For type II censoring, [n, U, $N_f = r = r^*$], we have n units on test and our objective is to test exactly a priori fixed number r < n of them to failure in order to obtain the failure times $t_1, t_2, ..., t_r$ and then use these observed failure instances to obtain the MLEs of θ and β. We should first obtain the proper LF as provided below, where ${}_nP_r$ is the permutations of n units taken r at a time.

$$L\theta,\beta) = \left[({}_nP_r)\prod_{i=1}^{r}\frac{\beta}{\theta}(t_i/\theta)^{\beta-1}e^{-(t_i/\theta)^{\beta}}\,dt_i\right]\times\left[e^{-(t_r/\theta)^{\beta}}\right]^{(n-r)}$$

$$= {}_nP_r\,(\frac{\beta}{\theta})^r\left[\prod_{i=1}^{r}(t_i/\theta)^{\beta-1}\,dt_i\right]\times e^{-\sum_{i=1}^{r}(t_i/\theta)^{\beta}}\times\left[e^{-(n-r)(t_r/\theta)^{\beta}}\right] \quad (121a)$$

For type I censoring (i.e., testing-time is censored), the above likelihood function modifies to

$$L\theta,\beta) = (\frac{\beta}{\theta})^r\left[\prod_{i=1}^{r}(t_i/\theta)^{\beta-1}\right]\times e^{-\sum_{i=1}^{r}(t_i/\theta)^{\beta}}\times\left[e^{-(n-r)(T/\theta)^{\beta}}\right] \quad (121b)$$

For multiple censoring the LF becomes

$$L(\theta,\beta) = \prod_{i=1}^{S_F}\frac{\beta}{\theta}(t_i/\theta)^{\beta-1}e^{-(t_i/\theta)^{\beta}}dt_i\prod_{j=1}^{S_C}e^{-(t_j^{+}/\theta)^{\beta}} = (\frac{\beta}{\theta})^r e^{-\sum_{i=1}^{r}t_i/\theta}\,e^{-\sum_{j=1}^{n-r}t_j^{+}/\theta}\prod_{i=1}^{r}(t_i/\theta)^{\beta-1}dt_i\,, \quad (121c)$$

where $r = S_F$ represents the failed units and $S_C = n-r$ represents the censored units.

For type II censoring, taking the natural logarithm of Eq. (121a) yields

$$L(\theta,\beta) = \ln({}_nP_r) + r\,(\ln\frac{\beta}{\theta}) + (\beta-1)\sum_{i=1}^{r}\ln(t_i/\theta) + \sum_{i=1}^{r}\ln(dt_i) - \sum_{i=1}^{r}(t_i/\theta)^{\beta} - (n-r)(t_r/\theta)^{\beta}$$

Because the derivative of 1st and 4th terms on the above RHS will vanish, the above log-likelihood function reduces to

$$L(\theta,\beta) = r\ln(\beta) - r\ln(\theta) + (\beta-1)\sum_{i=1}^{r}[\ln(t_i)-\ln(\theta)] - \sum_{i=1}^{r}(t_i/\theta)^{\beta} - (n-r)(t_r/\theta)^{\beta}$$

$$= r\times\ln(\beta) + (\beta - 1)\sum_{i=1}^{r}\ln(t_i) - \beta r\ln(\theta) - \sum_{i=1}^{r}(t_i/\theta)^{\beta} - (n - r)(t_r/\theta)^{\beta} \qquad (122a)$$

We 1st take the partial derivative of $L(\theta, \beta)$ in (122a) wrt θ and will require it to be zero.

$$\frac{\partial L}{\partial \theta} = -\frac{\beta r}{\theta} + \beta\theta^{-\beta-1}\sum_{i=1}^{r} t_i^{\beta} + \beta(n - r)\theta^{-\beta-1}(t_r^{\beta}) = -\frac{\beta r}{\theta} + \beta\theta^{-\beta-1}\left[\sum_{i=1}^{r} t_i^{\beta} + (n - r)t_r^{\beta}\right] \xrightarrow{\text{Set to}} 0$$

(123a)

In order to solve for $\hat{\theta}$ from equation (123a), we multiply throughout by $\theta^{\beta+1}/\beta$, which will reduce (123a) to $-r\theta^{\beta} + \sum_{i=1}^{r} t_i^{\beta} + (n - r)(t_r^{\beta}) = 0 \rightarrow$ Thus for

$$\textbf{Type II Censoring [n, U, r]: } \hat{\theta} = \left[\left[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n - r)t_r^{\hat{\beta}}\right]/r\right]^{1/\hat{\beta}} \qquad (123b)$$

Next we partially differentiate $L(\theta, \beta)$ in equation (122a) wrt β and set it equal to zero in order to obtain the MLE of β for type II censoring.

$$\frac{\partial L}{\partial \beta} = \frac{r}{\beta} + \sum_{i=1}^{r}\ln(t_i) - r\ln(\theta) - \sum_{i=1}^{r}\left[(t_i/\theta)^{\beta}\times\ln(t_i/\theta)\right] - (n - r)(t_r/\theta)^{\beta}\times\ln(t_r/\theta) \xrightarrow{\text{Set to}} 0$$

This last equation will simplify to

$$\frac{r}{\hat{\beta}} + \sum_{i=1}^{r}\ln(t_i) - r\ln(\hat{\theta}) - \sum_{i=1}^{r}(t_i/\hat{\theta})^{\hat{\beta}}\ln(t_i/\hat{\theta}) - (n - r)(t_r/\hat{\theta})^{\hat{\beta}}\times\ln(t_r/\hat{\theta}) = 0 \qquad (124a)$$

Combining Eqs. (124a), (123b), and using the fact that $\hat{\theta}^{-\hat{\beta}} = \dfrac{r}{\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n - r)t_r^{\hat{\beta}}}$,

and simplifying results in $\frac{r}{\hat{\beta}} + \sum_{i=1}^{r}\ln(t_i) - \hat{\theta}^{-\hat{\beta}}\sum_{i=1}^{r} t_i^{\hat{\beta}}\ln(t_i) - (n - r)\hat{\theta}^{-\hat{\beta}}(t_r^{\hat{\beta}})\times\ln(t_r) = 0$; thus, for

$$\textbf{Type II Censoring: } \frac{1}{\hat{\beta}} + \frac{1}{r}\sum_{i=1}^{r}\ln(t_i) - \frac{\sum_{i=1}^{r} t_i^{\hat{\beta}}\ln(t_i) + (n\text{-}r)t_r^{\hat{\beta}}\ln(t_r)}{\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n\text{-}r)t_r^{\hat{\beta}}} = 0 \qquad (124b)$$

This last equation (124b) and Eq. (123b) have to be solved simultaneously for $\hat{\beta}$ and $\hat{\theta}$ in order to obtain the ML estimates of the two Weibull parameters for type II censoring. However, there will never exist a closed-form solution for the two MLEs $\hat{\beta}$ and $\hat{\theta}$, and hence the approximate solutions have to be found using similar procedure that I have outlined on pages 232-243 of my notes for the case of no censoring.

For Type I censoring (i.e., testing-time is censored), similar procedure as above will lead to 2 equations with 2 unknowns that will have to be solved simultaneously in order to obtain the MLEs $\hat{\theta}$ and $\hat{\beta}$; the procedure is outlined in my summary at the end of this Chapter.

For multiple censoring the log-likelihood function is

$$\mathcal{L}(\theta, \beta) = r\,(\ln\frac{\beta}{\theta}) + (\beta - 1)\sum_{i=1}^{r}\ln(t_i/\theta) + \sum_{i=1}^{r}\ln(dt_i) - \sum_{i=1}^{r}(t_i/\theta)^{\beta} - \sum_{j=1}^{n-r}(t_j^{+}/\theta)^{\beta} \tag{125a}$$

For multiple censoring, the partial derivative of $\mathcal{L}(\theta, \beta)$ in (125) wrt θ is given by

$$\frac{\partial \mathbf{L}}{\partial \theta} = -\frac{r\beta}{\theta} + \beta\theta^{-\beta-1}\Big[\sum_{i=1}^{r} t_i^{\beta} + \sum_{j=1}^{n-r}(t_j^{+})^{\beta}\Big] \xrightarrow{\text{Set to}} 0 \tag{126a}$$

The solution to Eq. (126a) will lead to the following MLE of the parameter θ for the case of

Multiple Censoring:

$$\hat{\theta} = \left[\Big[\sum_{i=1}^{r} t_i^{\hat{\beta}} + \sum_{j=1}^{n-r}(t_j^{+})^{\hat{\beta}}\Big]/r\right]^{1/\hat{\beta}} \tag{126b}$$

Note that the MLE of θ given in (126b) reduce to equation (111a) for the case of no censoring when all n units are tested to failure. For Multiple Censoring, the log-likelihood function in Eq. (125a) reduces to

$$\mathcal{L}(\theta, \beta) = r(\ln\beta - \ln\theta) + (\beta - 1)\sum_{i=1}^{r}\ln(t_i/\theta) - \sum_{i=1}^{r}(t_i/\theta)^{\beta} - \sum_{j=1}^{n-r}(t_j^{+}/\theta)^{\beta} \tag{125b}$$

and its partial derivative wrt β is given by

$$\frac{\partial L(\theta,\beta)}{\partial \beta} = \frac{r}{\beta} + \sum_{i=1}^{r} \ln(t_i/\theta) - \sum_{i=1}^{r} (t_i/\theta)^{\beta} \ln(t_i/\theta) - \sum_{j=1}^{n-r} (t_j^{+}/\theta)^{\beta} \ln(t_j^{+}/\theta) \xrightarrow{\text{Set to}} 0$$

Multiple Censoring: $$\sum_{i=1}^{r}\left[\frac{1}{\hat{\beta}} + \ln(t_i/\hat{\theta}) - (t_i/\hat{\theta})^{\hat{\beta}} \ln(t_i/\hat{\theta})\right] - \sum_{j=1}^{n-r} (t_j^{+}/\hat{\theta})^{\hat{\beta}} \ln(t_j^{+}/\hat{\theta}) = 0 \quad (126c)$$

For multiple censoring the two Eqs. (126b) and (126c) have to be solved numerically in order to obtain their simultaneous solutions for $\hat{\beta}$ and $\hat{\theta}$. To illustrate the procedure, I will go through the Example 5.19 on pages 310-314 of E. A. Elsayed that involves type II censoring.

The Example 5.19 on page 310 of E. A. Elsayed. In this experiment n = 30 identical units are put on test at time zero W/O replacement and testing is stopped at the instant of t_{22} = 33 time units. The instances of failure are 18.5, 20, 20.5, 21.5, 22, 22.5, 23.5, 24, 24.3, 24.6, 25, 25.3, 25.6, 26, 26.3, 26.7, 27, 28, 29, 30, 32, 33. From the problem statement I could not surmise if the units of measurements were in hours, days, or possibly weeks. If this were an accelerated testing procedure, we could easily assume that the failure times were measured in hours; otherwise, the time unit could easily be in days. For simplicity, I will assume that our TTF in this example is measured in hours. It is clear that 8 units were right-censored at the instant t_{22} = 33 hours, i.e., the most conservative lives that we can assign to the 8 surviving units is the value of the 22[nd] order-statistic $t_{r=22}$ = 33 hours. As in other examples, we 1[st] try to obtain a rough value of the sample cv of TTF. Note that it is impossible to compute the exact sample cv because we do not know the exact values of t_{23}, t_{24}, ..., t_{30}, but we do know that their times TF are greater than 33 hours. To this end, let $\bar{t}_u$ and s_u represent the mean and standard deviation of the uncensored part of our data, i.e., $\bar{t}_u = \frac{1}{r}\sum_{i=1}^{r=22} t_i$ = 25.2409 and s_u = 3.7203 → cv_u = 0.1474. Now reference to Table 1 on page 10 of my Chapters 2, 3&4 will reveal that the value of $\hat{\beta}$ will roughly lie within the interval [5, 10]. Please note that $\bar{t} = \frac{1}{n}\sum_{i=1}^{n=30} t_i > \bar{t}_u$ but also the standard deviation, s, of all the 30 times TF, if the last 8 order statistics were available, would also be larger than s_u because the corresponding complete sample would have larger range (however, note that a larger range does not always guarantee a larger s value). This implies that the value of $cv_u = 0.1474$ should be fairly close to $cv = s/\overline{t}$.

To obtain the ML estimates, I will start by assuming that $\hat{\beta} \cong 5.0$ at which equation (123b) yields $\hat{\theta} = 30.5811$. Inserting these initial estimates into the LHS of equation (124b) yields the value of 0.1288 > 0 for the LHS. Several trial and errors, using the MS Excel-Solver, yield $\hat{\beta}$ = 5.1055565545 and $\hat{\theta}$ = 30.5764557476. Note that these MLEs are almost identical to those of Elsayed's given in the middle of his page 310, and also exactly match those of Minitab's.

Reducing the Amount of Bias in the MLEs of Weibull Parameters

As I have repeatedly mentioned before, the MLEs of Weibull parameters β and θ are biased, i.e., $E(\hat{\beta}) > \beta$ and only when $n \geq r > 30$ the amount of overestimation in $\hat{\beta}$ is almost negligible. As Elsayed mentions on his page 310, Bain and Engelhardt (B&E, Statistical Analysis of Reliability and Life-Testing Models, Theory and Methods, 1991, 2nd Edition, Marcel Dekker) provide a bias reducing factor for $\hat{\beta}$ in their Chapter 4, which can be approximated by Elsayed's empirical formula given below.

$$G(n) = 1.00 - 1.346/n - 0.8334/n^2 \qquad \text{(5.39 of Elsayed)}$$

Because I have not done extensive research in this area, I cannot asses how accurate equation (5.39 of Elsayed) for reducing the amount of bias $B(\hat{\beta}) = E(\hat{\beta} - \beta)$ is. However, Elsayed's apparent regression result (5.39) seems to ignore the observed number of failures, r, which has a higher impact in determining the accuracy of the MLEs than the actual number of units, n, put on test at time zero. Clearly, the larger the value of n, the higher the failure intensity level of a testing process, but if only r < 10 failures are observed, the MLEs may not be very precise (i.e., they may have large standard errors). Further, for r < 10, the MLEs will most likely be fairly biased. Just to check the accuracy of the model (5.39), I regressed the biasing factors, B_n, of Bain and Engelhardt (B&E) given in their Table 2 on page 221 of their text, using Minitab and obtained the following model (with $R^2_{Model} = 100\%$).

```
The Minitab regression equation is
```

$\hat{B}_n$ = 1.00104 – 1.41518 /n + 1.2288/n² – 18.1081/n³ + 47.86/n⁴ (5.39Magh)

I also had Minitab compute the fitted values of B_n and all of them were identical to 3 decimals to the tabular values of Bain & Engelhardt's Table 2 on their page 221.

It seems that Minitab does not adjust the MLEs for bias and uses their value and the corresponding elements of the Fisher's information-matrix in order to obtain the standard-errors of the estimates, which are described below.

Approximating the $se(\hat{\beta})$ for Type II Censoring

The exact variance of $\hat{\beta}$ cannot be computed, but as a 1st step we can compute the Cramer-Rao's glb for the $V(\hat{\beta})$ from $\text{glb}[V(\hat{\beta})] = \frac{1}{I(\beta)}$, where $I(\beta) = -E[\partial^2 L(\theta, \beta)/\partial\beta^2] = \frac{r}{\beta^2} + E$ $\left[\sum_{i=1}^{r}[(t_i/\theta)^{\beta} \times \ln^2(t_i/\theta)] + (n-r)\times(t_r/\theta)^{\beta}\times\ln^2(t_r/\theta)\right]$. Computing the exact expectation of the rv inside the large brackets is probably too complicated and seems not possible to this author, and thus, an approximate value of the $V(\hat{\beta})$ is given by

$$\hat{V}(\hat{\beta}) \doteq \frac{\hat{\beta}^2}{r + \hat{\beta}^2\left[\sum_{i=1}^{r}[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_i/\hat{\theta})] + (n-r)(t_r/\hat{\theta})^{\hat{\beta}}\ln^2(t_r/\hat{\theta})\right]} \tag{127}$$

Substitution of $\hat{\beta}$ = 5.1055565545 and $\hat{\theta}$ = 30.5764557476 into equation (127) and the use of Matlab computation yields $\text{glb}[V(\hat{\beta})] \cong 0.8334754$ and thus the $se(\hat{\beta}) \geq 0.91295$. Due to the fact that our censoring ratio p = 22/30 = 0.733333, then from Table 5.11 of Elsayed simple interpolation yields c_{22} = 1.0577. Therefore, the $se(\hat{\beta}) \cong 0.91295\times\sqrt{1.058} = 0.9391 \rightarrow$ HCIL (half CI length) = 1.96×0.9391 = 1.84054 → β_L = 5.10556 – 1.84054 = 3.26502, and β_U = 5.10556 +1.84054 = 6.94610. My Excel file shows how Minitab obtain the 95% CI for β using the $se(\hat{\beta})$ = 0.913278, and the 95% CI on β: 3.59568 ≤ β ≤ 7.24945. The details of obtaining the two CIs, that account for the $\text{cov}(\hat{\beta}, \hat{\theta})$ are provided on pp. 267-268 of these notes..

Approximating the $se(\hat{\theta})$ for Type II Censoring

As in the case of $\hat{\beta}$, computation of the exact $V(\hat{\theta})$ is intractable mainly due to the fact that there is no closed-form solution for the MLE $\hat{\theta}$. So, the 1st step is to obtain the glb for $V(\hat{\theta})$ from $glb[V(\hat{\theta})\} = \frac{1}{I(\theta)}$, i.e., the I_{11} of the Fisher's Info-Matrix, where $I(\theta) = -E[\partial^2 L(\theta, \beta)/\partial\theta^2] =$

$= -\frac{r\beta}{\theta^2} + \beta(\beta+1)\theta^{-\beta-2} E\left[\sum_{i=1}^{r} t_i^{\beta} + (n-r)t_r^{\beta}\right]$. Again taking the exact mathematical expectation of the rv inside this last large brackets seems almost impossible to carry out (due to the fact that it will require the knowledge of the pdf of the i^{th}–order, i = 1, 2, …., r, statistics from a Weibull base-line distribution), and thus an estimate of $I(\theta)$ is given by

$$I(\theta) \cong \hat{\beta}(\hat{\beta}+1)(\hat{\theta})^{-\hat{\beta}-2}\left[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)t_r^{\hat{\beta}}\right] - \frac{r\hat{\beta}}{\hat{\theta}^2} \qquad (128a)$$

Since the $glb[V(\hat{\theta})\} = \frac{1}{I(\theta)}$, then from (128a) we obtain the approximate asymptotic variance

$$V(\hat{\theta}) \cong \frac{\hat{\theta}^2/\hat{\beta}}{(\hat{\beta}+1)(\hat{\theta})^{-\hat{\beta}}\left[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)t_r^{\hat{\beta}}\right] - r} \qquad (128b)$$

Combining (123b) with equation (128b) results in $V(\hat{\theta}) \cong \frac{\hat{\theta}^2}{r\hat{\beta}^2}$. (128c)

The above approximate asymptotic variance for $\hat{\theta}$ is fairly close to what Minitab would provide to 2 decimals, but it totally disregards the correlation between $\hat{\beta}$ and $\hat{\theta}$. For the data of his Example 5.19, equation (128c) yields $V(\hat{\theta}) \cong 1.6303$ and $se(\hat{\theta}) = 1.27683$, where Minitab gives 1.27729. My Excel file provides all the necessary computations. Further, this last answer should be fairly close to the value of θ_L if we approximate the $se(\hat{\theta})$ from $se(\hat{\theta}) = \sqrt{c_{11}\hat{\theta}^2/(n\hat{\beta}^2)} =$

$\sqrt{1.4473 \times 30.576456^2/(30 \times 5.105556^2)} = 1.3154$ (see Table 5.11 of Elsayed on page 272 of my notes at this chapter's end). Note that in the denominator of this last formula for the $se(\hat{\theta}) \cong$

$\sqrt{c_{11}\hat{\theta}^2 / (n\hat{\beta}^2)}$ you must use n and not r.

Bain and Engelhardt (1991) state that the asymptotic SMD of $\hat{\mathbf{a}} = \hat{\beta} \times \ln(\hat{\theta}/\theta)$ is approximately Gaussian with zero mean and asymptotic variance equal to c_{11}/n, where the values of c_{11} are listed in Table 5.11 of Elsayed and reproduce herein on p. 272. Using this approximation we obtain $V(\hat{\mathbf{a}}) = 1.057695/30 = 0.0352565 \rightarrow se(\hat{\mathbf{a}}) = 0.187767143 \rightarrow P[\hat{\mathbf{a}} \le 1.645\times 0.187767143) = 0.95 \rightarrow P[\hat{\beta} \times \ln(\hat{\theta}/\theta) \le 0.30887695) = 0.95 \rightarrow P[\ln(\hat{\theta}/\theta) \le 0.30887695 /5.10556) = 0.95 \rightarrow P[\hat{\theta}/\theta \le \mathbf{e}^{\mathbf{0.30887695/5.10556}}) = 0.95 \rightarrow P[\hat{\theta}\, e^{-0.30887695/5.10556} \le \theta) = 0.95 \rightarrow \theta_L = 28.7814793$. If we use the MLEs of β and θ in this last equation, i.e., $\hat{\beta} = 5.1055565545$ and $\hat{\theta} = 30.5764557476$, we would obtain $\theta_L = 28.6191$, which is in good agreement with the value of $\theta_L = 28.7814793$ that I calculated using the fact that $\hat{a} = \hat{\beta} \times \ln(\hat{\theta}/\theta)$ is approximately $N(0, c_{11}/n)$.

Before discussing how to use χ^2 to obtain a more accurate CI for β, we must state that a point MLE of the RE function for the Example 5.19 of E. A. Elsayed is given by $\hat{R}(t) = e^{-(t/30.576456)^{5.10556}}$, but obtaining a lower one-sided confidence limit for R(t) is not a simple task because R(t) is a monotonically increasing function of θ but not of β. Recall that if β = 1, i.e., the TTF is exponential, then $R_L(t) = e^{-t/\theta_L}$, where I discussed the development of exponential θ_L and the 95% lower bound for RE in Chapters 12&13 of my notes. However, in the Example 5.19 of Elsayed the TTF distribution is Weibull [β > 1 and thus an IFR (increasing failure rate)] and hence we need the exact SMD of the statistic $\hat{R}(t) = e^{-(t/\hat{\theta})^{\hat{\beta}}}$ in order to obtain the exact lower 95% confidence limit for R(t). I am not well-read in this part of the literature of RE engineering, but I surmise that the exact SMD of $e^{-(t/\hat{\theta})^{\hat{\beta}}}$, where $\hat{\theta}$ is the MLE of θ and $\hat{\beta}$ is the MLE of β, is not mathematically tractable. However, all MLEs in the universe have the following nice properties under certain regularity conditions, which are generally met in life testing situations.

(1) Suppose the statistic $\hat{\theta}$ is the MLE of the parameter θ and let g(θ) be any function of the parameter θ ; then it can be proven that $g(\hat{\theta})$ is also a MLE of the parameter g(θ). For the example

5.19 of Elsayed, this property shows that $e^{-(t/\hat{\theta})^{\hat{\beta}}}$ is a MLE of the parameter $R(t) = e^{-(t/\theta)^{\beta}}$. Note that unbiased estimators, unless the function g(θ) is linear, do not possess this property.

(2) All MLEs in the universe, under certain regularity conditions (see Cramer H., Mathematical Methods of Statistics, Princeton University Press, 1946) are asymptotically unbiased. For practical applications, the size of the sample n must exceed 50 before the amount of bias in the MLE becomes small relative to its se of the estimate. For life testing situations, as I have repeatedly pointed out, the number of observed failures, r, also plays a very important role in the accuracy of the MLEs. For practical applications, r should exceed 20 (or perhaps at least 15) before the amount of bias in a MLE is small relative to the *se* of the estimate.

(3) The SMD of all MLEs approach normality under certain regularity condition (see the 2nd volume of Kendall and Stuart, page 43). However, the approach to normality is agonizingly slow unless the parent population is Gaussian, and unfortunately sometimes so slow that the practical application of this property may be rendered useless! As before, I would refrain from using the normal approximation unless the value of r is reasonably large. Let us relax the required value of r to at least 15 for practical applications, but be cognizant of the fact that the larger r values yield much better normal approximations of confidence intervals.

(4) MLEs are generally consistent and asymptotically efficient if the range of the frequency function $f(t \mid \theta)$ does not depend on θ. By consistency we mean that Lim of $\hat{\theta}$ as $n \to \infty$ is equal to θ, and by asymptotic efficiency we mean that $V(\hat{\theta})$ attains its Cramer-Rao's glb as $n \to \infty$.

(5) Unless the underlying population is Gaussian, the exact SMD of ML estimators for most underlying distributions (specifically the Weibull) are not known. Even if the underlying distribution of the data is Gaussian, the exact SMD of a general $g(\hat{\theta})$, is to my knowledge, not tractable. For example, it is well known that the sample mean $\bar{\mathbf{x}}$ is the MLE of the population mean E(X) = μ when the parent population is $N(\mu, \sigma^2)$ and the SMD of $\bar{\mathbf{x}}$ has been known for a long time to be $N(\mu, \sigma^2/n)$. However, this does not imply that the exact SMD of $g(\bar{\mathbf{x}}) = \mathbf{e}^{-(\bar{\mathbf{x}})^2}$ is

also normal just because $\bar{\mathbf{x}}$ is normally distributed. However, since $\mathbf{e}^{-(\bar{x})^2}$ is the MLE of $\mathbf{e}^{-\mu^2}$, then its asymptotic SMD should be close to normal. I hope the reader understands the spirit of what I am trying to convey from a statistical point of view? Getting now back to the problem at hand, we do know that the Weibull MLEs $\hat{\theta}$ and $\hat{\beta}$ are asymptotically normal, but this does not imply that the SMD of the MLE $e^{-(t/\hat{\theta})^{\hat{\beta}}}$ approaches normality as fast as $\hat{\theta}$ and $\hat{\beta}$ do as n becomes increasingly large. A second obvious problem is the fact that the MLE $\hat{\mathbf{R}}(\mathbf{t}) = \mathbf{e}^{-(t/\hat{\theta})^{\hat{\beta}}}$ contains a bivariate vector $\begin{bmatrix}\hat{\theta}\\ \hat{\beta}\end{bmatrix}$ whose components are highly correlated (see the information matrix inverse on page 207) and we have to obtain our lower confidence bound on R(t) without ignoring this correlation (i.e., we must take this correlation into account).

The Bonferroni CI for the Two-Parameter Weibull W(0, δ, β)

The 95% CIs for θ and β that we have obtained thus far have ignored the correlation between $\hat{\theta}$ and $\hat{\beta}$, i.e., the correct confidence band for the vector parameter $\begin{bmatrix}\theta\\ \beta\end{bmatrix}$ is an ellipsoid that with 95 % confidence contains the true vector $\begin{bmatrix}\theta\\ \beta\end{bmatrix}$. To avoid such a complicated multivariate analysis, we use the Bonferroni method of obtaining a 95% rectangular confidence region that has at least 95% chance (prior to sampling) of containing the true parameters θ and β simultaneously. This method is valid regardless of the correlation structure of the estimators and allows the experimenter to control the overall error rate α. It can be shown that $\alpha \le \sum_{i=1}^{m} \alpha_i$, where for our case m = 2 parameters, $1 - \alpha_1$ is the confidence coefficient for θ and $1 - \alpha_2$ is the confidence coefficient for the parameter β. This implies that if we wish to obtain a joint 95% rectangular confidence region for $\begin{bmatrix}\theta\\ \beta\end{bmatrix}$ so that α = 0.05, then the individual error rates should be set at $\alpha_1 = \alpha_2 \cong 0.025$ because the overall confidence pr is given by $(0.975)^2 = 0.950625 > 0.95$. If we had three parameters for which we would like to build a cubic 95% confidence region, then the Bonferroni method of obtaining joint CIs tells us that we should set the individual error rates α_i at approximately 0.05/3 = 0.016667 because $(1 - 0.016667)^3 = 0.95083$. Please note that the larger the value of m is, the more conservative (or smaller) the overall error rate, α, becomes; further, this method will work regardless of the correlation structures behind the confidence statements.

The above discussions tell us that in order to obtain a simultaneous Bonferroni 95% CI for θ and β, we should really obtain the 97.5% individual CI for θ and β. Therefore, the Bonferroni 95% CIs for θ and β are as follows:

$$\hat{\beta} \pm Z_{0.0125} \times se(\hat{\beta}) = 5.10556 \pm 2.2414 \times 0.9391 \rightarrow 3.00066 \le \beta \le 7.21046$$

$$\theta_L = \hat{\theta} - Z_{0.025} \times se(\hat{\theta}) = 30.57646 - 1.96 \times 1.27683 = 28.07387 \rightarrow 28.07387 \le \theta < \infty.$$

Due to the fact that the Weibull reliability, $R(t) = e^{-(t/\theta)^{\beta}}$, is an increasing function of both parameters θ and β up to the characteristic life θ, the Bonferroni 95% lower confidence bound for R(t) is given by $e^{-(t/\theta_L)^{\beta_L}} = e^{-t^{3.00066}/22175.055041}$, which is valid only for $0 \le t \le \theta$ $\rightarrow$

$\mathbf{e^{-t^{3.00066}/22175.055041}} \le R(t) < 1,\ 0 \le t \le \theta$. For example, the 95% lower confidence limit for R(t) at t = 22.5 hours is given by $R_L(22.5) = 0.597664$ ($22.5 < \hat{\theta}$). Note that the number of survivors beyond 22.5 hours for the data of Example 5.19 on page 299 is $N_s = 24$ so that a direct point estimate of R(22.5) would be roughly equal to $\hat{R}(22.5) = 24/30 = 0.80$. It seems that the Bonferroni confidence limit, $R_L(22.5) = 0.597664$, for R(t) is quite conservative, and perhaps not very useful!

Before using the χ^2 for CI estimation, let's try to obtain the 95% greatest lower bound on RE which is valid for the class of IFR distributions (such as the Weibull with $\beta > 1$) that generally will work even if the underlying distribution is not Weibull. The (95%) glb on RE at a specific $t_0 < \tau/n$ is given by

$$R_L(t_0) = \mathbf{Exp}\left[-\frac{(\lambda_{0.05;r-1})}{\tau}\mathbf{t_o}\right], \qquad (129a)$$

where τ represents the total testing time for all n units and $\lambda_{0.05;\,r-1} = \mu$ is the solution to

$$\sum_{x=0}^{r-1} \frac{\mu^x}{x!} e^{-\mu} \le \alpha = 0.05 \qquad (129b)$$

For the Example 5.19 on page 310 of E. A. Elsayed, $\tau = \sum_{i=1}^{22} t_i + 8\times t_r = 555.3000 + 8\times 33 =$ 819.3000, and $\mu = \lambda_{0.05;\,21}$ is the solution to $\sum_{x=0}^{21} \frac{\mu^x}{x!} e^{-\mu} \le 0.05$. Matlab computations yields $\mu = \lambda_{0.05;\,21} = 30.24045$. Then at $t_0 \equiv 22.5$, we have from (129a), $R_L(22.5) = \mathbf{e^{-(30.24045/819.30)(22.5)}} = 0.4358412$. Note that this 95% glb on RE is too conservative relative to 0.597664 obtained from the Bonferroni method, and perhaps almost useless. The main reason behind this is the fact that the glb in (119a) does not make the assumption of the Weibull baseline distribution; it just provides a glb on RE no matter what the underlying failure distribution is as long as the HZF is of increasing rate. Note that Eq. (119a) is disallowed if $\mu\times t_0 > \tau$.

Bain and Engelhardt (1991) also state on their page 220 that the asymptotic SMD of $\hat{R}(t)$ (for n > 20) is normal with asymptotic mean R(t) and asymptotic variance equal to

$$V[\hat{R}(t)] \cong \mathbf{\hat{R}^2 [(\ln(1/\hat{R})]^2 \left[c_{11} - 2c_{12} \ln(\ln(1/\hat{R})) + c_{22} [\ln(\ln(1/\hat{R}))]^2 \right] / n} \qquad (130)$$

From Table 5.11 on page 308 of Elsayed at p = 0.73333, interpolation yields $c_{11} = 1.38237665$, $c_{12} = -0.080121$, and $c_{22} = 1.057695$. Inserting these c values and $\hat{R}(\text{at } t = 22.5) = e^{-(22.5/30.576456)^{5.10556}} = 0.81148505$ into (120) yields $V[\hat{R}(t)] \cong 0.003567906$ and $se[\hat{R}(t)] = 0.05973195$; as a result, $R_L(22.5) = 0.81148505 - 1.645\times 0.05973195 = 0.713226$. This 95% glb on RE is more meaningful than the previous three, as all three were too conservative.

Minitab's Computations of CIs for the Two Weibull-Parameters

Again, for type II censoring Minitab first obtains the Local-Fisher's Information-matrix as follows:

$$\mathbf{I} = \begin{bmatrix} \mathbf{I_{11}} & \mathbf{I_{12}} \\ \mathbf{I_{21}} & \mathbf{I_{11}} \end{bmatrix} = \begin{bmatrix} \mathbf{I(\theta)} & \mathbf{I(\theta,\beta)} \\ \mathbf{I(\theta,\beta)} & \mathbf{I(\beta)} \end{bmatrix} = \begin{bmatrix} \mathbf{-E(\partial^2 L/\partial\theta^2)} & \mathbf{-E(\partial^2 L/\partial\theta\partial\beta)} \\ \mathbf{-E(\partial^2 L/\partial\theta\partial\beta)} & \mathbf{-E(\partial^2 L/\partial\beta^2)} \end{bmatrix}$$

where $\mathbf{I(\theta) = -E[\partial^2 L(\theta, \beta)/\partial\theta^2] = -\frac{r\beta}{\theta^2} + \beta(\beta + 1)\theta^{-\beta-2}\times E\left[\sum_{i=1}^{r} t_i^{\beta} + (n - r)\, t_r^{\beta}\right] = I_{11}}$,

$\mathbf{I(\beta) = -E[\partial^2 L(\theta, \beta)/\partial\beta^2] = \frac{r}{\beta^2} + E[\sum_{i=1}^{r}[(t_i/\theta)^{\beta} \times \ln^2(t_i/\theta)] + (n-r)(t_r/\theta)^{\beta} \times \ln^2(t_r/\theta)] = I_{22}}$, **and**

$\mathbf{I(\theta,\beta)} = -E(\partial^2 \mathbf{L}/\partial\theta\partial\beta) = \mathbf{-r/\theta + \theta^{-(\beta+1)} E[\sum_{i=1}^{r}[t_i^{\beta} + (n-r)t_r^{\beta}] \times (1-\beta\ln\theta) + \beta\theta^{-(\beta+1)} \times}$

$\mathbf{E[\sum_{i=1}^{r}[t_i^{\beta}\ln(t_i) + (n-r)t_r^{\beta}\ln(t_r)] = I_{12}}$. For the estimated elements of the matrix I, see my summary at the end of this chapter. Minitab uses the inverse of the above, however approximated, Fisher's matrix to obtain the *se*'s of $\hat{\theta}$ and $\hat{\beta}$, which takes the correlation between the 2 Weibull estimators into account. For the Example 5.19-Elsayed, we have $\hat{\mathbf{I}}_{11} \cong$ $-E[\partial^2 L(\theta, \beta)/\partial\theta^2] \cong 0.61338703$, $\hat{\mathbf{I}}_{12} = -0.02301614 = \hat{\mathbf{I}}_{21}$, and $\hat{\mathbf{I}}_{22} \cong \mathbf{1}.199795402$; $\hat{\mathbf{I}}$ $= \begin{bmatrix} 0.613387029 & -0.02301614 \\ \hat{I}(\theta,\beta) & 1.199795402 \end{bmatrix}$. Upon inverting this last Local-Fisher's matrix, we obtain

$\hat{\mathbf{I}}^{-1} = \begin{bmatrix} 1.631466416 & 0.031297052 \\ 0.031297052 & 0.834075823 \end{bmatrix}$. Thus, the *se*($\hat{\theta}$)= $\sqrt{1.631466416}$ = 1.277288679 and the *se*($\hat{\beta}$) = 0.913277517, which are identical to those of Minitab's to 5 decimals.

More Exact Confidence Interval for the Weibull slope β

For censored data Bain and Engelhardt (1991, p. 223) have established that the SMD of $2r\beta/\hat{\beta}$ approximately follows a χ^2 with an approximate df = 2(r – 1), while for complete samples the SMD of $\beta^2/\hat{\boldsymbol{\beta}}^2$ approximately follows a χ^2 with n – 1 df. Further, Bain and Engelhardt state that a better approximation for censored and complete samples is that the SMD of $c \times r \times (\beta/\hat{\beta})^{1+p^2}$ approximately follows a χ^2 with c(r –1) df, where the constant $c = \dfrac{\mathbf{2}}{\mathbf{p(1+p^2)^2 \times c_{22}}}$. Table 5.11 on page 308 of Elsayed provides the values of c_{22} for different values of failed fraction p = r/n. For his Example 5.19 data on pages 310-314, the failed fraction p = 22/30 = 0.7333 = , c_{22} = 1.0577 and thus c = 1.0901 and as a result df = 1.0901(22 – 1) = 22.8915. In order to obtain the 95% CI for β, I will interpolate for the $\chi^2_{0.025,22.8915}$ and $\chi^2_{0.975,22.8915}$ between the percentage points of χ^2 with 22 and 23 degrees of freedom. Given that $\chi^2_{0.025,22}$ = 36.7807, $\chi^2_{0.025,23}$ = 38.0756, then $\chi^2_{0.025,22.8915} \cong$ 37.9351; similarly, $\chi^2_{0.975,22}$ = 10.9823, and $\chi^2_{0.975,23}$ = 11.6886 yield $\chi^2_{0.975,22.8915} \cong 11.6120$. Letting $\nu = 22.8915$, then the 95% confidence pr statement using χ^2 distribution is given by $P(11.6120 \le \chi^2_\nu \le 37.9351) = 0.95$, or $P[11.6120 \le cr(\beta/\hat{\beta})^{1+p^2} \le 37.9351] = 0.95 \rightarrow$

$$P[(\frac{11.6120}{cr})^{1/(1+p^2)} \le \beta/\hat{\beta} \le (\frac{37.9351}{cr})^{1/(1+p^2)}] = 0.95 \rightarrow$$

$$P[\hat{\beta}(\frac{11.6120}{cr})^{1/(1+p^2)} \le \beta \le \hat{\beta}(\frac{37.9351}{cr})^{1/(1+p^2)}] \rightarrow \beta_L = \hat{\beta}(\frac{11.6120}{cr})^{1/(1+p^2)}$$ and

$\beta_U = \hat{\beta}(\frac{37.9351}{cr})^{1/(1+p^2)}$.

For the data of Example 5.19 of Elsayed, n = 30, r = 22, $\hat{\beta}$ = 5.105557 so that p = 22/30 = 0.73333, $c = \frac{2}{p(1+p^2)^2 \times c_{22}} = \frac{2}{0.73333(1+0.73333^2)^2 \times 1.058}$ = 1.090074, and $\beta_L = \hat{\beta}(\frac{11.6120}{cr})^{1/(1+p^2)}$ = 3.185816 and $\beta_U = \hat{\beta}(\frac{37.9351}{cr})^{1/(1+p^2)}$ = 6.87950, i.e., the 95% confidence interval is $3.185816 \le \beta \le 6.87950$.

Finally, a more exact 95% lower confidence bound on θ can be obtained from $\theta_L = \hat{\theta}\, e^{-u_{0.95}/(\hat{\beta}\sqrt{n})}$, where the 95th percentiles of $U = \hat{\beta}\sqrt{n} \times \ln(\hat{\theta}/\theta)$ are given in equation (5.50) of Elsayed (p. 313) for a censored sample for different values of p = r/n and is reproduced below.

$$U_{0.95} = 4.08 - 4.76p + 2.43p^2 + 11.41/n - 9.85/(np) + 10.46/(np)^2 \qquad \text{(5.50 of Elsayed)}$$

The Approximate *se*'s for Multiple Censoring

The exact variance of $\hat{\beta}$ cannot be computed, but as a 1st step we can compute the Cramer-Rao's glb for the $V(\hat{\beta})$ from $glb[V(\hat{\beta})] = \frac{1}{I(\beta)}$, where $I(\beta) = -E[\partial^2 L(\theta, \beta)/\partial\beta^2] = \frac{\mathbf{r}}{\boldsymbol{\beta}^2} +$

$E[\sum_{i=1}^{r}[(t_i/\theta)^{\beta} \times \ln^2(t_i/\theta)] + \sum_{j=1}^{n-r}[(t_j^+/\theta)^{\beta} \times \ln^2(t_j^+/\theta)]]$. Computing the exact expectation of the rv inside the large brackets is probably too complicated and seems not possible to this author, and thus, an approximate value of the $V(\hat{\beta})$ for Multiple Censoring is given by

$$\hat{V}(\hat{\beta}) \geq \frac{\hat{\beta}^2}{r + \hat{\beta}^2\left[\sum_{i=1}^{r}[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_i/\hat{\theta})] + \sum_{j=1}^{n-r}[(t_j^+/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_j^+/\hat{\theta})]\right]} \tag{131}$$

A better approximation than the above glb can be obtained from $\hat{V}(\hat{\beta}) \cong \frac{c_{22}\hat{\beta}^2}{n}$.

As in the case of $\hat{\beta}$, computation of the exact $V(\hat{\theta})$ for multiple censoring is intractable mainly due to the fact that there is no closed-form solution for the MLE $\hat{\theta}$. So, the 1st step is to obtain the glb for $V(\hat{\theta})$ from $glb[V(\hat{\theta})\} = \frac{1}{I(\theta)}$, where $I(\theta) = -E[\partial^2 L(\theta, \beta)/\partial\theta^2] = -r\beta\theta^{-2} +$

$\beta\beta+1\theta^{-\beta-2}E[\sum_{i=1}^{r} t_i^{\beta} + \sum_{j=1}^{n-r}(t_j^+)^{\beta}]$. Again taking the exact mathematical expectation of the rv inside the last large brackets seems almost impossible to carry out (due to the fact that it will require the knowledge of the pdf of the powers of ith –order, i = 1, 2, …., r, statistics from a Weibull base-line distribution), and thus an estimate of $I(\theta)$ is given by

$$I(\theta) \cong \hat{\beta}(\hat{\beta}+1)(\hat{\theta})^{-\hat{\beta}-2}[\sum_{i=1}^{r} t_i^{\hat{\beta}} + \sum_{j=1}^{n-r}(t_j^+)^{\hat{\beta}}] - \frac{r\hat{\beta}}{\hat{\theta}^2} = \frac{r\hat{\beta}^2}{\hat{\theta}^2} \tag{132a}$$

Since the $glb[V(\hat{\theta})\} = \frac{1}{I(\theta)}$, then from (132a) we obtain the approximate glb of the asymptotic variance from

$$V(\hat{\theta}) \geq \frac{\hat{\theta}^2/\hat{\beta}}{(\hat{\beta}+1)(\hat{\theta})^{-\hat{\beta}}[\sum_{i=1}^{r} t_i^{\hat{\beta}} + \sum_{j=1}^{n-r}(t_j^+)^{\hat{\beta}}] - r} \tag{132b}$$

Combining (132b) with equation (126b) results in

$$V(\hat{\theta}) \geq \frac{\hat{\theta}^2}{\mathbf{r}\hat{\beta}^2}. \tag{132c}$$

Thus an approximate more conservative $V(\hat{\theta})$ is given by $V(\hat{\theta}) = c_{11}\frac{\hat{\theta}^2}{n\hat{\beta}^2}$.

Summary of Weibull with Censoring

1. For type **I** censoring, numerically solve the following two Eqs. in 2 unknowns.

$$\hat{\theta} = \left[\left[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)T^{\hat{\beta}}\right]/r\right]^{1/\hat{\beta}} \quad \text{and} \quad \frac{1}{\hat{\beta}} + \frac{1}{r}\sum_{i=1}^{r}\ln(t_i) - \frac{\sum_{i=1}^{r} t_i^{\hat{\beta}}\ln(t_i) + (n\text{-}r)T^{\hat{\beta}}\ln(T)}{\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n\text{-}r)T^{\hat{\beta}}} = 0$$

$$V(\hat{\theta}) \geq \frac{\hat{\theta}^2/\hat{\beta}}{(\hat{\beta}+1)(\hat{\theta})^{-\hat{\beta}}[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)T^{\hat{\beta}}] - r}, \quad se(\hat{\theta}) \cong \sqrt{c_{11}\hat{\theta}^2/(n\hat{\beta}^2)}$$

$$\hat{V}(\hat{\beta}) \geq \frac{\hat{\beta}^2}{r + \hat{\beta}^2\left[\sum_{i=1}^{r}[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_i/\hat{\theta})] + (n-r)(T/\hat{\theta})^{\hat{\beta}}\ln^2(T/\hat{\theta})\right]}$$

$$se(\hat{\beta}) \cong \hat{\beta}\sqrt{c_{22}/n}$$

It is paramount to note that the *se*'s of $\hat{\theta}$ and $\hat{\beta}$ both diminish with increasing n and r. The decreasing relationship wrt n is obvious. However, when r increases, then p increases and Table 5.11 of E. A. Elsayed shows that both c_{11} and c_{22} decrease.

A more accurate estimate of $se(\hat{\beta})$ can be obtained from

$$se(\hat{\beta}) \cong \hat{\beta}_{Unb}\sqrt{C_n/n}$$

where $C_n = 0.617 + 1.8/n + 78.25/n^3$ (5.40Elsayed)

and $$\hat{\beta}_{Unb} = \hat{\beta}_{MLE} \times G_n$$

where $G_n = 1 - 1.346/n - 0.8334/n^2$ (5.39Elsayed)

From Bain & Engelhardt $\hat{a} = \hat{\beta} \times \ln(\hat{\theta}/\theta)$ is approximately Gaussian with zero mean and asymptotic variance equal to c_{11}/n, and $c \times r(\beta/\hat{\beta})^{1+p^2}$ approximately follows a χ^2 with c(r –1) df, where the constant c = $\dfrac{2}{p(1+p^2)^2 \times c_{22}}$, where p = sample failed fraction.

In order to obtain Minitab's answers CI limits, you must invert the local Fisher's Information-Matrix in order to obtain their *se*'s and the resulting CI limits from are $\beta_L = \hat{\beta} e^{-Z_{0.025} \times cv(\hat{\beta})}$, $\beta_u = \hat{\beta} e^{Z_{0.025} \times cv(\hat{\beta})}$ **and similarly for θ.**

2. **Type II Censoring:** $\hat{\theta} = \left[\left\{\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r) t_r^{\hat{\beta}}\right\} / r\right]^{1/\hat{\beta}}$; $\dfrac{1}{\hat{\beta}} + \dfrac{1}{r}\sum_{i=1}^{r} \ln(t_i) + \dfrac{\sum_{i=1}^{r} t_i^{\hat{\beta}} \ln(t_i) + (n\text{-}r) t_r^{\hat{\beta}} \times \ln(t_r)}{r(\hat{\theta})^{\hat{\beta}}}$ = 0. The elements of Fisher's I-matrix are $\hat{I}_{11} = \dfrac{r\hat{\beta}^2}{\hat{\theta}^2}$,

$\hat{I}_{22} = \dfrac{r}{\hat{\beta}^2} + \sum_{i=1}^{r}\left[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_i/\hat{\theta})\right] + (n-r)(t_r/\hat{\theta})^{\hat{\beta}} \ln^2(t_r/\hat{\theta})$, and $\hat{I}_{12} = r(\hat{\beta}/\hat{\theta})\ln(\hat{\theta}) - \hat{\beta}\hat{\theta}^{-(\hat{\beta}+1)}\left\{\sum_{i=1}^{r}\left[t_i^{\hat{\beta}} \ln(t_i)\right] + (n-r) t_r^{\hat{\beta}} \ln(t_r)\right\}$.

In order to obtain Minitab's answers on CI limits, you must invert the local Fisher's Information-Matrix, $\hat{I}$, in order to obtain their *se*'s and the resulting CI limits.

3. Multiple Censoring: $\hat{\theta} = \left[\left\{\sum_{i=1}^{r} t_i^{\hat{\beta}} + \sum_{j=1}^{n-r} (t_j^{+})^{\hat{\beta}}\right\} / r\right]^{1/\hat{\beta}}$ and the constraint is

$$\frac{r}{\hat{\beta}} + \sum_{i=1}^{r} \ln(t_i/\hat{\theta}) - \sum_{i=1}^{n}\left[(t_i/\hat{\theta})^{\hat{\beta}} \ln(t_i/\hat{\theta}) + (t_i^{+}/\hat{\theta})^{\hat{\beta}} \ln(t_i^{+}/\hat{\theta})\right] = 0,$$

The estimated elements of Fisher's Info-matrix are $I_{11} = I\theta) \cong \frac{r\hat{\beta}^2}{\hat{\theta}^2}$, $\hat{I}_{22} = \frac{r}{\hat{\beta}^2} +$

$\sum_{i=1}^{n}[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_i/\hat{\theta}) + (t_i^+/\hat{\theta})^{\beta} \times \ln^2(t_i^+/\hat{\theta})]$, and $\hat{I}_{12} = r(\hat{\beta}/\hat{\theta})\ln(\hat{\theta})$ −

$\hat{\beta}\hat{\theta}^{-(\hat{\beta}+1)}\sum_{i=1}^{n}[(t_i)^{\hat{\beta}}\ln(t_i) + (t_i^+)^{\hat{\beta}}\ln(t_i^+)]$.

4. In all cases $\hat{R}(t) = e^{-(t/\hat{\theta})^{\hat{\beta}}}$, $V[\hat{R}(t)] \cong$

$\hat{R}^2[(\ln(1/\hat{R})]^2\left[c_{11} - 2c_{12}\ln(\ln(1/\hat{R})) + c_{22}[\ln(\ln(1/\hat{R}))]^2\right]/n$, and the 95% glb on R(t) is given

by $$R_L(t) \cong e^{-(t/\hat{\theta})^{\hat{\beta}}} - 1.645 \times se[\hat{R}(t)]$$

Table 5.11 of E. A. Elsayed (on his page 308, 1−p = proportion of the sample that is censored; p = failed proportion)

p	1	0.9	0.8	0.7	0.6	0.5
c_{11}	1.108665	1.151684	1.252617	1.447258	1.811959	2.510236
c_{22}	0.607927	0.767044	0.928191	1.122447	1.372781	1.716182
c_{12}	0.257022	0.176413	0.049288	− 0.144825	− 0.446603	− 0.935766

p	0.4	0.3	0.2	0.1
c_{11}	3.933022	7.190427	16.478771	60.517110
c_{22}	2.224740	3.065515	4.738764	9.744662
c_{12}	− 1.785525	− 3.438610	−7.375310	− 22.187207