&G_3&G_2\end{bmatrix}\end{array},
$$

$$
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_1&G_2&G_2&G_3&G_3\\G_3&G_3&G_2&G_2&G_1\\G_2&G_1&G_3&G_2&G_3\\G_3&G_2&G_3&G_1&G_2\\G_2&G_3&G_1&G_3&G_2\end{bmatrix}\end{array},
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_2&G_1&G_3&G_2&G_3\\G_1&G_2&G_2&G_3&G_3\\G_2&G_3&G_1&G_3&G_2\\G_3&G_3&G_2&G_2&G_1\\G_3&G_2&G_3&G_1&G_2\end{bmatrix}\end{array},
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_2&G_1&G_3&G_2&G_3\\G_1&G_2&G_2&G_3&G_3\\G_3&G_3&G_2&G_2&G_1\\G_2&G_3&G_1&G_3&G_2\\G_3&G_2&G_3&G_1&G_2\end{bmatrix}\end{array},
$$

$$
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_2&G_2&G_3&G_3&G_1\\G_2&G_3&G_2&G_1&G_3\\G_1&G_3&G_3&G_2&G_2\\G_3&G_1&G_2&G_3&G_2\\G_3&G_2&G_1&G_2&G_3\end{bmatrix}\end{array},
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_1&G_3&G_2&G_3&G_2\\G_2&G_3&G_1&G_2&G_3\\G_2&G_2&G_3&G_3&G_1\\G_3&G_2&G_2&G_1&G_3\\G_3&G_1&G_3&G_2&G_2\end{bmatrix}\end{array},
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_1&G_3&G_2&G_2&G_3\\G_3&G_1&G_2&G_3&G_2\\G_3&G_2&G_3&G_2&G_1\\G_2&G_2&G_1&G_3&G_3\\G_2&G_3&G_3&G_1&G_2\end{bmatrix}\end{array},
$$

$$
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_3&G_2&G_2&G_1&G_3\\G_3&G_1&G_3&G_2&G_2\\G_2&G_2&G_3&G_3&G_1\\G_2&G_3&G_1&G_2&G_3\\G_1&G_3&G_2&G_3&G_2\end{bmatrix}\end{array},
\begin{array}{c}\begin{array}{ccccc}(4)&(5)&(6)&(7)&(8)\end{array}\\ \begin{bmatrix}G_1&G_1&G_1&G_1&G_1\\G_2&G_2&G_3&G_1&G_3\\G_3&G_2&G_1&G_3&G_2\\G_3&G_1&G_2&G_2&G_3\\G_2&G_3&G_2&G_3&G_1\\G_1&G_3&G_3&G_2&G_2\end{bmatrix}\end{array}
$$

modates the controllable factors, while the noise (or uncontrollable) factors are embedded into the outer orthogonal array. The objectives of a PDE (parameter design experiment) for a nominal dimension is three-fold, the last two of which are optimization steps:

1. To classify the design factors into three categories of Control, Signal, and Weak factors. A Control factor is one that impacts process variability and may or may not impact the process mean response. A Signal factor significantly impacts the mean response but has no (or trivial) impact on the variability of the response. A Weak factor has no impact on the mean or variability of the response.

2. To use the levels of the Control factors to reduce process variability.
3. To use the levels of Signal factors to move the mean response toward the ideal target m.

When the response, y, is either STB or LTB, QI can be accomplished in one step by increasing the signal-to-noise ratio (which in turn lowers the signal for an STB and heightens it for an LTB QCH), and as a result the experimenter can accomplish objectives (2) and (3) above in one step by setting the process conditions at those levels of influential factors that maximize Taguchi's S/N ratio, measured in decibels as defined below.

$$\eta_{db} = -10\log_{10}(\text{MSD}) = \begin{cases} -10\log_{10}\left(\sum_{i=1}^{n} y_i^2 / n\right), & \text{if y is STB} \\ -10\log_{10}\left[\frac{1}{n}\sum_{i=1}^{n}\left(1/y_i^2\right)\right], & \text{if y LTB} \end{cases} \tag{4}$$

It should be highlighted that classical design of experiments until the mid 1970s emphasized methods that improve only the mean response, and hence one OA would generally be sufficient, and in some cases more efficient, to carry out QI only when the response y is STB or LTB. However, when the response is of nominal type, the variability of response plays a very important role in QI, and hence an outer OA is needed to embed the noise factors that cause process variability. In a PDE, the experimenter intentionally induces noise into the response y through the use of an outer array and then takes advantage of the interactions between the noise factors in the outer array with the controllable factors in the inner array to assess and then diminish the impact of noise on the response y (using appropriate S/N ratios). The impact of noise factors is diminished by selecting those levels of the controllable influential factors embedded in the inner OA that are less sensitive to noise factors embedded in the outer array, thereby producing a more robust product.

To illustrate Taguchi's PDE, we make use of a quality engineering experiment published by the American Supplier Institute (ASI), Inc. (1984) from its Seminar Series B. The nominal response variable, y, is the pull-force of ignition cables measured in pounds (lb). The specifications on y are 40 ± 15 lb, that is, the ideal target for pull-force is m = 40 lb and the manufacturing (semi-) tolerance is $\Delta = 15$ lb. The experimental layout and the resulting data are displayed in *Table 10*. The controllable factors are A = extrusion tooling (two levels: type 1 and type 2), B = line speed (three levels: slow, medium, and fast), C = water through temperature (low, medium, high), D = insulation material (types 1, 2, and 3), E = CV stream pressure (low, medium, and high), F = CV speed (slow, medium, fast), G = braid tension (low, medium, high), and H = release coating (types 1, 2, and 3). The noise factors are Sample (two cables selected at random), P = position within each of the two sampled cables.

Note that we have borrowed, with permission, the experimental layout and the data therein from the American Supplier Institute, but all the analyses performed in Microsoft Excel, Matlab, and a Taguchi software belong to the authors. *Table 10* shows that run number 1, where all factors were at their low levels, yielded much better results than run 2 because all four measurements of pull-force in run 1 were within the spec interval (LSL = 25, USL = 55 lb). At run 2, where factors A and B were at low while the other six factors were at their medium levels, the sample 1 at both positions gave two nonconforming measurements (10 and 15 lb). *Table 11* gives the summary statistics for each run (i = 1, 2, …, 18), where the grand total $y_{..} = \sum_{i=1}^{18}\sum_{j=1}^{4} y_{ij} = 3779$. In *Table 11*, the first-row statistics were computed as $\bar{y}_{1.} = (30 + 40 + 38 + 49)/4 = 39.25$, $S_1 = \sqrt{\frac{1}{3}\sum_{j=1}^{4}\left(y_{1j} - 39.25\right)^2} = 7.80491$, and $\eta_1 = 10\times\log_{10}\left(\frac{\bar{y}_{1.}^2}{S_1^2} - \frac{1}{4}\right) = 10\times\log_{10}(25.0397) = 13.9863$, where the subscript i extends over the FLCs 1 through 18 and j extends over the four observations in the outer L_4 OA. The last column gives the natural logarithm of the i-th run standard deviation.

We next use the summary statistics in *Table 11* to obtain a response table (RT) for total S/N ratios that will help identify the factors that control process variation. Microsoft Excel was used to obtain the RT for S/N ratios, which is presented in *Table 12a*. Because this is a mixed-level design, only MSs (mean squares) can be compared against each other to assess their relative influence on the S/N ratio of the response y. *Table 12a* clearly shows that factor H has the largest impact on the S/N ratio of the pull-

Table 10
PDE Layout from ASI, Inc. (reprinted with permission)

Taguchi's	L_{18} Inner OA								L_4 Outer OA			
L_{18} OA	A	B	C	D	E	F	G	H	Sample 1		Sample 2	
Columns	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	P_1	P_2	P_1	P_2
Run No.												
1	1	1	1	1	1	1	1	1	30 lb	40	38	49
2	1	1	2	2	2	2	2	2	10	15	25	25
3	1	1	3	3	3	3	3	3	49	53	53	55
4	1	2	1	1	2	2	3	3	62	58	52	68
5	1	2	2	2	3	3	1	1	30	50	49	62
6	1	2	3	3	1	1	2	2	10	25	29	36
7	1	3	1	2	1	3	2	3	58	42	41	50
8	1	3	2	3	2	1	3	1	28	29	32	31
9	1	3	3	1	3	2	1	2	110	74	94	115
10	2	1	1	3	3	2	2	1	76	88	66	103
11	2	1	2	1	1	3	3	2	52	37	54	59
12	2	1	3	2	2	1	1	3	55	79	62	98
13	2	2	1	2	3	1	3	2	5	35	16	42
14	2	2	2	3	1	2	1	3	52	96	79	91
15	2	2	3	1	2	3	2	1	50	70	56	65
16	2	3	1	3	2	3	1	2	15	20	18	21
17	2	3	2	1	3	1	2	3	51	62	59	70
18	2	3	3	2	1	2	3	1	77	83	66	74

force and thus was assigned a rank of 1, and factors A, C, and E have relatively trivial impact on the S/N of the pull-force. The MSs in *Table 12a* indicate that factors H, B, D, and G are, in that order, relatively the most influential from the standpoint of process variation, and factor F moderately influences S/N ratio. Thus, the Control factors in the order of their strength are H, B, D, G, and F. Our experience indicates that most quantitative Control factors have a quadratic impact on the response and most Signal factors have a linear impact on y (but not always).

Before deciding where to set the levels of these five Control factors, the reader should be reminded that the L_{18} OA provides sufficient df to study only the interaction (1)×(2) = A×B. Factors A and B are crossed to exhibit their interaction in *Table 12b*. From *Table 12b*, it is deduced that the effect of factor B on the S/N of y at A_1 is positively quadratic (i.e., convex upward) given by Contrast(B_{A1}) = 48.15217 – 2(36.98029) + 54.34445 = 28.5360, and the quadratic effect of B at A_2 is given by Contrast(B_{A2}) = 40.40491 – 2(31.19984) + 55.08833 = 33.0936. Because these two quadratic contrasts are quite similar, factors A and B do not much interact in impacting the S/N ratio of the response y. From *Table 12b*, the SS of A×B is given by $SS(A\times B)_\eta =$

$$\frac{48.15217^2+36.98029^2+54.34445^2+40.40491^2+31.19984^2+55.08833^2}{3} - \frac{266.1700^2}{18} - SS(A_\eta) - SS(B_\eta) = 6.58524 \rightarrow MS(A\times B)_\eta$$

= 3.292620, which again confirms that A and B do not materially interact in impacting the variability of pull-force on a relative basis [see *Table 12a* that shows $MS(H_\eta)$ = 81.7904].

Table 11
Summary Statistics for the 18 Runs of *Table 10*

L_{18} Inner OA Run No.	Mean of Run i $\bar{y}_{i.}$	Standard Deviation S_i	Signal-to-Noise Ratio η_i	Natural log of S_i $\ln(S_i)$
1	39.25	7.804913	13.9863	2.05475
2	18.75	7.5	7.7815	2.0149
3	52.5	2.516611	26.3844	0.92291
4	60	6.733003	18.9852	1.90702
5	47.75	13.22561	11.0671	2.58215
6	25	10.98484	6.9281	2.39652
7	47.75	7.932003	15.5617	2.07091
8	30	1.825742	24.3096	0.60199
9	98.25	18.48197	14.4731	2.9168
10	83.25	15.94522	14.3151	2.76916
11	50.5	9.469248	14.5012	2.24805
12	73.5	19.19201	11.5887	2.95449
13	24.5	17.0196	2.6060	2.83437
14	79.5	19.67232	12.0632	2.97921
15	60.25	8.958236	16.5306	2.19257
16	18.5	2.645751	16.8702	0.97296
17	60.5	7.852813	17.7163	2.06087
18	75	7.071068	20.5019	1.95601

We are now in a position to show the only interaction that an L_{18} OA allows to be studied is $(1)\times(2)$. The total SS from the standpoint of S/N ratios is given by $SS(\text{Total}_\eta) = 13.9863^2 + 7.7815^2 + 26.3844^2 + \ldots + 20.5019^2 - \frac{266.1700^2}{18} = 583.77100$ (with 17 df). However, $SS(A_\eta) + SS(B_\eta) + SS(C_\eta) + SS(D_\eta) + SS(E_\eta) + SS(F_\eta) + SS(G_\eta) + SS(H_\eta) = 577.18575$ (with 15 df), showing that the value of the difference $SS(\text{Total}_\eta) - 577.18575 = 6.58524$ is exactly equal to $SS[(1)\times(2)]$. If we compute the $SS((1)\times(j))$ for any $j \neq (2)$, we will not obtain the value of 6.58524 required for the orthogonality of the L_{18} design matrix.

Because maximizing the S/N ratio is equivalent to minimizing variance (and simultaneously maximizing the mean), the subtotals L_i ($i = 1, 2, 3$) for the five Control factors B, D, F, G, and H in response *Table 12a* show that their optimal levels are given by $B_3D_{3(1)}F_3G_3H_{3(1)}$. If $(A\times B)_\eta$ were relatively significant, then we would have to simultaneously choose the optimal levels of factors A and B from the interaction *Table 12b*. The optimum levels $B_3D_{3(1)}F_3G_3H_{3(1)}$ provide only four choices, $B_3D_3F_3G_3H_3$, $B_3D_1F_3G_3H_3$, $B_3D_3F_3G_3H_1$, and $B_3D_1F_3G_3H_1$ for the five Controls B, D, F, G, and H at which the cables should be manufactured to minimize process variation. The reader should note that some authors use graphical methods to identify the influential effects and their optimal levels, but graphical methods work only for quantitative factors while in the present experiment factors A, D, and H are qualitative. The RT method used herein (first recommended by Dr. Taguchi and ASI) works well in all cases and is also our recommended choice. Our next step consists of determining which one of the remaining three de-

Table 12a
Response Table for S/N Ratio in dB

Effects	A	B	C	D	E	F	G	H
L_1	139.4769	88.5571	82.3244	96.1926	83.5424	77.1349	80.0485	100.7105
L_2	126.6931	68.1801	87.4389	69.1068	96.0658	88.1199	78.8333	63.16
L_3	N/A	109.4328	96.4067	100.8706	86.5618	100.9152	107.2882	102.2994
SS	9.0792	141.822	16.9383	98.0253	14.2376	47.216	86.2863	163.5809
MS	9.0792	70.911	8.4692	49.0127	7.1188	23.608	43.1431	81.7904
Ranks	6	2	7	3	8	5	4	1

Table 12b
A×B Interaction for η_{dB}

	B_1	**B_2**	**B_3**
A_1	48.15217	36.98029	54.34445
A_2	40.40491	31.19984	55.08833

sign factors, A, C, and E, have large impacts on the mean of y and using their levels to adjust the mean toward the ideal target of m = 40 lb.

To ascertain which design factors have nontrivial impact on the mean, we have made a RT for the mean of pull-force, which is provided in *Table 13a*. In *Table 13a*, rank 5 is missing because the ANOVA table from Minitab, given below, shows that A×B is the fifth most statistically significant effect that affects the mean response, as illustrated in *Table 13b*. *Table 13b* clearly shows that the impact of factor B on the mean response is positively linear at the low level of A, but its impact has a negative slope at A_2, and hence factors A and B interact in impacting the mean response.

General Linear Model: y vs. A, B, A*B, C, D, E, F, G, H, Sample, Position, Sample*Position

```
Factor   Type    Levels     Values
A        fixed     2       1,  2,
B        fixed     3       1,  2,  3
C        fixed     3       1,  2,  3
D        fixed     3       1,  2,  3
E        fixed     3       1,  2,  3
F        fixed     3       1,  2,  3
G        fixed     3       1,  2,  3
H        fixed     3       1,  2,  3
Sample   random    2       1,  2
Position random    2       1,  2
```

```
Analysis of Variance for y, using Adjusted SS for Tests

 Source    DF  Seq SS  Adj SS  Adj MS     F      P
 A          1  2508.7  2508.7  2508.7  26.97  0.000
 B          2   371.0   371.0   185.5   1.99  0.147
 A*B        2  4715.9  4715.9  2357.9  25.35  0.000
 C          2  4904.9  4904.9  2452.4  26.36  0.000
 D          2  2898.8  2898.8  1449.4  15.58  0.000
 E          2  3732.0  3732.0  1866.0  20.06  0.000
 F          2 10166.8 10166.8  5083.4  54.64  0.000
 G          2  1753.0  1753.0   876.5   9.42  0.000
 H          2  6794.7  6794.7  3397.3  36.52  0.000
 Sample     1   715.7   715.7   715.7   6.51  0.238
 Position   1  1810.0  1810.0  1810.0  16.45  0.154
 Sample*    1   110.0   110.0   110.0   1.18  0.282
  Position
 Error     51  4744.5  4744.5    93.0
 Total     71 45226.0

S = 9.64522     R-Sq = 89.51%   R-Sq(adj) = 85.40%
```

The above ANOVA table from Minitab verifies that the impact of factor B on the mean response is not significant, but factors A, C, and E do significantly impact (all three P-values < 0.0001) the mean pull-force. Therefore, A, C, and E qualify as Signal (or adjustment) factors, and their levels can be used to adjust the mean response toward the ideal target of 40 lb. Note that even if the impact of factors D, F, G, and H on the mean response are highly statistically significant, their levels cannot be adjusted because these four factors are Controls and their levels must be set according to maximizing the S/N ratio of the process to reduce variance.

Further, the above Minitab output can be used to verify the fact that the only interaction that can be studied with an L_{18} OA is (1)×(2). In fact, we ran Minitab's GLM and requested the Model terms A, B, C, A×C, D, E, F, G, H, Sample, Position, and Sample×Position, but Minitab would not even provide a meaningful ANOVA table due to the loss of orthogonality. The corresponding Minitab output is provided below.

Table 13a
Response Table for the Mean in Pounds

Effects	A	B	C	D	E	F	G	H
L_1	46.58333	52.95833	45.54167	61.45833	52.83333	42.125	59.45833	55.91667
L_2	58.38889	49.5	47.83333	47.875	43.5	69.125	49.25	39.25
L_3	N/A	55	64.08333	48.125	61.125	46.20833	48.75	62.29167
Ranks	3	9	4	7	6	1	8	2

Table 13b
A×B Interaction Table Based on Means

	B_1	B_2	B_3
A_1	36.83333	44.25000	58.66667
A_2	69.08333	54.75000	51.33333

General Linear Model: y vs. A, B, C, A×C, D, E, F, G, H, Sample, Position

```
Factor    Type    Levels      Values
A         fixed      2       1,  2,
B         fixed      3       1,  2,  3
C         fixed      3       1,  2,  3
D         fixed      3       1,  2,  3
E         fixed      3       1,  2,  3
F         fixed      3       1,  2,  3
G         fixed      3       1,  2,  3
H         fixed      3       1,  2,  3
Sample    random     2       1,   2
Position  random     2       1,   2
```

```
Analysis of Variance for y, using Adjusted SS
for Tests

 Source          Model  DF  Reduced  DF     Seq  SS
 A                   1          1          2508.7
 B                   2          2           371.0
 A*C                 2          2          4395.0
 C                   2          2          4904.9
 D                   2          2           364.7
 E                   2          2          8665.8
 F                   2          2         10166.8
 G                   2          2          1753.0
 H                   2          0+            0.0
 Sample              1          1           715.7
 Position            1          1          1810.0
 Sample*Position     1          1           110.0
 Error              51         53          9460.4
 Total              71         71         45226.0

+ Rank deficiency due to empty cells, unbal-
anced nesting, collinearity, or an undeclared
covariate. No storage of results or further
analysis will be done.

S = 13.3603   R-Sq = 79.08% R-Sq(adj) = 71.98%
```

The above Minitab output clearly indicates that A×B = (1)×(2) is the only interaction that can be studied with a Taguchi L_{18} OA and that the request for any other interaction will lead to a non-orthogonal (or non-additive) SS, most of which will be erroneous.

Before searching for the optimum, it must be mentioned that G. Taguchi defines a measure called Sensitivity for a nominal dimension as $Sm_{dB} = 10\times\log_{10}(CF) = 10\times\log_{10}\left[n(\bar{y})^2\right] = 20\times\log_{10}\left(\bar{y}\sqrt{n}\right) = 10\times\log_{10}\left[\left(\sum_{j=1}^{n} y_j\right)^2 \Big/ n\right]$, which is clearly a logarithmic transformation of the sample mean. The main utility of Sensitivity is to use its ANOVA table to identify the Signal factors. We would recommend against the use of Taguchi's Sm_{dB} because the RT for the mean and the corresponding ANOVA (see below *Table 12b*) not only identify Signal factors but the RT for the mean has the added utility of aiding the experimenter in adjusting the mean toward the ideal target m. The variance-reduction logarithmic transformation, $Sm_{dB} = 10\times\log_{10}\left[n(\bar{y})^2\right]$, recommended by Taguchi is sometimes needed when data are not normally distributed, but the F tests in ANOVA are fairly robust and can withstand moderate departures of data from normality.

8. Identifying the Optimal Condition X_O

We have partially identified the optimal levels of some of the process parameters such as Line Speed at its third level, Insulation Material either type 1 or 3, CV Speed at the level Fast, Braid Tension at High, and Release Coating that must be set at either Low (type 1) or High (type 3). To identify the optimal levels of the Signal factors A, C, and E, we have a two-fold problem:

1. At what FLC was the process being run before PDE and thus what is the mean of the existing process condition and the resulting societal QLs?

2. What are the best levels of A, C, and E that will force the mean of the process closest to 40 lb to minimize societal QLs?

For example, if the existing process condition is FLC_1, then the process mean is close to $\overline{y}_{1.} = 157/4 = 39.25$ (see *Table 11*) and thus the mean does not need adjustment, but if the process is being run at FLC_2 for which $\overline{y}_{2.} = 18.75$, then the mean has to be adjusted upward with the aid of factors A, C, and E. In the absence of this information about the existing process condition (EPC), we will assume that the *default* value of the process mean is at the grand average of all 72 observations of the pull-force from the PDE, namely $\mu_{EPC} \cong \overline{\overline{y}} = 3779/72 = 52.4861\overline{1}$ lb. Before we proceed to adjust the mean downward from the default value of 52.48611 toward 40 lb, we first examine what the value of the mean would be without the use of the Signal factors A, C, and E. For example, if the Control factor B is set at its high level, then from *Table 13a* the adjustment to the process mean would amount to roughly (55.00000 – 52.48611) = 2.51389, that is, setting Line speed at B_3 adjusts the mean further upward to 52.48611 + 2.51389 = 55.0000. Thus, from *Table 13a*, based only on our Controls (B, D, F, G, and H), the value of the process mean (assuming additivity of individual adjustments) is in the vicinity of $\hat{\mu}_{CT} = 52.48611 + (55.00000 - 52.48611) + (48.12500 - 52.48611) + (46.20833 - 52.48611) + (48.75 - 52.48611) + 62.29167 - 52.48611) = \overline{y}_{B3} + \overline{y}_{D3} + \overline{y}_{F3} + \overline{y}_{G3} + \overline{y}_{H3} - 4 \times 52.48611 = 50.43056$.

We are now in a position, based on the above value of $\hat{\mu}_{CT} = 50.43056$ (CT denotes Control), to set the levels of the adjustment factors A, C, and E to lower the process mean. *Table 13a* again reveals that setting A and C at their low levels and E at its middle level should adjust the mean downward to the value given below.

$\hat{\mu}(A_1B_3C_1D_3E_2F_3G_3H_3) = 46.58333 + 55.00 + 45.54167 + 48.125 + 43.50 + 46.20833 + 48.750 + 62.29167 - 7 \times 52.48611 = 28.59723$ lb

Note that in the above calculation we have erroneously ignored the interaction between factors A and B in impacting the mean response. To include the interaction effect in computing $\hat{\mu}(A_1B_3C_1D_3E_2F_3G_3H_3)$, we replace the values of 46.58333 + 55.00 with their joint value of 58.66667 from *Table 13b*. Thus,

$\hat{\mu}(A_1B_3C_1D_3E_2F_3G_3H_3) = 58.66667 + 45.54167 + 48.125 + 43.50 + 46.20833 + 48.750 + 62.29167 - 6 \times 52.48611 = 38.16668$ lb

We now use *Table 12a* to evaluate the estimated value of the S/N ratio at the FLC $A_1B_3C_1D_3E_2F_3G_3H_3$.

$\hat{\eta}_{dB}(A_1B_3C_1D_3E_2F_3G_3H_3) = 139.4769/9 + 109.4328/6 + 82.3244/6 + 100.8706/6 + 96.0658/6 + 100.9152/6 + 107.2882/6 + 102.2994/6 - 7 \times 266.1700/18 = 28.5196$ dB

which compares favorably against the *default* value of $\overline{\eta}_{dB} = 266.1700/18 = 14.78722$ dB. We are now faced with the question "is the FLC $A_1B_3C_1D_3E_2F_3G_3H_3$ truly optimal?"—that is, does it produce nearly the least amount of variation and a process mean very close to the ideal target of 40 lb. To answer this question, a complete search is needed using a computer because our five Control factors provide four possibilities, while our three Signal factors, A, E, and C, provide 3×3×3 = 27 additional possibilities, adding to a total of 4×27 = 108 FLCs whose S/N ratios and means have to be computed to pinpoint the location of the estimated optimal process condition X_O. Further, the location of the optimum always depends on the PEC S/N ratio and the mean. We used the Taguchi software written by Hung-Hsiang (Kevin) Hsu to make a search of X_O that yielded the result $X_O = A_1B_3C_2D_3E_2F_3G_3H_3$ with $\hat{\eta}_O = \hat{\eta}(A_1B_3C_2D_3E_2F_3G_3H_3) = 29.37202$ dB and $\hat{\mu}_O = 40.45834$ lb. A complete search to pinpoint the true optimum would be impossible because not all possible $2 \times 3^7 = 4374$ FLCs were studied, but only a 1/243rd fraction were experimentally tested.

9. Loss Function Analysis After a PDE

To estimate the percent reduction in societal QLs, we must first compute the amount of QL at the PEC (present existing condition) given by

$$\overline{L}_{PEC} = k\left[S_n^2(PEC) + \left(\overline{y}_{PEC} - m\right)^2\right] \quad (5)$$

where the value of $k = A_c/\Delta^2$. Although k does not play any role in determining the value of percent reduction in QLs, it does play a part in the actual value of average quality loss per unit. For the sake of illustration, it is assumed that the amount of quality loss at either the LSL = 25 lb or USL = 55 lb is equal to \$11.25. This assumption yields $k = 11.25/15^2 = 0.05$. Substitution into Eq. (5) yields $\overline{L}_{PEC}$. There are two choices for the estimate of process variance of the existing condition:

1. Use of $MS_{Error} = 93.0302 = \hat{\sigma}_y^2$ from the Minitab ANOVA table and then multiplying this estimate by $(n - 1)/n = 3/4$, which yields S_n^2(PEC) = 69.7726.
2. Because we are assuming that the existing process mean is 52.48611 and the existing S/N ratio is 14.78722, then by the definition of S/N ratio we must have:

$$14.78722 = 10 \times \log_{10}\left[\frac{52.48611^2}{S^2(\text{PEC})} - \frac{1}{4}\right] \tag{6}$$

Dividing Eq. (6) by 10 and exponentiating both sides using base 10 yields S^2(PEC) = 90.7352, which gives the final result of S_n^2(PEC) = (3/4) × 90.7352 = 68.0514. Because the value of σ_O^2 has to be estimated using the S/N equation similar to (6), we settle on S_n^2(PEC) = 68.0514. Hence, $\bar{L}_{PEC}$ = 0.05[68.0514 + (52.48611 – 40)^2] = \$11.1977 ≅ \$11.20. Note that $\bar{L}_{PEC}$ = \$11.20 is very close to the amount of QL at a spec limit of A_c = \$11.25 and thus is perhaps a pessimistic overestimate.

The amount of QL at the optimal condition, X_O, is given by

$$\bar{L}_O = k\left[S_n^2(O) + (\bar{y}_O - m)^2\right] \tag{7}$$

To compute S_n^2(O), Eq. (6) is used, replacing the present existing values with those at the optimum, that is, $29.3720 = 10 \times \log_{10}\left[\frac{40.45834^2}{S^2(\text{Optimal})} - \frac{1}{4}\right]$. Solving for $S_O^2 = S^2$(Optimal) from this last equality yields S_O^2 = 1.89099 and S_n^2(O) = $(n - 1)S_O^2/n$ = 1.41824. The use of Eq. (7) results in $\bar{L}_O$ = 0.05[1.41824 + (40.45834 – 40)^2] = \$0.0814. Thus, the expected percent reduction in QLs is given by [(11.1977 – 0.08142)/11.1977] × 100% = 99.2729%. A word of caution is quite necessary at this juncture because a percent reduction in QLs of 99.2729% is quite unrealistic and most probably impossible to achieve by one set of experiments. The reader should bear in mind we assumed that the location of process mean before optimization was at 52.48611 lb, which is quite off target and pessimistic relative to the nominal desired value of m = 40 lb, and further we made the assumption (perhaps unrealistic and optimistic) that the overall change in the mean $\Delta\mu$ = 52.48611 – 40.45834 = 12.0278 lb is the sum of individual improvements from each factor (that is, we are assuming an additive model), and so was the improvement in the S/N ratio $\Delta\eta$ = 29.37202 – 14.78722 = 14.58480 dB. We will use the confidence intervals developed in the following section to obtain a more realistic and plausible percent reduction in societal QLs.

10. Confidence Intervals for $\mu(X_O)$ and $\sigma^2(X_O)$

Because the expression for $\hat{\mu}(X_O) = \hat{\mu}_O$ is a linear combination of means of normal random variables, the sampling distribution of $\hat{\mu}_O = \hat{\mu}(X_O) = \hat{\mu}(A_1B_3C_2D_3D_2F_3G_3H_3)$ is normal with expectation $E(\hat{\mu}_O) = \mu(X_O)$ and variance $V(\hat{\mu}_O)$ yet to be determined. Thus, a 97.5% confidence interval (CI) for $\mu(X_O)$ is given by $\hat{\mu}_O \pm t_{0.0125;\,\nu} \times \text{se}(\hat{\mu}_O)$, where ν is the degrees of freedom of se($\hat{\mu}_O$) given by se($\hat{\mu}_O$) = $\sqrt{V(\hat{\mu}_O)}$. We are obtaining 97.5% CIs for both process mean and variance to be on the safe side so that the joint Bonferroni confidence level will be $(0.975)^2$ ≅ 0.95. It is well known from statistical theory that for a Gaussian (or normal) process, the sample mean and standard deviation are stochastically independent so that the Bonferroni CIs are not needed because the Bonferroni procedure accounts for the correlation structure between two responses. Thus, we are developing Bonferroni intervals just in case the independence assumption between $\bar{y}$ and S is not quite tenable.

To compute the $V(\hat{\mu}_O)$, we first apply the variance operator to the expression for $\hat{\mu}_O$.

$$V(\hat{\mu}_O) = V\left[\bar{y}_{A\times B_{13}} + \bar{y}_{C_2} + \bar{y}_{D_3} + \bar{y}_{E_2} + \bar{y}_{F_3} + \bar{y}_{G_3} + \bar{y}_{H_3} - 6\times\bar{\bar{y}}\right] \tag{8}$$

Second, it is noted that the means that comprise $\hat{\mu}_O$ are positively correlated and thus the covariance of any pairs of means in Eq. (8) is larger than zero and must be taken into account in applying the variance operator in Eq. (8). It is well known from statistical theory that if $Y = \sum_{i=1}^{n} c_i X_i$ is a linear combination of random variables, where c_i's are known constants, then

$$V(Y) = \sum_{i=1}^{n} c_i^2\sigma_i^2 + 2\sum_{j>i}\sum_{i=1}^{n-1} c_i c_j \sigma_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \sigma_{ij} \tag{9}$$

where σ_{ij} = the covariance between X_i and X_j = $E[(X_i - \mu_i) \times (X_j - \mu_j)]$, $i \neq j$, $\sigma_{ii} = \sigma_i^2 = V(X_i)$, and $\mu_i = E(X_i)$. Further, if X_i's that comprise the sum Y are all normally distributed, then the sampling distribution of Y itself is Gaussian regardless of the correlation structure among all distinct pairs of X_i's. For a proof of Eq. (9), the interested reader is referred to *www.eng.auburn.edu/~maghsood/homepage.html*, STAT 3600, Chapter 5, pp. 88-89.

Comparing Eq. (8) with (9), it is noted that $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 1$ and $c_8 = -6$. We now compute the variances and covariances of the eight means in Eq. (8) term by term.

$$V\left(\bar{y}_{A\times B_{13}}\right) = V\left(\sum_{j=1}^{12}\frac{1}{12}y_{A_1B_3,j}\right) = \frac{\sigma_y^2}{12};$$

$$V\left(\bar{y}_{C_2}\right) = V\left(\frac{1}{24}\sum_{j=1}^{24}y_{C_2,j}\right) = \frac{\sigma_y^2}{24};$$

similarly, $V\left(\bar{y}_{D_3}\right) = V\left(\bar{y}_{E_2}\right) = V\left(\bar{y}_{F_3}\right) = V\left(\bar{y}_{G_3}\right) = V\left(\bar{y}_{H_3}\right) = \frac{\sigma_y^2}{24}$, and $V(-6\times\bar{\bar{y}}) = 36\left(\frac{\sigma_y^2}{72}\right)$. The covariance between $\bar{y}_{A\times B_{13}}$ and $\bar{y}_{C_2}$ is $Cov\left(\bar{y}_{A\times B_{13}}, \bar{y}_{C_2}\right) = Cov\left(\sum_{j=1}^{12}\frac{1}{12}y_{A_1B_3,j}, \frac{1}{24}\sum_{j=1}^{24}y_{C_2,j}\right)$. To carry out this last covariance operation, it is noted that out of the 12 y_{ij}'s that comprise $\bar{y}_{A\times B_{13}}$ exactly four of them are the same observations as the 24 observations that comprise $\bar{y}_{C_2}$, and thus $Cov\left(\bar{y}_{A\times B_{13}}, \bar{y}_{C_2}\right) = (1/12) \times (1/24) \times 4\sigma_y^2 = \frac{\sigma_y^2}{72}$; similarly, the covariance of $\bar{y}_{A\times B_{13}}$ with $\bar{y}_{D_3}$, $\bar{y}_{E_2}$, $\bar{y}_{F_3}$, $\bar{y}_{G_3}$, and $\bar{y}_{H_3}$ is also equal to $\frac{\sigma_y^2}{72}$.

The covariance between $\bar{y}_{A\times B_{13}}$ and $-6\times\bar{\bar{y}}$ is given by $Cov\left(\bar{y}_{A\times B_{13}}, -6\times\bar{\bar{y}}\right) = Cov\left(\sum_{j=1}^{12}\frac{1}{12}y_{A_1B_3,j}, \frac{-6}{72}\sum_{i=1}^{18}\sum_{j=1}^{4}y_{ij}\right) = (1/12)(-6/72)(12)\,\sigma_y^2 = (-6/72)\,\sigma_y^2$. Similarly, $Cov\left(\bar{y}_{C_2}, -6\times\bar{\bar{y}}\right) = \frac{-6\sigma_y^2}{72}$. Combining Eqs. (8) and (9), we obtain the result $V(\hat{\mu}_O) = \frac{\sigma_y^2}{12} + 6 \times \frac{\sigma_y^2}{24} + 36 \times \frac{\sigma_y^2}{72} + 2\left[{}_7C_2 \times \frac{\sigma_y^2}{72} + 7 \times \frac{-6\sigma_y^2}{72}\right] = \frac{10\sigma_y^2}{12} + 2\left[\frac{7\times 6}{2}\times\frac{\sigma_y^2}{72} + 7\times\frac{-6\sigma_y^2}{72}\right] = \frac{10\sigma_y^2}{12} + 2\left(\frac{-21\sigma_y^2}{72}\right) = \frac{\sigma_y^2}{4}$.

This last result is intuitively appealing because X_0 is an FLC and there were n = 4 observations at each FLC, and it is well known that $V\left(\bar{y}\right) = \frac{V(y)}{n} = \frac{\sigma_y^2}{4}$. Because σ_y^2 is unknown, it must be estimated from the gathered data. We have two choices, namely the MS_{Error} = 93.0302 from the Minitab ANOVA table with 51 df, or the estimate of the optimal variance S_O^2 = 1.89099 with 3 df. Because $\hat{\mu}_O$ is the estimated mean at the optimum, the estimated variance at the optimum is the more logical choice, that is, $\hat{\sigma}_y^2 = S_O^2$ = 1.89099. Therefore, the standard error is given by $se\left(\hat{\mu}_O\right) = \sqrt{\frac{\sigma_y^2}{4}} \cong \frac{1}{2}\sqrt{S_O^2} = \frac{\sqrt{1.89099}}{2} = 0.6876$, and the 97.5% confidence interval for μ_O is given by 40.45834 ± 4.1765 × 0.6876 = (37.5867, 43.3300), where 4.1765 = $t_{0.0125;\,3}$ = the inverse of W.S. Gosset's t-distribution with 3 df at a cumulative of 0.9875.

Dr. Taguchi highly recommends that once the optimal condition X_O is identified, the experimenter should conduct several confirmation runs at the optimal FLC because of the fact that only 18 out of the possible 4374 FLCs were experimentally tested. Before conducting confirming experiments, one should note if the optimal condition $X_O = A_1B_3C_2D_3E_2F_3G_3H_3$ is one of the FLCs that was studied using the L_{18} OA. A comparison of X_O with the 18 FLCs of the L_{18} design matrix reveals that $A_1B_3C_2D_3E_2F_3G_3H_3$ is not one of the FLCs that was included in the L_{18}. Note that the 18 runs that were studied are just a 1/243rd fraction of all 2×3^7 = 4374 possible distinct FLCs. Thus, in this case, Dr. Taguchi recommends that we set the process at $X_O = A_1B_3C_2D_3E_2F_3G_3H_3$ and make (at least) two cables, measure the pull-force at two positions, and compute the mean of the four measurements denoted by $\bar{y}_{CN}$, where CN represents confirmation. We may use either the inexact decision rule that if $\bar{y}_{CN}$ falls inside the 97.5% CI (37.5867, 43.3300), then there should be no concern about X_O from the standpoint of the mean; or we may compare $\bar{y}_{CN}$ against the decision interval 40.45834 ± $t_{0.025;\,\nu}$ × se($\bar{y}_{CN}$). The experimenter should also be cognizant of the fact that the confidence band against which the future confirmation mean is to be compared should actually be wider than (37.5867, 43.3300) because the confirmation runs will also have experimental error that is not included in the computation of the CI (37.5867, 43.3300), which includes only the model error.

To arrive at an upper 97.5% CI for $\sigma^2(X_O) = \sigma_O^2$, we make use of the fact that the sampling distribution of $(n-1)S_O^2/\sigma_O^2$ is chi-squared with 3 df, assuming that n = 4. Thus, the upper 97.5% confidence limit for σ_O^2 is given by $\sigma_O^2(U) = \frac{(n-1)S_O^2}{\chi^2_{0.975;3}} = \frac{3\times1.89099}{0.2158} = 26.2887$, that is, $0 < \sigma_O^2 \leq 26.2887$ at the 97.5% confidence level. Again, if the variance found through confirmation experiments lies within the CI, $0 < \sigma_O^2 \leq 26.2887$, then the optimal condition is practically (not exactly from a statistical viewpoint) confirmed from the standpoint of variability because this CI includes only the model error.

11. Further Loss Function Analysis

In section 9, we computed an unrealistic percent reduction in QLs as 99.2729% that was, to say the least, very optimistic. We now use the results of the CIs from section 10 to obtain a more realistic value for the percent reduction in QLs. To accomplish this objective, we use the worst-case scenarios for the mean within the CI (37.5867, 43.3300), namely $\mu_O = 43.3300$, and also for the variance we will use the worst-case $\sigma_O^2 = 26.2887$. These yield an expected QL at the optimal condition as $L_O(U) = 0.05\,[26.2887 + (43.3300 - 40)^2] = \$1.8689$. This leads to a percent reduction of $[(11.1977 - 1.8689)/11.1977] \times 100\% = 83.31\%$. Because the two confidence bands each were at the 97.5% level, we would have a Bonferroni confidence level of $(0.975)^2 \cong 95\%$ in the percent QL reduction interval (83.31, 99.273%). Again, even the expected 83.31% reduction in QLs is overly optimistic and is due to the fact that we are using the default value of $\bar{\bar{y}} = 52.48611$ lb for the existing mean and the default value of $S_n^2(\text{PEC}) = 68.0514$. This brings us to the point that is important experimentally, and that is, if at all possible, one of the experimental runs should generally be the FLC at which the process is presently being run so that the experimenter would have a good idea about the mean and S/N ratio of the existing process before optimization. Most design matrices afford sufficient flexibility to embed the existing condition in the design matrix, but this is sometimes impossible. Our 11 alternatives to Taguchi's L_{18} in section 6 should help in this regard. For example, if run number 7 (see *Table 11*) in the L_{18} OA is the PEC of the process, then from *Table 11* we obtain $\bar{L}_{PEC} = 0.05\,[62.9167 + (47.75 - 40)^2] = \$6.1490$, which leads to a more realistic percent reduction in QLs of 69.61%. From the above discussions, if the optimal condition X_O is verified through confirmation experiments for the design matrix of *Table 10*, then we should expect more than a 50% reduction in QLs from the above optimization procedure and no further analysis is needed. Otherwise, we will also need to reexamine our analyses according to the following section, and we should ponder the possibility that there may be active interactions in the process that were left out in the design phase of the experiment.

12. Further Analysis of Data from a PDE

One warranted criticism of Taguchi methods by most Western statisticians is the use of the S/N ratio for analyzing data from a DOE. Because the sample mean and variance, when sampling a normal process, are stochastically independent, the recommendation from the statistical community is to analyze the mean and variance separately. This is the reason that in *Table 11* we provided a column for the $\ln(S_i)$. *Table 14a* provides the RT for the ln(S). Unfortunately, the RT for ln(S) in *Table 14a* does not yield rankings that are totally consistent with those of the S/N ratio in *Table 12a* because now the factor H has very minimal (if any) impact on process variability while H was the number-one factor in impacting the S/N ratio as shown in *Table 12a*. This can sometimes happen because the S/N ratio measures the simultaneous impact of mean and variance. However, factors B, G, and D are still influential in impacting process variability, and factors A, E, and F may also affect process variation, while from *Table 12a* factors A and E did not influence the S/N ratio. From *Tables 14a* and *13a*, factors C and H now seem to be the only Signal factors. *Tables 14a* and *14b* show that the optimal condition based only on the Control factors is $A_1B_3D_3E_2F_3G_3$, $A_1B_1D_3E_2F_3G_3$, or $A_2B_3D_3E_2F_3G_3$, and the tables also indicate that factors B and A are relatively more influential than (A×B) on the variability of pull-force. Testing the mean at the presumed optimum $A_1B_3D_3E_2F_3G_3$ yields (see *Table 13a*) $\hat{\mu}(A_1B_3D_3E_2F_3G_3) = 46.58333 + 55.00 + 48.125 + 43.5 + 46.20833 + 48.75 - 5 \times 52.48611 = 25.7361$. To increase this expected mean toward 40 lb, we set the Signal factors at C_3H_1. This yields $\hat{\mu}(A_1B_3C_3D_3E_2F_3G_3H_1) = 40.7639$, which

Table 14a
Response Table for ln(S)

Effects	A	B	C	D	E	F	G	H
L_1	17.4679	12.96427	12.60916	13.38007	13.70545	12.90299	14.46037	12.15664
L_2	20.9677	14.89184	12.48718	14.41283	10.64393	14.54310	13.50493	13.38358
L_3	N/A	10.57953	13.33930	10.64274	14.08626	10.98955	10.47035	12.89542
SS	0.6805	1.5555	0.0708	1.2652	1.1871	1.0544	1.4468	0.1272
MS	0.6805	0.7777	0.0354	0.6326	0.5935	0.5272	0.7234	0.0636
Ranks	3	1	9	4	5	6	2	8

Table 14b
Response Table for A×B for ln(S)

	$\mathbf{B_1}$	$\mathbf{B_2}$	$\mathbf{B_3}$
$\mathbf{A_1}$	4.99257	6.885692	5.589688
$\mathbf{A_2}$	7.971703	8.006151	4.989838
SS(A×B) = 1.067958		**Rank(A×B) =** 7	

again assumes that the process mean is at the default value of 52.48611. Because the average of ln(S) in *Table 14a* is $\overline{\ln(S)} = 2.1353$, the estimated ln(S) at $X_O = A_1B_3C_3D_3E_2F_3G_3H_1$ is given by $\widehat{\ln(S)}$ = 17.4679/9 + (10.57953 + 13.3393 + 10.64274 + 10.64393 + 10.98955 + 10.47035 +12.15664)/6 – 7 * 2.1353 = 0.1307. This yields $S_O = e^{0.1307}$ = 1.1396 and $\bar{L}_O = 0.05[0.9740 + (40.7639 - 40)^2]$ = \$0.0779. This compares favorably against $\bar{L}_O$ = \$0.0814 obtained in section 9.

It is paramount to emphasize that the optimal levels of the Signal factors always depend on the value of the existing process mean. In all the above analyses, we assumed the default value of μ_{PEC} = 52.48611; had we assumed any other value of μ_{PEC}, the optimal levels of the signal factors would alter accordingly.

13. Taguchi's Tolerance Design

A tolerance design is performed always after a PDE only if sufficient reduction in process variability is not attained after parameter design. A good example for tolerance design is provided by Barker and Clausing (1984). To describe the objective of a tolerance design, we quote from the excellent reference *An Introduction to Off-Line Quality Control Methods* by Kackar and Phadke (1984). These two authors state that, "Whenever the reduction in the output variation achieved by parameter design is not enough, the last alternative is tolerance design. Narrower tolerance ranges are specified for those production process factors whose variations impart large influence on the output variation. To meet the tighter process specifications, better grade material and better equipment are needed. Thus tolerance design increases the production costs."

Because the article by Barker and Clausing may not be easily accessible, we present their work almost in its entirety to illustrate Taguchi's tolerance design. In the tolerance design experiment by Barker and Clausing, the objective is to improve a friction welding process, where the response y represents the tensile strength of weld. Presently the company is experiencing heavy losses due to breakage at weld strength below 160 ksi (in consultation with Professor J T. Black we have assumed ksi for the units as the authors do not provide the units). Although the QCH is of the LTB type, it seems that the authors use a modification of the NTB type loss function to perform their loss function analyses. Further, the two authors report that the company is presently losing \$350/engine due to field failure and the resulting warranty service claims. Because y is an LTB type QCH, it seems that from a quality standpoint 160 ksi may be the value of the LSL. However, Figure 5 on Barker and Clausing's p. 40 shows that the value of the loss function is nearly zero at 160 ksi (perhaps this is why the authors did not assume that LSL = 160 ksi because the value of A_c cannot equal to zero at the LSL for an LTB type QCH); further, their Figure 5 on p. 40 shows that L(y) = \$500.00 at y = 100 ksi. It seems (but we are not certain) that the modified loss function the authors are using is given by

Table II of Barker and Clausing—Process Factors and Their Low-Cost (or Wide-Range) Tolerances

Process Parameters	Range of Interest	Low-Cost (or Wide-Range) Tolerances
Speed	1000 – 1400 rpm	±10%
Heating pressure = HTPRS	4000 – 4800 psi	±15%
Upset pressure = UPPRS	8500 – 9500 psi	±15%
Length	–30 to +30 thous.	±10%
Heating time = HTTIME	2.8 – 3.6 sec.	±20%
Upset time = UPTIME	3.2 – 4.0 sec.	±20%

$$L(y) = \begin{cases} k(y-160)^2, \text{ if } y \le 160 \text{ ksi} \\ 0, \text{ if } y > 160 \text{ ksi} \end{cases} \tag{10}$$

To compute their loss function constant, k, Barker and Clausing inserted the point (y =100 ksi, L = \$500.00) into their loss function as 500 = k(100 – 160)^2 and obtained k = 500/3600 = 0.1388889. Thus, L(y) = 0.1388889(y – 160)^2 if y ≤ 160 and L(y) = 0 if y > 160 ksi. It should be emphasized that the two authors do not specify L(y) = 0 when y > 160, but we deduced this information from their loss function graph of Figure 5 on their p. 40.

Table II of Barker and Clausing on their p. 41, reproduced above, lists the six process parameters that may impact the tensile strength of the weld along with their low-cost (or wide-range) tolerances. The PDE started with the Taguchi L_{27} as the inner OA with the factor Speed at three levels, 1000, 1200, and 1400 rpm, embedded on column (1) of L_{27}; Heating Pressure (HTPRS) at three levels, 4000, 4400, and 4800 psi, on column (2) of L_{27}; Upset Pressure (UPPRS) at 8500, 9000, and 9500 on column (5); Length at –30, 0, 30 on column (9); Heating Time (HTTIME) at levels 2.8, 3.2, and 3.6 on (10); and Upset Time (UPTIME) also at three levels, 3.2, 3.6, and 4.0 seconds, embedded on column (12) of L_{27}. The reader should refer to our *Table 9* to follow the logic behind the above column assignments that forbids aliasing a main factor with a two-way interaction; our *Table 9* shows that column (13) of L_{27} could also have been used for one of the factor assignments. The inner OA for the PDE is reproduced below and shown as Table III of Barker and Clausing. Because this PDE is the first step in a tolerance design, there is an outer array for each FLC of the L_{27} inner array consisting of the wide-range (or low-cost) tolerances of process factors. The first FLC of the inner array is (1000, 4000, 8500, –30, 2.8, 3.2) for which columns (2) through (7) of Taguchi's L_{18} OA is the outer array as shown in Table IV of Barker and Clausing. (Because Table V of Barker and Clausing provides the responses for their Table IV, the two tables are combined and reproduced as Tables IV and V below.) Because the low-cost tolerances on Speed are ±10% and at the first FLC of Table III the factor "Speed" is 1000 rpm, then ±0.10 × 1000 = ±100, so the three levels of Speed in the L_{18} outer array are 1000 – 100 = 900, 1000, and 1000 + 100 = 1100 rpm. Similarly, the low-cost tolerances for Heating Pressure are ±15%, so for the first FLC of the inner array the third column of the outer array consists of three levels, 4000 – 0.15 × 4000 = 3400, 4000, and 4000 + 0.15 × 4000 = 4600. Tables IV and V of Barker and Clausing (on their p. 45) exhibit the L_{18} outer array for the first FLC of the Taguchi L_{27} inner OA and the corresponding values of tensile strength y. Because each FLC of the inner array has a Taguchi L_{18} as its outer array, there are a total of 27 × 18 = 486 runs, requiring one week (as reported by the two authors) of experimentation.

Each L_{18} outer array yields one mean ($\bar{y}_i$), one standard deviation (S_i), and one S/N ratio (η_i) for i = 1, 2, 3, …, 27. We used Tables IV and V of Barker and Clausing to compute $\bar{y}_1$ = 85.2391 ksi, S_1 = 38.0393 ksi, and type B S/N ratio $\eta_1 = -10 \times \log_{10}\left[\frac{1}{18}\sum_{j=1}^{18}\left(1/y_{1j}\right)^2\right]$ = 34.2211 db, where by type B S/N ratio Barker and Clausing mean bigger is better (the same as larger the better). Our values of S_1 = 38.0393 and η_1 = 34.2211 exactly match those of Barker and Clausing to four decimals, which they list near the bottom of their p. 45, but we could not obtain their targeted strength value of 104.345 ksi. From the information provided in the article, we could

Table III of Barker and Clausing—Taguchi's L_{27} Used as Inner OA

Speed (1)	HTPRS (2)	UPPRS (5)	Length (9)	HTTIME (10)	UPTIME (12)
1000	4000	8500	–30	2.8	3.2
1000	4000	9000	0	3.2	3.6
1000	4000	9500	30	3.6	4.0
1000	4400	8500	0	3.2	4.0
1000	4400	9000	30	3.6	3.2
1000	4400	9500	–30	2.8	3.6
1000	4800	8500	30	3.6	3.6
1000	4800	9000	–30	2.8	4.0
1000	4800	9500	0	3.2	3.2
1200	4000	8500	0	3.6	3.6
1200	4000	9000	30	2.8	4.0
1200	4000	9500	–30	3.2	3.2
1200	4400	8500	30	2.8	3.2
1200	4400	9000	–30	3.2	3.6
1200	4400	9500	0	3.6	4.0
1200	4800	8500	–30	3.2	4.0
1200	4800	9000	0	3.6	3.2
1200	4800	9500	30	2.8	3.6
1400	4000	8500	30	3.2	4.0
1400	4000	9000	–30	3.6	3.2
1400	4000	9500	0	2.8	3.6
1400	4400	8500	–30	3.6	3.6
1400	4400	9000	0	2.8	4.0
1400	4400	9500	30	3.2	3.2
1400	4800	8500	0	2.8	3.2
1400	4800	9000	30	3.2	3.6
1400	4800	9500	–30	3.6	4.0

not ascertain how the authors determined that the nominal (or targeted) value of tensile strength for the FLC #1 was 104.345 while the realized experimental value of the mean computed from their Tables IV and V was $\overline{y}_1 = 85.2391$ ksi. Our concern here is the fact that the targeted signal value of 104.345, although not achieved experimentally, was also used along with 26 other presumably nominal strength values in their Table VI on their p. 46 to perform an analysis of variance to identify the Signal factors. Because the raw data for the remaining 26 outer arrays of FLCs 2 through 27 of L_{27} were not provided, we are not presently sure how to interpret the strength values provided by Barker and Clausing in their Table VI, which is also reproduced below. Further, Barker and Clausing also provide the type N (nominal the best) S/N ratio near the bottom of their p. 45 as $\eta_1 = 8.76478$, which they obtained from $\eta_1 = 10 \times \log_{10}\left[(\overline{y}_1/S_1)^2\right]$ by using the nominal value of 104.345 for their $\overline{y}_1$, while as mentioned above, the sample mean for the L_{18} outer array of FLC 1 is $\overline{y}_1 = 85.2391$. If we use the realized value of $\overline{y}_1 = 85.2391$ in our NTB type S/N ratio, we would obtain $\eta_1 = 10 \times \log_{10}\left[(\overline{y}/S)^2 - 1/18\right] = 6.95982$.

Tables IV and V of Barker and Clausing—Using Taguchi's L_{18} as Outer Array for First FLC of Taguchi's L_{27} Inner Orthogonal Array

Speed (2)	HTPRS (3)	UPPRS (4)	Length (5)	HTTIME (6)	UPTIME (7)	Tensile Strength (y)
900	3400	7225	–27	2.24	2.56	26.7038
900	4000	8500	–30	2.8	3.2	90.9161
900	4600	9775	–33	3.36	3.84	133.1830
1000	3400	7225	–30	2.8	3.84	84.7627
1000	4000	8500	–33	3.36	2.56	40.8376
1000	4600	9775	–27	2.24	3.2	120.489
1100	3400	8500	–27	3.36	3.2	94.7072
1100	4000	9775	–30	2.24	3.84	146.1440
1100	4600	7225	–33	2.8	2.56	29.4095
900	3400	9775	–33	2.8	3.2	111.1420
900	4000	7225	–27	3.36	3.84	72.2078
900	4600	8500	–30	2.24	2.56	22.4529
1000	3400	8500	–33	2.24	3.84	124.7730
1000	4000	9775	–27	2.8	2.56	72.4566
1000	4600	7225	–30	3.36	3.2	93.8594
1100	3400	9775	–30	3.36	2.56	58.6379
1100	4000	7225	–33	2.24	3.2	88.8731
1100	4600	8500	–27	2.8	3.84	122.7480

It should be noted that the authors did use the S/N ratios obtained from the experimental results to obtain their MSs (mean squares) in their Table VII for S/N, but as far as we could ascertain they used the targeted strength values to obtain their MSs for the strength values. We could not resolve this troublesome issue from all the data that were provided in their article. To explain the values in Table VII of Barker and Clausing, we are providing RTs for both S/N ratio and Strength in our *Tables 15* and *16*, respectively. In *Table 15*, one asterisk on an F statistic implies significance at the 5% level and two asterisks on an F implies statistical significance at the 1% level. Using the S/N ratios from Table VI of Barker and Clausing, the $SS(Total_\eta) = USS_\eta - CF_\eta = 34935.110 - 33376.685 = 1558.4252 \rightarrow SS(Residual_\eta) = 345.5563 \rightarrow MS(RES_\eta) = 345.5563/14 = 24.683$, which differs a bit from Barker and Clausing's $MS_{RES} = 26.00$ in their Table VII (p. 46). The discrepancy could be due to the fact that the authors provide only one-decimal accuracy in their Table VI for both S/N and Strength values. In our *Table 15*, we obtained the linear and quadratic contrasts for the six factors by applying the orthogonal polynomial coefficients $P_L = [-1 \quad 0 \quad 1]'$ and $P_Q = [1 \quad -2 \quad 1]'$. Note that these two 3-D vectors are contrasts (that is, the components of P_L and P_Q add to zero) and they are perpendicular (or orthogonal) because their vector dot product is zero. In our *Table 15*, the linear contrast for speed was computed from $Speed_L = -1 \times 298.4 + 0 \times 334.2 + 1 \times 306.7 = 8.3$, and the quadratic contrast was computed from $Speed_Q = 1 \times 298.4 - 2 \times 334.2 + 1 \times 306.7 = -83.3$ (concave downward). Because the factor "Speed" is at three levels with 2 df, each of these two contrasts carry exactly 1 df, and thus the SS and MS of contrasts are the same, while the MS(Speed) = SS(Speed)/2. The SS of these two contrasts were computed from $SS(Speed_L) = \frac{8.3^2}{2\times 9}$ and $SS(Speed_Q) = \frac{(-83.3)^2}{6\times 9}$. Further, few of the MSs that Barker and Clausing provide in their Table VII are actually the SS as shown in *Table 15*. To check our answers, we ran Minitab's GLM (general linear model) on the S/N ratio of the

Table VI of Barker and Clausing—Using Taguchi's L_{27} as Inner Array

Speed	HTPRS	UPPRS	Length	HTTIM	UPTIME	S/N	STRENGTH	STD. DEV.
1000	4000	8500	–30	2.8	3.2	34.2	104.3	38.04
1000	4000	9000	0	3.2	3.6	39.9	135.1	27.89
1000	4000	9500	30	3.6	4	12.3	128.6	45.16
1000	4400	8500	0	3.2	4	25.6	123.8	42.41
1000	4400	9000	30	3.6	3.2	37.5	134.6	45.59
1000	4400	9500	–30	2.8	3.6	40	134.7	27.06
1000	4800	8500	30	3.6	3.6	40.6	150.6	38.88
1000	4800	9000	–30	2.8	4	28.6	116.2	43.24
1000	4800	9500	0	3.2	3.2	39.7	151.2	45.03
1200	4000	8500	0	3.6	3.6	39.6	134.2	31.73
1200	4000	9000	30	2.8	4	35.4	134.1	41.28
1200	4000	9500	–30	3.2	3.2	39.1	132	40.67
1200	4400	8500	30	2.8	3.2	37.9	125.8	38.47
1200	4400	9000	–30	3.2	3.6	40.5	140.9	28.67
1200	4400	9500	0	3.6	4	37.8	158.5	46.85
1200	4800	8500	–30	3.2	4	30.9	129.6	44.86
1200	4800	9000	0	3.6	3.2	41.3	164.5	50.00
1200	4800	9500	30	2.8	3.6	41.7	156.1	29.91
1400	4000	8500	30	3.2	4	12.5	111.7	43.96
1400	4000	9000	–30	3.6	3.2	31.1	109.6	40.74
1400	4000	9500	0	2.8	3.6	40.8	146.7	30.66
1400	4400	8500	–30	3.6	3.6	38.3	125.6	37.59
1400	4400	9000	0	2.8	4	34.6	128.3	44.40
1400	4400	9500	30	3.2	3.2	39.3	139.1	44.84
1400	4800	8500	0	2.8	3.2	35.2	119.9	44.07
1400	4800	9000	30	3.2	3.6	40.8	148	36.84
1400	4800	9500	–30	3.6	4	34.1	150.1	53.13

Table VII of Barker and Clausing—ANOVA for S/N and Strength

S/N: Source	MS	F_0	Strength: Source	MS	F_0
Speed_q	128.7	4.9	Speed	693	15.8
HTPRS_L	126.8	4.8	HTPRS	1248	28.5
UPPRS	78	3.0	UPPRS	1633	37.3
Length	75	2.9	Length_L	407	9.3
HTTIME	25	1.0	Length_q	434	9.9
UPTIME_L	388	14.9	HTTIME	452	10.3
UPTIME_q	350	13.5	UPTIME	611	14.0
Residual	26		Residual	44	

Table 15
Response Table for S/N Ratio and Corresponding F Statistics

Effects	Speed	HTPRS	UPPRS	Length	HTTIME	UPTIME
L_1	298.4	284.9	294.8	316.8	328.4	335.3
L_2	344.2	331.5	329.7	334.5	308.3	362.2
L_3	306.7	332.9	324.8	298.0	312.6	251.8
SS =	132.325	165.834	79.3341	74.0363	24.8941	736.445
MS =	66.163	82.917	39.667	37.018	12.447	368.223
F_0 =	2.681	3.359	1.607	1.500	0.504	14.918**
Linear Contrast	8.3	48	30	–18.8	–15.8	–83.5
SS_L =	3.82722	128.00	50.00	19.6356	13.8689	387.347
F_0 =	0.15506	5.186*	2.02572	0.7955	0.56189	15.693**
Quadratic Contrast	–83.3	–45.2	–39.8	–54.2	24.4	–137.3
SS_Q =	128.498	37.8341	29.3341	54.4007	11.0252	349.098
F_0 =	5.206*	1.53282	1.18845	2.204	0.44668	14.144**

$F_{0.05;\,2,\,14} = 3.74$; $F_{0.05;\,1,\,14} = 4.60$; $F_{0.01;\,2,\,14} = 6.51$; $F_{0.01;\,1,\,14} = 8.86$

Table 16
Response Table for Strength and the Corresponding F Statistics

Effects	Speed	HTPRS	UPPRS	Length	HTTIME	UPTIME
L_1	1179.1	1136.3	1125.5	1143.0	1166.1	1181.0
L_2	1275.7	1211.3	1211.3	1262.2	1211.4	1271.9
L_3	1179.0	1286.2	1297.0	1228.6	1256.3	1180.9
SS =	691.943	1248.334	1634.014	839.4430	452.005	612.7341
MS =	345.971	624.167	817.007	419.721	226.003	306.367
F_0 =	7.914**	14.277**	18.688**	9.601**	5.169*	7.008**
Linear Contrast	–0.1	149.9	171.5	85.6	90.2	–0.1
SS_L =	0.0005556	1248.334	1634.014	407.0756	452.002	0.00056
F_0 =	1.271E-05	28.5538**	37.3757**	9.3113**	10.3389**	1.3E-05
Quadratic Contrast	–193.3	–0.1	–0.1	–152.8	–0.4	–181.9
SS_Q =	691.94241	0.000185	0.000185	432.3674	0.00296	612.7335
F_0 =	15.82718**	4.24E-06	4.24E-06	9.890**	6.8E-05	14.0154**

$F_{0.05;\,2,\,14} = 3.74$; $F_{0.05;\,1,\,14} = 4.60$; $F_{0.01;\,2,\,14} = 6.51$; $F_{0.01;\,1,\,14} = 8.86$

authors' Table VI, whose output is given below, and Minitab's GLM verified our answers of *Table 15*. The entries at the bottom of *Tables 15* and *16* are inverse functions of Fisher's F distribution at the cdf values of 0.95 and 0.99, respectively.

ANOVA: SN vs. Speed, HTPRS, UPPRS, Length, HTTIME, UPTIME

General Linear Model: SN versus Speed, HTPRS, ...

Factor	Type	Levels	Values
Speed	fixed	3	1000, 1200, 1400
HTPRS	fixed	3	4000, 4400, 4800
UPPRS	fixed	3	8500, 9000, 9500
Length	fixed	3	-30, 0, 30
HTTIME	fixed	3	2.8, 3.2, 3.6
UPTIME	fixed	3	3.2, 3.6, 4.0

Analysis of Variance for SN, Using Adjusted SS for Tests

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Speed	2	132.33	132.33	66.16	2.68	0.103
HTPRS	2	165.83	165.83	82.92	3.36	0.064
UPPRS	2	79.33	79.33	39.67	1.61	0.235
Length	2	74.04	74.04	37.02	1.50	0.257
HTTIME	2	24.89	24.89	12.45	0.50	0.614
UPTIME	2	736.45	736.45	368.22	14.92	0.000
Error	14	345.56	345.56	24.68		
Total	26	1558.43				

S = 4.96816   R-Sq = 77.83%   R-Sq(adj) = 58.82%

The above Minitab output shows that the most influential factor that impacts the S/N ratio of Strength is Upset Time, while Heating Pressure is significant at the 0.064 level and Speed's effect is significant at the 10.3% level. Thus, the only factors that impact process variability in the order of their influence are Upset Time, Heating Pressure, and Speed. From our RT 15, Speed is optimal at L_2 = 1200 rpm, Heating Pressure attains maximum S/N ratio at either L_2 = 4400 psi or L_3 = 4800 psi, and Upset Time is optimal at its middle level, L_2 = 3.60 seconds. We will break the tie between L_2 = 4400 psi and L_3 = 4800 psi of Heating Pressure from the RT of Strength. The RT for Strength values of Barker and Clausing's Table VI is given in *Table 16*, and the associated Minitab file, which also verified our answers, is available on request. *Table 16* clearly shows that the impact of Upset Pressure and Length are highly significant on the mean Strength, while the effect of Heating Time is statistically significant at the 5% level ($F_0 = 5.169 > F_{0.05;\,2,\,14} = 3.74$). Thus, these three process parameters (UPPRS, Length, and HTTIME) are Signal factors because they significantly influence the mean response but have minimal impact on variability of y. *Table 16* further shows that the optimal levels of these Signal factors to maximize Strength are $UPPRS_3(L_3$ = 9500 psi), $Length_2(L_2 = 0)$, and $HTTIME_3(L_3$ = 3.60 seconds). Further, we had a choice of two levels for the Control factor HTPRS (L_2 = 4400 psi or L_3 = 4800 psi). Because Heating Pressure has the maximum signal at 4800 psi, its optimal level is L_3 = 4800 psi. Therefore, our optimum settings from our PDE for low-cost tolerances are $X_O = Speed_{1200}\ HTPRS_{4800}\ UPPRS_{9500}\ Length_0\ HTTIME_{3.6}\ UPTIME_{3.6}$.

The article by Barker and Clausing (1984) then proceeded with a confirmation experiment at the above optimum condition X_O as a tolerance design with low-cost tolerances, and the result is given in their Table IX and reproduced below. The value of the mean for the confirmation run of Table IX is $\bar{y}_{CN}$ = 159.7312 and the larger-the-better S/N ratio is $\eta_{CN} = -10 \times \log_{10}(MSD) = 43.2496$, and the confirmation standard deviation S_{CN} = 40.1526. Our values of mean, S/N, and standard deviation match those of the authors listed near the bottom of their p. 49, but again we could not determine how Barker and Clausing arrived at their nominal strength value of 184.152 for the confirmation experiment. The reader should observe that the optimal levels from the PDE have improved the mean strength to $\bar{y}_{CN}$ = 159.7312 (barely below 160 ksi) but better than 26 out of the 27 means listed in their original design Table VI. However, the variability measured by S_{CN} = 40.1526 is still too large (coefficient of variation is still more than 25%), yielding a quality loss based only on variability as $\bar{L} = kS^2 = 0.138889 \times 40.1526^2$ = \$223.921, which is consistent with the authors' value of \$224 listed in their Table IX (p. 49). We have a slight problem with using k = 0.138889 for the loss function $\bar{L} = kS^2$ because the value of 0.138889 was computed for a modified form of the NTB loss function given in Eq. (10). Because Strength is LTB, we used the Taguchi LTB loss function $L(y) = k/y^2$ to compute the constant k using the point (100 ksi, \$500), which yields k = 5,000,000. Then, $\bar{L} = k(MSD) = 5 \times$

$$10^6\left[\frac{1}{18}\sum_{i=1}^{18}(1/y_i)^2\right] = 5 \times 10^6(0.00004732) = \$236.5961,$$

which is in slight disagreement with the value of \$224 reported near the bottom of the authors' p. 49.

Barker and Clausing (1984) next ran an ANOVA on the 18 strength values of their Table IX and provided their results in Table X, which is reproduced below. There is a slight problem with Barker and

Table IX of Barker and Clausing—Confirmation Run at X_0 with Wide-Range Tolerances
Speed: 1200±10%; HTPRS: 4800±15%; UPPRS: 9500±15%; Length: 0± 10%; HTTIME: 3.6±20%; UPTIME: 3.6±20%

Speed (2)	HTPRS (3)	UPPRS (4)	Length (5)	HTTIME (6)	UPTIME (7)	Strength (y)
1080	4080	8075	0	2.88	2.88	93.8254
1080	4800	9500	0	3.6	3.6	180.285
1080	5520	10925	0	4.32	4.32	247.502
1200	4080	8075	0	3.6	4.32	96.4075
1200	4800	9500	0	4.32	2.88	173.245
1200	5520	10925	0	2.88	3.6	194.133
1320	4080	9500	0	4.32	3.6	151.143
1320	4800	10925	0	2.88	4.32	152.789
1320	5520	8075	0	3.6	2.88	148.251
1080	4080	10925	0	3.6	3.6	179.554
1080	4800	8075	0	4.32	4.32	142.253
1080	5520	9500	0	2.88	2.88	130.376
1200	4080	9500	0	2.88	4.32	124.819
1200	4800	10925	0	3.6	2.88	178.513
1200	5520	8075	0	4.32	3.6	229.883
1320	4080	10925	0	4.32	2.88	145.504
1320	4800	8075	0	2.88	3.6	131.302
1320	5520	9500	0	3.6	4.32	175.376

Table X of Barker and Clausing—Analysis of Sources of Variation

Source	Sum of Squares (SS)	% Contribution
Speed	772	2.0%
Heat pressure	9318	34.0%
Upset pressure	5595	20.0%
Length	535	2.0%
Heat time	5733	21.0%
Upset time	3312	12.0%
Residual	No value was provided	9.0%

Clausing's Table X, which represents the ANOVA output from their Table IX design matrix. That is, in their Table IX, the level of Length stays at zero throughout the matrix and thus no SS (sum of squares) attributed to Length can be computed because levels of Length have to change at least once before a SS can be computed. Our Minitab output with Length at zero at all 18 FLCs is provided below.

General Linear Model: Strength vs. Speed, HTPRS, ...

```
Factor    Type    Levels       Values
Speed     fixed      3     1080, 1200, 1320
HTPRS     fixed      3     4080, 4800, 5520
UPPRS     fixed      3     8075, 9500,10925
HTTIME    fixed      3     2.88, 3.60, 4.32
UPTIME    fixed      3     2.88, 3.60, 4.32
```

```
Analysis of Variance for Strength, Using Adjusted SS
for Tests
```

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Speed	2	774.5	774.5	387.2	1.01	0.411
HTPRS	2	9311.3	9311.3	4655.6	12.17	0.005
UPPRS	2	5598.4	5598.4	2799.2	7.32	0.019
HTTIME	2	5732.8	5732.8	2866.4	7.49	0.018
UPTIME	2	3313.0	3313.0	1656.5	4.33	0.060
Error	7	2678.0	2678.0	382.6		
Total	17	27407.9				

S = 19.5594 = $(382.6)^{1/2}$ R-Sq = 90.23% R-Sq(adj) = 76.27%

The only way that we could generate a SS for Length was to replace the zeros in column (5) of Barker and Clausing's design matrix IX on their p. 49 with column (5) of the Taguchi L_{18} OA, which is

$$[1\ \ 2\ \ 3\ \ 2\ \ 3\ \ 1\ \ 1\ \ 2\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 1\ \ 2\ \ 2\ \ 3\ \ 1]'$$

We again ran Minitab on the resulting design matrix and obtained the following output.

General Linear Model: Strength vs. Speed, HTPRS, ...

Factor	Type	Levels	Values
Speed	fixed	3	1080, 1200, 1320
HTPRS	fixed	3	4080, 4800, 5520
UPPRS	fixed	3	8075, 9500,10925
Length	fixed	3	1, 2, 3
HTTIME	fixed	3	2.88, 3.60, 4.32
UPTIME	fixed	3	2.88, 3.60, 4.32

```
Analysis of Variance for Strength, Using Adjusted SS
for Tests
```

Source	DF	Seq SS	Adj SS	Adj MS	F	P	%Contrib.
Speed	2	774.5	774.5	387.2	0.90	0.462	2.826%
HTPRS	2	9311.3	9311.3	4655.6	10.87	0.015	33.973%
UPPRS	2	5598.4	5598.4	2799.2	6.53	0.040	20.426%
Length	2	535.6	535.6	267.8	0.62	0.572	1.954%
HTTIME	2	5732.8	5732.8	2866.4	6.69	0.039	20.917%
UPTIME	2	3313.0	3313.0	1656.5	3.87	0.097	12.088%
Error	5	2142.4	2142.4	428.5			7.816%
Total	17	27407.9					

S = 20.6998 R-Sq = 92.18% R-Sq(adj) = 73.42%

Note that Minitab's GLM does not provide a percent contribution column, and as a result we used Microsoft Excel to compute the last column of the above output. Further, the last column of the above Minitab output is in good agreement with Table X of Barker and Clausing.

As stated earlier, although the mean strength has improved to $\bar{y}_{CN}$ = 159.7312, variability for low-cost tolerances measured by standard deviation S_{CN} = 40.1526 is still too large, causing 10 out of the 18 confirmation runs to have strength values well below 160 ksi. The only option left for further variance reduction is to find high-cost (or narrow-range) tolerances for the six process factors to reduce standard deviation from S_{CN} = 40.1526 to a desired value of, say S_d = 20 ksi, where the desired value of S_d = 20 was selected by the authors. Thus, the desired variance of high-cost (or narrow-range) tolerances is $S_d^2 = 400\ \text{ksi}^2$, resulting in $400/40.1526^2 = 0.2481$, or simply 24.81% of the low-cost variance of $40.1526^2 = 1612.2313$. To determine the high-cost tolerances of the six factors, Barker and Clausing set up a tolerance reduction formula based on the percent contribution of their Table X. Their trade-off tolerance reduction formula, based on Table X, is duplicated below and designated as Eq. (11).

$$0.2481 = \underset{\text{Speed}}{1^2(0.02)} + \underset{\text{HTPRS}}{\left(\frac{1}{hp}\right)^2 \times 0.34} + \underset{\text{UPPRS}}{\left(\frac{1}{up}\right)^2 \times 0.20} + \underset{\text{Length}}{1^2(0.02)} + \underset{\text{HTTIME}}{\left(\frac{1}{ht}\right)^2 \times 0.21} + \underset{\text{UPTIME}}{\left(\frac{1}{ut}\right)^2 \times 0.12} + \underset{\text{Residual}}{0.09} \tag{11}$$

As Barker and Clausing state on their p. 48, Speed and Length had minimal impact on variation (roughly a total of 4%) and thus it is not worthwhile tightening their tolerances. As a result, they kept the Speed and Length tolerances at the low cost of ±10%. The authors' tolerance reduction Eq. (11) has an infinite number of solutions for hp, up, ht, and ut to reach a goal of 24.81%. Although there are an infinite number of solutions to the rational tolerance reduction Eq. (11), in practice the values of hp, up, ht, and ut should be determined not only to reach the desired goal of 0.2481 but also to minimize the cost of tolerance reduction. One possible solution that Barker and Clausing (1984) provide at the bottom of their p. 50 is to reduce the low-cost tolerances ±15% of Heating Pressure by a factor of hp = 3 to high-cost tolerances ±5%; to reduce the low-cost tolerances ±15% of Upset Pressure by a factor of up = 2 to the high-cost tolerances ±7.5%; to reduce the low-cost tolerances Heating Time by a factor of ht = 4 from ±20% to ±5%; and to reduce the low-cost tolerances of Upset Time by a factor of ut = 4 from ±20% to ±5%. We inserted these values of hp = 3, up = 2, ht = 4, and ut = 4 into Eq. (11), and the left-hand side of the Eq. (11) becomes 0.2384 so that the resulting variance reduction is a bit more than the desired amount of 0.2481. Another possible solution, out of infinite, is hp = 2.5, up = 3, ht = 3, and ut =

Table XI of Barker and Clausing—Confirmation Experiment at X_O Using High-Cost or Tight-Range Tolerances Embedded in the Taguchi L_{18} OA

Speed (2)	HTPRS (3)	UPPRS (4)	Length (5)	HTTIME (6)	UPTIME (7)	Strength (y)
1080	4440	8787.5	0	3.42	3.42	148.175
1080	4800	9500	0	3.6	3.6	180.285
1080	5160	10212.5	0	3.78	3.78	214.085
1200	4440	8787.5	0	3.6	3.78	154.614
1200	4800	9500	0	3.78	3.42	187.569
1200	5160	10212.5	0	3.42	3.6	203.285
1320	4440	9500	0	3.78	3.6	168.928
1320	4800	10212.5	0	3.42	3.78	186.335
1320	5160	8787.5	0	3.6	3.42	178.602
1080	4440	10212.5	0	3.6	3.6	179.920
1080	4800	8787.5	0	3.78	3.78	170.139
1080	5160	9500	0	3.42	3.42	183.808
1200	4440	9500	0	3.42	3.78	165.605
1200	4800	10212.5	0	3.6	3.42	195.667
1200	5160	8787.5	0	3.78	3.6	192.875
1320	4440	10212.5	0	3.78	3.42	180.443
1320	4800	8787.5	0	3.42	3.6	161.258
1320	5160	9500	0	3.6	3.78	192.165

Table XII of Barker and Clausing—Summary of Experimental Work

START			END	
Speed	Wide Range	10% Tolerances	1200 rpm	10% Tolerances
HTPRS	Wide Range	15% Tolerances	4800 psi	5% Tolerances (1/3 of original)
UPPRS	Wide Range	15% Tolerances	9500 psi	7.5% Tolerances (cut in half)
Length	Wide Range	10% Tolerances	Zero	10% Tolerances
HTTIME	Wide Range	20% Tolerances	3.6 sec.	5% Tolerances (1/4 of original)
UPTIME	Wide Range	20% Tolerances	3.6 sec.	5% Tolerances (1/4 of original)
Strength Response	Wide Range causing a big loss LOSS: $350/engine		Strength Loss reduced	Reduced range LOSS: $22/engine

2.57, for which the left-hand side of Eq. (11) becomes 0.2481.

Before presenting the authors' Final Design Section, we must present an excellent quote from Barker and Clausing (1984) from their p. 51, which is reproduced below:

"Based on the rational reduction of variation we set up a final design to verify our new tolerances. We use another 18 treatment Taguchi L_{18} OA and the results show that the variation has been reduced as predicted to a much more tolerable level. The S/N is now 45 db which is 2.13 times better than the best combination of our original design and far better than anything we would have achieved by trial and error. More importantly, we have come to a logical end point. There is no more searching for that 'pot of gold' at the end of the rainbow that often goes on in a nonsystematic searches for optimum conditions."

There may be a minor typographical error in their S/N assessment in the above quote because we could not verify the value "2.13 times better" because the best S/N ratio in the original design occurs at run number 18, whose S/N is $\eta_{18} = 41.7$ db.

As stated in the above quote, Barker and Clausing (1984) then conducted another set of confirmation experiments at the optimal condition $X_O = \text{Speed}_{1200}\ \text{HTPRS}_{4800}\ \text{UPPRS}_{9500}\ \text{Length}_0\ \text{HTTIME}_{3.6}\ \text{UPTIME}_{3.6}$ with high-cost tolerances Speed: 1200 ± 10%, HTPRS: 4800 ± 5%, UPPRS: 9500 ± 7.5%, Length: 0 ± 10%, HTTIME: 3.6 ± 5%, and UPTIME: 3.6 ± 5% using a Taguchi L_{18} OA. The results are given in their Table XI (p. 51) and duplicated above. Table XI of Barker and Clausing may have a minor error because the high-cost (or tight-range) tolerances of Heating Pressure are 4800 ± 5% = 4800 – 0.05 × 4800, 4800, 4800 + 0.05 × 4800 = 4560, 4800, 5040, while they are listing 4440, 4800, and 5160 as the three levels of HTPRS in their L_{18} outer OA. We do not know if this minor error is typographical or if 4440, 4800, and 5160 are the actual levels used in the final tolerance design confirmation runs. If indeed the HTPRS levels were set at 4440, 4800, and 5160, then the high-cost tolerances that are being used for HTPRS are 4800 ± 7.5% (not the required amount of ±5%), so the actual value of hp = 2 and not the required amount of 3. With hp = 2, up = 2, ht = 4, and ut = 4, the LHS of Eq. (11) becomes 0.2856 (not satisfying the desired amount of 0.2481).

We computed the mean, standard deviation, and the larger-the-better S/N ratio of Table XI of Barker and Clausing (1984), obtaining $\bar{y}_{XI} = 180.2088$, $S_{XI} = 16.8246$, and $\eta_{XI} = 45.00522$, which are identical to those of the authors' to three-decimal accuracy, which they list near the bottom of their p. 51. The authors then state that the amount of QLs has been reduced to \$22/engine in their Table XII, which is reproduced above. We tried to use the data of Table XI of Barker and Clausing to reproduce the amount of reduced QLs that Barker and Clausing report as \$22/engine at the bottom of their Table XII. We attempted three different methods to obtain the average quality loss/engine of \$22, which are summarized below.

1. $\bar{L} = k \times S_{XI}^2 = 0.138889 \times 16.8246^2 = 39.3149$ (not too far from \$22/engine)

2a. $\bar{L} = k \times \text{MSD}$, where $\text{MSD} = \frac{1}{18}\sum_{i=1}^{18}(1/y_i)^2 = 0.000568526/18 = 0.0000315848 \rightarrow \bar{L} = 5000000 \times 0.0000315848 = \$157.924$ (very far from the stated \$22/engine)

2b. Or, since $\eta_{db} = -10 \times \log_{10}(\text{MSD})$, $\bar{L} = k \times (10^{-\eta_{db}/10}) = 5000000 \times (10^{-4.500522}) = \$157.924$ (identical to 2(a) as expected)

3. Using the modified loss function $L(y) = \begin{cases} k(y-160)^2, & \text{if } y \le 160 \text{ ksi} \\ 0, & \text{if } y > 160 \text{ ksi} \end{cases}$, we obtained (see Table XI of Barker and Clausing) $\bar{L} = k \times [(148.175 - 160)^2 + (154.614 - 160)^2]/18 = 0.138889 \times [(-11.825)^2 + (-5.386)^2]/18 = \$1.3028$ (not close to \$22/engine)

Presently, we are not sure how Barker and Clausing arrived at their $\bar{L}$ = \$22/engine. However, it is quite clear that Taguchi's tolerance design has led to a tremendous amount of reduction in societal QLs. Even if we take the average of the above three distinct computed quality losses, we obtain an $\bar{L}$ = (39.3149 + 157.924 + 1.3028)/3 = \$66.1806, which is a quality loss reduction of (\$350 – \$66.1806)/\$350 = 81.09% (a remarkable achievement).

14. Summary and Conclusions

In this article, each of Taguchi's major contributions has been presented and described in sufficient detail to allow the reader to understand and use the tools presented. Taguchi's contributions were identified as:

- The quantification of quality through Gauss's Quality Loss Function*
- Orthogonal Arrays to simplify the use of Design of Experiments (DOE)
- Robust Designs (Parameter and Tolerance Designs) to identify optimum settings to reduce process variability and to get the mean on target*
- Definition and use of the Signal-to-Noise (S/N) Ratio, which combines the mean and standard deviation into one measure

where * implies Taguchi's major contributions. Each of the above contributions was reviewed in detail with an objective review of accolades and criticisms of each contribution and examples of its use.

The Quality Loss Function was introduced as one of Taguchi's two greatest contributions. By applying Gauss's quadratic loss, Taguchi was able to quantify quality as deviation from the ideal target (instead of the traditional view that cost is incurred only when results do not meet design or consumer specifications). This contribution of Taguchi induced a change in philosophy by manufacturers, who now realize that to minimize total cost they must reduce variability around a customer-defined target instead of just meeting customer specifications.

Although Orthogonal Arrays (OAs) were not "invented" by Dr. Taguchi (many are actually classical (fractional) factorial designs credited to Sir R.A. Fisher, Box and Hunter, Kempthorne, Yates, and other notables), he can certainly be credited with popularizing their use by simplifying their format. This article explained how OAs are constructed, the importance of their alias structure and design resolution, and provided examples of the construction and use of some of the more popular Taguchi's OAs, specifically the L_8, L_{16}, L_{27}, and L_{18} designs. The contents presented herein can also aid those who wish to use the Taguchi and Konishi (1987) book on orthogonal arrays.

Taguchi's Robust (Parameter and Tolerance) Designs are the other most significant contributions; both were reviewed with examples of each. Parameter Design Experiments use inner and outer OAs to identify the Control and Signal factors through the use of Signal-to-Noise Ratio (S/N). The optimum levels are identified with a resulting cost reduction (using Taguchi's Loss Function). The authors used an example from Barker and Clausing (1984) to show how Taguchi's Parameter Design is used to identify optimum process condition, followed by Tolerance Design to tighten factor tolerances to attain sufficient reduction in process variation. As a general rule, we would recommend to the process or manufacturing engineer to use a single classical FFD with maximum resolution if the QCH of interest is STB or LTB and if the CV is known to be < 20%. However, if the QCH is a nominal dimension, then it is best to use Taguchi's crossed-array (or inner and outer OAs) to perform DOE.

Throughout the discussions on Robust Designs, the Signal-to-Noise Ratio was used. This is perhaps one of Taguchi's most controversial contributions because some Western statisticians believe that sample mean and variance should be analyzed separately since they are stochastically independent for a Gaussian process. The example used for Parameter Design was also presented by analyzing the mean and variance separately (as compared to the analysis using S/N), and results were presented that were in slight disagreement with the results obtained from the use of the S/N ratio. The dilemma of when to use Taguchi's S/N ratios and when to analyze the mean and ln(S) separately is an interesting problem that should be investigated in the future.

Finally, all manufacturing and process engineers should be well aware of the fact that Sir Ronald A. Fisher is the founder of the field of DOE, and Dr. Taguchi is one of the pioneers of its applications to process and manufacturing engineering. Taguchi facilitated the use of the very same methodology that Sir Fisher started in the early 1920s and on which many prominent Western 20th-century statisticians continued to expand and build.

Nomenclature

ANOVA	Analysis of Variance
CV	Coefficient of Variation
CF	Correction Factor
CIs	Confidence Intervals
cdf	Cumulative Distribution Function
df	Degrees of Freedom
DOE	Design of Experiments
FF	Fractional Factorial
FFD	Fractional Factorial Design
FLC	Factor Level Combination
FNC	Fraction Nonconforming
GLM	General Linear Model
k	Loss function coefficient and the number of factors in a factorial or FFD
$\bar{L}$	Mean (or Average) Quality Loss
LHS	Left Hand Side
LTB	Larger the Better
LSL	Lower Specification Limit
L(y)	Quality Loss Function for a single item
mod	Modulus
MSD	Mean Squared Deviation
MS	Mean Square

N_f	Number of Distinct Factor Level combinations
NTB	Nominal the Best
OA	Orthogonal Array
PCR	Process Capability Ratio
PDE	Parameter Design Experiment
PEC	Present Existing Condition
QCH	Quality Characteristic
QL	Quality Loss
QLF	Quality Loss Function
QD	Quality Difference
QI	Quality Improvement
R	Resolution
RT	Response Table
RHS	Right Hand Side
S/N	Signal to Noise
SS	Sum of Squares
STB	Smaller the Better
SPC	Statistical Process Control
TOIBTC	Table of Interactions Between Two Columns
USL	Upper Specification Limit
$\overline{y}_{CN}$	Mean of Confirmation runs

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