

Statistical Inference (SI) When the Underlying Distribution is Weibull

Recall that by SI we mean estimation and test of hypothesis. We will cover only the case of minimum life $\delta = 0$, and therefore, the underlying TTF pdf is given by: $f(t) = \frac{\beta}{\theta^\beta} t^{\beta-1} e^{-(t/\theta)^\beta} =$

$$\frac{\beta}{\theta} (t/\theta)^{\beta-1} e^{-(t/\theta)^\beta} = (\beta\lambda)(\lambda t)^{\beta-1} e^{-(\lambda t)^\beta}. \text{ We consider two experimental life testing situations: (1)}$$

No Censoring, (2) Failure Data with Censoring. The experimenter should bear in mind that if data shows the minimum life is not zero (for example if the value of the 1st-order statistic were equal to 23000 cycles, then obviously the minimum life is not zero), then such data can easily be converted to $\delta = 0$ by subtracting, say $0.90 \times t_{(1)} = 20700$ cycles, from all times to failure and treating the resulting data with minimum life equal to zero in order to use the above 2-parameter Weibull pdf. The experimenter should bear in mind that if data shows that the minimum life is not zero (for example if the value of the 1st-order statistic were equal to 23000 cycles to failure, then obviously the minimum life is not zero), then such data can easily be converted to $t_0 = \delta = 0$ by subtracting, say $0.90 \times t_{(1)} = 20700$ cycles (you may even try $0.85 \times t_{(1)}$), from all times to failure and treating the resulting data with minimum life equal to zero in order to use the above 2-parameter Weibull pdf. For more accurate estimate of minimum life t_0 see pp. 412 and 413 of Ebeling, where for the exponential ($\beta = 1$) $\hat{t}_0 = \text{Max} [2t_{(1)} - t_{(2)}, 0]$, this is basically the Eq. 15.25 of Ebeling on p. 413. For $\beta > 1$ (IFR), use Eq. (15.23) of Ebeling on his page 412, $\hat{t}_0 = (t_1 t_n - t_j^2) / (t_1 + t_n - 2t_j)$, where $t_1 = t_{(1)}$, $t_n = t_{(n)}$ is the n th order-statistic, and t_j for the Weibull is j th-order statistic, where $j = \lceil np \rceil$ and np is always rounded up to the next higher positive integer. The value of p is given in Ebeling's Eq. (15.24) on his p. 413 as $p = 0.8829n^{-0.3437}$. In almost all cases, the results of these more accurate formulas come very close to $0.85 \times t_{(1)} \leq \hat{t}_0 \leq 0.91 \times t_{(1)}$. Further, Ebeling (p. 421) states that the MLE of $t_0 = \delta = \text{Minimum-life}$ is equal to $t_{(1)}$.

When TTF, T , has the 3-parameter-Weibull $W(\delta, \theta, \beta)$, my recommendation is to make a simple transformation of the data to $t = T - \delta$ in order to reduce the 3-parameter to the 2-parameter Weibull.

Failure Data Without Censoring (Complete Samples, i.e. $r = n$)

In this case we have n units on test and our objective is to test all of them to failure in order to obtain the ordered failure times t_1, t_2, \dots, t_n and then use these observed failure instances to obtain the MLEs of θ and β . Note that this testing situation is applicable only to components that have moderately large increasing hazard rate (function); otherwise, the testing duration will become cost prohibitive. I will provide the MLF and its log and the resulting two partial derivatives wrt θ and β , but I am doing it only for the record. You will be responsible only for knowing how to apply the results, which I will outline in a step-by-step procedure.

ML Estimation Procedure

Recall that the Pr element of the i^{th} failure time, t_i , is given by $\frac{\beta}{\theta^\beta} t_i^{\beta-1} e^{-(t_i/\theta)^\beta} dt_i$ and

hence the LF (likelihood function) is given by

$$L(\theta, \beta) = \prod_{i=1}^n \frac{\beta}{\theta^\beta} t_i^{\beta-1} e^{-(t_i/\theta)^\beta} = \left(\frac{\beta}{\theta^\beta}\right)^n \times \left[\prod_{i=1}^n t_i^{\beta-1} \right] \times e^{-\sum_{i=1}^n (t_i/\theta)^\beta} \quad (109a)$$

Taking the natural log of (109a) leads to

$$L(\theta, \beta) = n[\ln(\beta) - \beta \ln(\theta)] + (\beta - 1) \sum_{i=1}^n \ln(t_i) - \sum_{i=1}^n (t_i/\theta)^\beta \quad (109b)$$

The partial derivative of $L(\theta, \beta)$ wrt θ is given by

$$\partial L(\theta, \beta)/\partial \theta = -n\beta/\theta - \frac{\partial}{\partial \theta} \left[\theta^{-\beta} \sum_{i=1}^n (t_i)^\beta \right] = -n\beta/\theta + \beta \theta^{-\beta-1} \sum_{i=1}^n (t_i)^\beta \xrightarrow{\text{Set to}} 0 \quad (110)$$

The solution to equation (110) is the MLE of θ given below.

$$\hat{\theta} = \left[\frac{1}{n} \sum_{i=1}^n t_i^{\hat{\beta}} \right]^{1/\hat{\beta}} \quad (111a)$$

Equation (111a) is the result that we need to obtain the point MLE of the characteristic life $t_c = \theta$, but the difficulty lies in the fact that unless we 1st obtain the point ML estimate of the slope β , then we will not be able to compute $\hat{\theta}$ from Eq. (111a). Next we will obtain the MLE of β by partially differentiating $L(\theta, \beta)$ wrt β .

$$\begin{aligned} \xi_{\beta} &= \partial L(\theta, \beta) / \partial \beta = \frac{n}{\beta} - n \ln(\theta) + \sum_{i=1}^n \ln(t_i) - \left[\sum_{i=1}^n \frac{\partial}{\partial \beta} (t_i / \theta)^{\beta} \right] \\ &= \frac{n}{\beta} - n \ln(\theta) + \sum_{i=1}^n \ln(t_i) - \sum_{i=1}^n \left[(t_i / \theta)^{\beta} \times \ln(t_i / \theta) \right] \xrightarrow{\text{Set to}} 0 \end{aligned} \quad (111b)$$

Equations (111a&b) will have to be solved simultaneously in order to obtain the ML estimates of θ and β . Unfortunately, no closed-form solution will ever exist for $\hat{\theta}$ and $\hat{\beta}$. Therefore, the solutions have to be obtained thru trial/and error that will make both partial derivatives $\partial L(\theta, \beta) / \partial \theta$ in (110) and $\partial L(\theta, \beta) / \partial \beta$ in (111b) almost equal to zero. I will now go thru the Example 5.17 on pages 304-307 of E. A. Elsayed (2nd Ed.) in a step-by step procedure, and compare answers with his solution and that of Minitab's.

Step 1: For the Example 5.17 on pages 304-307 of E. A. Elsayed (2nd Ed.), $n = 10$ diodes were tested to failure under accelerated conditions. The instances to failure in his example are $t_i = 31000, 36000, 40000, 44000, 50000, 51000, 51500, 54000, 57000,$ and 63000 minutes ($i = 1, 2, \dots, 10$); however, no information about the load factor A_f is provided. Matlab computations yield \bar{x}_S

$$= \frac{1}{10} \sum_{i=1}^{n=10} t_i = 47750 \text{ minutes}, S = \sqrt{\frac{1}{9} \sum_{i=1}^{10} (t_i - \bar{x})^2} = \sqrt{\frac{1}{9} \text{CSS}} = \sqrt{\frac{1}{9} (879625000)} =$$

9886.1575504 minutes. (Notice that Elsayed lists hours for the units of \bar{x} and s.) Thus the sample coefficient of variation of TTF is given by $cv = S / \bar{x} = 20.704\% \ll 1$, which clearly implies that the underlying TTFs distribution is not exponential.

Step 2. Since the hazard function of a Weibull is decreasing only during the early-life cycle of products (i.e., only during the RE growth cycle or burn-in period the slope lies in the interval $0 < \beta < 1$), then for our example $\beta > 1$. Equation (5.32) on page 304 of E. A. Elsayed (2nd Ed.) gives a rough approximation of the Weibull slope in terms of the corresponding sample cv.

$$\hat{\beta} = \frac{1.05}{cv} \quad (\text{Equation 5.32 of Elsayed})$$

The above equation gives us a starting point for ML estimation, i.e., we estimate our initial slope as $\hat{\beta} = 1.05 / 0.20704 = 5.0715$. I have used the values in my Table 1 in my Chapters 2,3&4, p.10, to

obtain a better regression model that relates the Weibull slope to the cv. The Minitab output is given below.

Regression Analysis: Beta versus x = 1/CV

The regression equation is

Beta = - 0.270 + 1.20 x, where x = 1/CV, $1 \leq \beta \leq 5$.

Predictor	Coef	SE Coef	T	P
Constant	-0.27015	0.02082	-12.97	0.000
x	1.19841	0.00715	167.71	0.000

S = 0.03010 R-Sq = 99.9% R-Sq(adj) = 99.9%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	25.486	25.486	28125.57	0.000
Residual Error	15	0.014	0.001		
Total	16	25.500			

Unusual Observations

Obs	x	Beta	Fit	SE Fit	Residual	St Resid
1	1.00	1.00000	0.92826	0.01435	0.07174	2.71R

R denotes an observation with a large standardized residual

Results for: WBeta.MTW

Regression Analysis: Beta versus x = 1/CV

The regression equation is

Beta = - 0.591 + 1.28 x, $5 \leq \beta \leq 20$

Predictor	Coef	SE Coef	T	P
Constant	-0.590770	0.008170	-72.31	0.000
x	1.27540	0.00075	1698.72	0.000

S = 0.01085 R-Sq = 100.0% R-Sq(adj) = 100.0%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	340.00	340.00	2.886E+06	0.000
Residual Error	14	0.00	0.00		
Total	15	340.00			

Unusual Observations

Obs	x	Beta	Fit	SE Fit	Residual	St Resid
1	4.4	5.0000	4.9762	0.0052	0.0238	2.50R
16	16.1	20.0000	19.9801	0.0052	0.0199	2.08R

R denotes an observation with a large standardized residual

The estimate of β from the above 2 regression models are, respectively, $\hat{\beta} = -0.27015 + 1.19841 / 0.20704 = 5.5182$, or $\hat{\beta} = -0.59077 + 1.2754 / 0.20704 = 5.5694$. These are fairly close to the value of 4.83 obtained from Cohen (1965), *Technometrics* 7, pp. 579-588, given in equation (5.32) of Elsayed. So, I will settle for a rough estimate of $\hat{\beta} = 5.00$ and will use this to obtain a rough value of $\hat{\theta}$ from my equation (111a), which is $\hat{\theta} = 50938.5117215225$ minutes. These two estimates ($\hat{\beta} = 5.00$) make the RHS of equation (111b) equal to 0.504003453. Since $\partial L(\theta, \beta) / \partial \beta$ in equation (111b) is a decreasing function of β , then we must increase the value of β in order to reduce the RHS of (111b) close to zero. Through the Excel-Solver, I found that at $\hat{\beta} = 5.996972780$ and $\hat{\theta} = 51559.62598478$ minutes, the RHS of (111b) is close to 10^{-15} . As a matter of fact, the estimate $\hat{\beta} = 5.996972780$ is very close to that of Elsayed's in the middle of his page 306. Therefore, the MLE of β and θ to 6 decimals are $\hat{\beta} = 5.996973$ and $\hat{\theta} = 51559.625985$, which lead to $\hat{R}(t) = e^{-(t/51559.625985)^{5.996973}}$. As a result, a point ML estimate of the RE function at 40000 minutes is given by $\hat{R}(40000 \text{ min}) = 0.8039725487$, which is consistent (to 3 decimals) with Elsayed's answer on his page 306.

We have completed point ML estimation for a $W(0, \theta, \beta)$ for a complete sample, and now it is time to perform further SI by obtaining a 95% CI for both parameters θ and β . Again since the Weibull t_c (characteristic life) is definitely a LTB type parameter (and thus no concern on the high side), my personal engineering preference would be to obtain the one-sided CI for θ , ($\theta_L \leq \theta < \infty$), and a 2-sided CI for the parameter β : $\beta_L \leq \beta \leq \beta_U$. The reader must be cognizant that throughout these notes, we are setting the confidence coefficient, for the sake of convenience, at $1 - \alpha = 0.95$; further, the statistical entities θ_L , β_L , and β_U are indeed (correlated) random variables before experimentation for obtaining failure data. However, after experimental data are gathered and the needed sample statistics are computed, then no longer θ_L , β_L , and β_U are random variables but they are simply real-valued numbers and hence the deterministic intervals no longer have 95% pr of containing the true values of θ and β . Once these two CIs are obtained, then they may be used to conduct tests of hypotheses about the parameters θ and β . For example, if the data of the example

5.17 of Elsayed provides the 95% CI : $3.5 \leq \beta \leq 8.5$, then we cannot reject the null hypothesis $H_0 : \beta = 6.5$ at the LOS $\alpha = 0.05$ because the hypothesized value of $\beta_0 \equiv 6.5$ lies inside the 95% CI : $3.5 \leq \beta \leq 8.5$.

Confidence Interval for the Weibull Slope (or Shape) β

It is well known that an exact CI for a parameter can be obtained only if the frequency function of the corresponding point estimator is exactly known. Equation (111b) clearly shows that no closed-form solution exists for the point estimator $\hat{\beta}$. I am almost certain that the exact SMD (or pdf) of $\hat{\beta}$ for small to moderate values of r ($r < 50$) is not known, but from statistical theory the frequency function of all ML estimators (MLEs) asymptotically approach normality with asymptotic mean equal to the corresponding parameter and the asymptotic variance equal to $1/I(\beta)$, where the sample information $I(\beta)$ is described below. Therefore, the asymptotic 95% CI for parameter β is given by $\hat{\beta} \pm Z_{0.025} \times se(\hat{\beta})$, where $Z_{0.025} = 1.959964$. I am also fairly certain that no exact closed-form formula exists for the $V(\hat{\beta})$ and hence the exact $se(\hat{\beta})$ cannot be computed, and as a result all of the following developments are simply approximations, and only for moderately large values of r failures the approximate CIs are expected to be fairly accurate. Thus, our next objective is to obtain a rough estimate of the $se(\hat{\beta})$. This will require understanding the concept of statistical efficiency.

Efficiency

We first show that $E[\partial L(t; \theta)/\partial \theta]$ is always zero for any parameter θ , where $\mathbf{t} = [t_1 \ t_2 \ \dots \ t_n]'$ represent a complete sample of n failure times. $E[\partial L(t; \theta)/\partial \theta] = \int [\partial L(\mathbf{t}; \theta)/\partial \theta] f(\mathbf{t}; \theta) d\mathbf{t} = \int \frac{1}{f(\mathbf{t}; \theta)} \left[\frac{\partial f(\mathbf{t}; \theta)}{\partial \theta} \right] f(\mathbf{t}; \theta) d\mathbf{t} = \frac{d}{d\theta} \int f(\mathbf{t}; \theta) d\mathbf{t} = \frac{d}{d\theta} (1) = 0 \rightarrow V[\partial L(t; \theta)/\partial \theta] = E\{[\partial L(t; \theta)/\partial \theta]^2\}$,

where we are using the fact that the joint pdf of $\mathbf{t} = [t_1 \ t_2 \ \dots \ t_n]'$ is given by $f(\mathbf{t}; \theta) = \prod_{i=1}^n f(t_i; \theta)$.

Let $\hat{\theta}$ be a MLE of θ (or any estimator of θ); then, the $COV[\hat{\theta}, \partial L(t; \theta)/\partial \theta] = E[\hat{\theta} \times \partial L(t; \theta)/\partial \theta] =$

$$\int_{\mathbf{R}_t} \hat{\theta} [\partial \mathbf{L}(\mathbf{t}; \theta) / \partial \theta] \mathbf{f}(\mathbf{t}; \theta) d\mathbf{t} = \int \hat{\theta} \left[\frac{d}{d\theta} \mathbf{f}(\mathbf{t}; \theta) \right] d\mathbf{t} = \frac{d}{d\theta} E(\hat{\theta}) = \frac{d}{d\theta} [\theta + B(\hat{\theta})] = 1 + B'(\hat{\theta}),$$

where $B(\hat{\theta})$ is the amount of bias in the estimator $\hat{\theta}$ and $B'(\hat{\theta}) = \frac{d}{d\theta} B(\hat{\theta})$. For convenience of notation, let $\xi_\theta = \partial \mathbf{L}(\mathbf{t}; \theta) / \partial \theta$ for which we just have shown that $E(\xi_\theta) = 0$ and $\text{COV}(\hat{\theta}, \xi_\theta) = E(\hat{\theta} \times \xi_\theta) = 1 + B'(\hat{\theta})$. It is well known that $0 \leq \rho^2 \leq 1$, where ρ is the correlation coefficient between any two random variables. Thus, $0 \leq \rho^2(\hat{\theta}, \xi_\theta) \leq 1 \rightarrow 0 \leq \frac{\text{COV}^2(\hat{\theta}, \xi_\theta)}{V(\hat{\theta})V(\xi_\theta)} \leq 1 \rightarrow$

$$\frac{[\text{COV}(\hat{\theta}, \xi_\theta)]^2}{V(\xi_\theta)} \leq V(\hat{\theta}) < \infty \quad (112a)$$

The quantity $V(\xi_\theta) = V[\partial \mathbf{L}(\mathbf{t}; \theta) / \partial \theta] = E(\xi_\theta^2) = I(\theta)$ in the denominator of (112a) is called the information in the sample (about θ); since, $\text{COV}(\hat{\theta}, \xi_\theta) = E(\hat{\theta} \times \xi_\theta) = 1 + B'(\hat{\theta})$, then (112a) reduces to

$$\frac{[1 + B'(\hat{\theta})]^2}{I(\theta)} \leq V(\hat{\theta}) < \infty \quad (112b)$$

The information inequality (112b) is called the Cramer-Rao inequality in the field of statistics. It provides the greatest lower bound (glb) for the variance of any estimator in the universe. Simply put, there exists no estimator in the universe whose variance is less than the glb $\frac{[1 + B'(\hat{\theta})]^2}{I(\theta)}$.

Further, only those estimators whose variance is equal to the Cramer-Rao's glb are called efficient. Before discussing an Example, we need to show that $V(\xi_\theta) = V[\partial \mathbf{L}(\mathbf{t}; \theta) / \partial \theta] = E(\xi_\theta^2) = I(\theta)$ is also equal to $-E(\partial \xi_\theta / \partial \theta)$ as illustrated below.

$$\frac{\partial \xi_\theta}{\partial \theta} = \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial \theta} \ln(f(\mathbf{t}; \theta)) \right] = \frac{\partial}{\partial \theta} \left[\frac{f'(\mathbf{t}; \theta)}{f(\mathbf{t}; \theta)} \right], \text{ where } f'(\mathbf{t}; \theta) = \frac{\partial}{\partial \theta} [f(\mathbf{t}; \theta)]. \text{ Thus,}$$

$$\frac{\partial \xi}{\partial \theta} = \frac{f f'' - (f')^2}{f^2} = \frac{f''}{f} - \xi^2 \quad (113)$$

Applying the expected-value operator to both sides of (113), and using the fact that

$$E\left(\frac{f''}{f}\right) = \int \left(\frac{f''}{f}\right) f d\mathbf{t} = \frac{d^2}{d\theta^2} \int f(\mathbf{t}; \theta) d\mathbf{t} = \frac{d^2}{d\theta^2} (1) = 0, \text{ we obtain } V(\xi) = -E(\partial \xi / \partial \theta).$$

This last result also shows that

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{V}(\boldsymbol{\xi}) = -\mathbf{E}\left[\frac{\partial^2}{\partial\boldsymbol{\theta}^2}\ln(\mathbf{f}(\mathbf{t}; \boldsymbol{\theta}))\right] = -\mathbf{E}\left[\frac{\partial^2}{\partial\boldsymbol{\theta}^2}\mathbf{L}(\mathbf{t}; \boldsymbol{\theta})\right] = \mathbf{I}_{11} \quad (114)$$

In general, if $\hat{\boldsymbol{\theta}}_i$ ($i = 1, 2, \dots, m$) are the MLEs of m parameters with the log likelihood function $\mathbf{L}(x; \theta_1, \theta_2, \dots, \theta_m)$, then the $(i, j)^{\text{th}}$ element of the information matrix, \mathbf{I} , is given by $I_{ij} = -\mathbf{E}[\partial^2\mathbf{L}(x; \theta_1, \theta_2, \dots, \theta_m)/\partial\theta_i\partial\theta_j]$. It can then be shown that the asymptotic covariance matrix of the vector $\hat{\boldsymbol{\theta}} = [\hat{\boldsymbol{\theta}}_1 \quad \hat{\boldsymbol{\theta}}_2 \dots \quad \hat{\boldsymbol{\theta}}_m]'$ is given by the inverse of the information matrix \mathbf{I} , i.e., $\text{COV}(\hat{\boldsymbol{\theta}}) = \text{COV}[\hat{\boldsymbol{\theta}}_1 \quad \hat{\boldsymbol{\theta}}_2 \dots \quad \hat{\boldsymbol{\theta}}_m]' = \mathbf{I}^{-1}$.

I will start the procedure by 1st obtaining the exact Cramer-Rao's glb for the $\mathbf{V}(\hat{\boldsymbol{\beta}})$; recall from inequality (112b) that $\frac{[1+B'(\hat{\boldsymbol{\beta}})]^2}{\mathbf{I}(\boldsymbol{\beta})} \leq \mathbf{V}(\hat{\boldsymbol{\beta}}) < \infty$. From statistical theory it is well known that MLEs are asymptotically unbiased, and hence for large n , $\frac{1}{\mathbf{I}(\boldsymbol{\beta})} \leq \mathbf{V}(\hat{\boldsymbol{\beta}}) < \infty$, where $\mathbf{I}(\boldsymbol{\beta}) = -\mathbf{E}[\partial^2\mathbf{L}(\boldsymbol{\theta}, \boldsymbol{\beta})/\partial\boldsymbol{\beta}^2]$. I partially differentiated equation (111b) to obtain $\partial^2\mathbf{L}(\boldsymbol{\theta}, \boldsymbol{\beta})/\partial\boldsymbol{\beta}^2$, which is provided below.

$$\begin{aligned} \partial^2\mathbf{L}(\boldsymbol{\theta}, \boldsymbol{\beta})/\partial\boldsymbol{\beta}^2 &= -n/\boldsymbol{\beta}^2 - \sum_{i=1}^n \left[(t_i / \boldsymbol{\theta})^\beta \times [\ln(t_i / \boldsymbol{\theta})]^2 \right] \rightarrow \\ -\partial^2\mathbf{L}(\boldsymbol{\theta}, \boldsymbol{\beta})/\partial\boldsymbol{\beta}^2 &= n/\boldsymbol{\beta}^2 + \sum_{i=1}^n \left[(t_i / \boldsymbol{\theta})^\beta \times [\ln(t_i / \boldsymbol{\theta})]^2 \right] \rightarrow \\ \mathbf{I}(\boldsymbol{\beta}) &= -\mathbf{E}[\partial^2\mathbf{L}(\boldsymbol{\theta}, \boldsymbol{\beta})/\partial\boldsymbol{\beta}^2] = n/\boldsymbol{\beta}^2 + \mathbf{E} \sum_{i=1}^n \left[(t_i / \boldsymbol{\theta})^\beta \times [\ln(t_i / \boldsymbol{\theta})]^2 \right] = \mathbf{I}_{22} \end{aligned} \quad (115)$$

Eq. (115) clearly shows that the information on $\boldsymbol{\beta}$ in the sample is directly proportional to the number of failures n because as n increases, the amount of information about $\boldsymbol{\beta}$ increases. Further, it involves a very complicated mathematical expectation on the RHS of equation (115), given by

$$\theta^\beta \sum_{i=1}^n \mathbf{E} \left[(t_i)^\beta \times [\ln(t_i / \boldsymbol{\theta})]^2 \right].$$

Applying the Expected-Value operator to the term inside the summation in order to obtain an exact result is impossible, at least to the capability of this author.

Even simulation may not help because both parameters β and θ are unknown and therefore the simulation procedure has to start with specified values of these parameters of the Weibull and hence the simulation result for $\theta^\beta \sum_{i=1}^n E \left[(t_i)^\beta \times [\ln(t_i / \theta)]^2 \right]$ would depend on the inputted values of β and θ . Having stated the simulation problems, there is hope because once failure data are obtained, then the MLEs of β and θ can be obtained, and hence the simulation can get started with these MLEs. This last computer simulation procedure in order to estimate the *se*(of any estimator) is called Bootstrapping in the field of Statistics.

Bootstrap Estimation Procedure

Step 1: Recall that the MLEs of β and θ (to 3 decimals) for the Example 5.17 are $\hat{\beta} = 5.997$ and $\hat{\theta} = 51559.626$ minutes. First assume that the underlying distribution of the failure data is

$W(0, \theta = 51559.626, \beta = 5.997)$, i.e., assume that the cdf is actually $F(t) = 1 - e^{-(t/51559.626)^{5.997}}$.

Recall from statistical theory that all continuous cdfs in the universe are uniformly distributed over

the interval $[0, 1]$, i.e., $F(t) \sim U(0, 1)$. This is because $\int_0^1 dF = 1$ for all continuous $F(t)$.

Step 2: Use the cdf from step 1 with the aid of a computer to generate a random bootstrap sample of r ($= n$ in this case) $t_{11}, t_{12}, \dots, t_{1n}$. Use these sample results to obtain the point MLE estimate of β and θ , denoted by $\hat{\beta}_1^*$ and $\hat{\theta}_1^*$, as outlined above.

Step 3: Repeat step 2 roughly $B = 500$ times (i.e., a simulation run size of at least 500, where $B = 500$ is my recommendation). By now we will have roughly 500 bootstrap estimates ($\hat{\beta}_1^*, \hat{\beta}_2^*, \dots,$

$\hat{\beta}_{500}^*$) and ($\hat{\theta}_1^*, \hat{\theta}_2^*, \dots, \hat{\theta}_{500}^*$). Next compute the bootstrap averages $\bar{\beta}^* = \frac{1}{B} \sum_{i=1}^B \hat{\beta}_i^*$ and $\bar{\theta}^*$.

Step 4: Then Bootstrap estimate of the *se*($\hat{\beta}$) is given by $S_{\hat{\beta}} = \sqrt{\frac{1}{B} \sum_{i=1}^B [\hat{\beta}_i^* - \bar{\beta}^*]^2}$ and similarly

for the $se(\hat{\theta})$, i.e., $S_{\hat{\theta}} = \sqrt{\frac{1}{B} \sum_{i=1}^B [\hat{\theta}_i^* - \bar{\theta}^*]^2}$. You may wish to replace B in the divisors by B – 1,

if that is your preference. Note that the above Bootstrap procedure can be applied to all estimates whether the underlying distribution family is known or not. When the underlying distribution family is unknown, it will become more difficult.

Now that I have outlined how to obtain reliable point estimates of the standard errors by a computer simulation, we have to get back to the problem at hand where we have to obtain rough estimate of $se(\hat{\beta})$ W/O the use of a computer. Eq. (112b & 115) show that the glb for the $V(\hat{\beta})$ is given by

$$\frac{[1 + \mathbf{B}'(\hat{\beta})]^2}{\frac{n}{\beta^2} + \sum_{i=1}^n E[(t_i / \theta)^\beta \times [\ln(t_i / \theta)]^2]} \leq V(\hat{\beta}) < \infty. \quad (116)$$

If we assume that $n = r$ is sufficiently large in order to ignore the amount of bias in the MLE $\hat{\beta}$, then the numerator of (116) reduces to 1. To get the procedure for estimation of $v(\hat{\beta})$ started from inequality (116), we first assume that $v(\hat{\beta}) \doteq \hat{\beta}^2 / n$, where the two dots on the equality imply very rough approximation, and $n = r$ when the failure data are not censored. Note that $v(\hat{\beta}) \doteq \hat{\beta}^2 / r$ is consistent with equation (5.37) atop page 308 of Elsayed with the element in the 2nd row and 2nd column of I^{-1} , except for the multiplier c_{22} , which for the case of $r = n$ is equal to 0.607927 from Table 5.11 on page 308 of Elsayed, reproduced below. Therefore, for the Example 5.17 data on pp. 304-307 of Elsayed, we have a better approximation for $v(\hat{\beta}) \cong c_{22} \times \hat{\beta}^2 / n = 0.608 \times (5.99697)^2 / 10 = 2.18633 \rightarrow se(\hat{\beta}) \cong 1.4786 \rightarrow cv(\hat{\beta}) = 0.246562$. Therefore, lower 95% confidence limit for the parameter β is given by $5.99697e^{-1.959964 \times 0.246562} = 3.69878$, which is fairly close to Minitab's answer of 3.65760. Another way to estimate the value of the $se(\hat{\beta})$ is simply computing the glb of the inequality (116) using the given data, which I will now proceed to do. From Eq. (116),

$$v(\hat{\beta}) \doteq \frac{1}{\frac{n}{\beta^2} + \sum_{i=1}^n E[(t_i / \theta)^\beta \times [\ln(t_i / \theta)]^2]} \doteq \frac{\hat{\beta}^2 / n}{1 + \frac{\hat{\beta}^2}{n} \sum_{i=1}^n [(t_i / \hat{\theta})^\beta \times [\ln(t_i / \hat{\theta})]^2]} =$$

Table 5.11 of E. A. Elsayed (on his page 308, 1-p = proportion of the sample that is censored; p = failed proportion)

p	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
c₁₁	1.1087	1.1517	1.2526	1.4473	1.8120	2.5102	3.9330	7.1904	16.4788	60.5171
c₂₂	0.6079	0.7670	0.9282	1.1225	1.3728	1.7162	2.22474	3.0655	4.7388	9.7447
c₁₂	0.2570	0.1764	0.0493	-0.1448	-0.4466	-0.9358	-1.7855	-3.4386	-7.3753	-22.1872

= 2.063142867; this Cramer-Rao glb (CRGLB) is very close to the actual $v(\hat{\beta}) \cong 2.18633$.

We can now use the 95% CI: $3.69878 \leq \beta \leq 9.72312$ to perform further SI on the parameter β . For example, suppose we wish to test the null 2-sided hypothesis $H_0 : \beta = 4.00$ at the preassigned LOS $\alpha = 0.05$. Since the hypothesized value of β , $\beta_0 \equiv 4$, is inside this 95% CI, then we cannot reject $H_0: \beta = 4.00$ at the % LOS. However, if we were to test $H_0 : \beta = 10.00$ at $\alpha = 0.05$, then we have sufficient evidence to reject $H_0: \beta = 10.00$ because $\beta_0 \equiv 10.00$ lies outside the 95% CI $3.0989 \leq \beta \leq 8.8950$. In other words, our 95% CI: $3.69878 \leq \beta \leq 9.72312$ has provided all possible 5%-level tests of hypotheses regarding the parameter β .

Our next objective is to use the MLE point estimator of θ , which is given by $\hat{\theta} =$

$$\left[\frac{1}{n} \sum_{i=1}^n t_i^{\hat{\beta}} \right]^{1/\hat{\beta}}, \text{ and its } se \text{ to obtain a lower one-sided 95\% CI for the characteristic life } t_c = \theta.$$

To this end, we 1st need to compute the value of the sample information on θ given by $I_{11} = I(\theta) =$

$$-E[\partial^2 L(\theta, \beta) / \partial \theta^2] = \beta(\beta + 1)\theta^{-\beta-2} E\left(\sum_{i=1}^n t_i^\beta\right) - \frac{n\beta}{\theta^2} = \frac{\beta}{\theta^2} \left[(\beta + 1)\theta^{-\beta} E\left(\sum_{i=1}^n t_i^\beta\right) - n \right]. \text{ Therefore,}$$

from Cramer-Rao's inequality the $glb[V(\hat{\theta})] = \frac{\theta^2}{\beta \left[(\beta + 1)\theta^{-\beta} E\left(\sum_{i=1}^n t_i^\beta\right) - n \right]}$. Equation (110)

shows that $\partial L(\theta, \beta) / \partial \theta = -n\beta/\theta + \beta \theta^{-\beta-1} \sum_{i=1}^n t_i^\beta \xrightarrow{\text{Set to } 0} 0$, or $-n + (\hat{\theta})^{-\hat{\beta}} \sum_{i=1}^n t_i^{\hat{\beta}} = 0$.

Inserting $(\hat{\theta})^{-\hat{\beta}} \sum_{i=1}^n t_i^{\hat{\beta}} = n$ into $\text{glb}[V(\hat{\theta})] = \frac{\theta^2}{\beta \left[(\beta + 1)\theta^{-\beta} E\left(\sum_{i=1}^n t_i^{\beta}\right) - n \right]}$, we obtain

$$v(\hat{\theta}) \doteq \frac{\hat{\theta}^2}{\hat{\beta} \left[(\hat{\beta} + 1)n - n \right]} = \frac{(\hat{\theta}/\hat{\beta})^2}{n} \quad (117a)$$

It is interesting to notice that $E \sum_{i=1}^n t_i^{\beta} = \sum_{i=1}^n E(t_i^{\beta}) = n \int_0^{\infty} t^{\beta} \frac{\beta}{\theta} (t/\theta)^{\beta-1} e^{-(t/\theta)^{\beta}} dt = n\theta^{\beta}$, which is

consistent with ML estimation. The $v(\hat{\theta})$ in Equation (117a) is identical to the term in the 1st row and column of the I^{-1} matrix given in equation (5.37) of Elsayed, except for the multiplier c_{11} , which is equal to 1.108665 listed in his Table 5.11. Thus, for the case of uncensored data (i.e., complete sample),

$$v(\hat{\theta}) \cong 1.109 \times \frac{(\hat{\theta}/\hat{\beta})^2}{n} \quad (117b)$$

For the failure data of the Example 5.17 of Elsayed, $v(\hat{\theta}) \cong 1.1087 \times (8597.6088104)^2/10 = 8195127.205356 \rightarrow se(\hat{\theta}) = 2862.7133$ hours. Hence, the value of the asymptotic 95% lower confidence limit is $\theta_L = 51559.6259832 - 1.645 \times se(\hat{\theta}) = 46850.4627 \rightarrow 46850.4627 \leq \theta < \infty$. This 95% CI provides uncountably infinite number of right-tailed tests on the parameter θ of the type $H_0: \theta = \theta_0$ versus the alternative $H_1: \theta > \theta_0$. For example, if we were to test $H_0: \theta = 50000$ hours vs $H_1: \theta > 50000$, then our CI does not provide sufficient evidence at the 5% level to reject H_0 and hence we will be unable to conclude that $\theta > 50000$. This is due to the fact that the hypothesized value $\theta_0 \equiv 50000$ hours lies inside the 95% CI: $46850.4627 \leq \theta < \infty$. On the other hand, our 95% CI will provide sufficient evidence to conclude, at the 5% level, that the value of $t_c = \theta$ exceeds 45000 hours.

Computing the Standard Error using Fisher's Information-Matrix

The Fisher's inverse of information matrix, I , for the 2-parameter Weibull is given by

$$I^{-1} \doteq \begin{bmatrix} V(\hat{\theta}) & \text{Cov}(\hat{\theta}, \hat{\beta}) \\ \text{Cov}(\hat{\theta}, \hat{\beta}) & V(\hat{\beta}) \end{bmatrix}$$

E. A. Elsayed recommends the following variance-covariance matrix in order to compute the variances of the two estimators $\hat{\theta}$ and $\hat{\beta}$

$$\hat{\mathbf{I}}^{-1} = \begin{bmatrix} \mathbf{c}_{11} \hat{\theta}^2 / (\mathbf{n} \hat{\beta}^2) & \mathbf{c}_{12} \hat{\theta} / \mathbf{n} \\ \mathbf{c}_{12} \hat{\theta} / \mathbf{n} & \mathbf{c}_{22} \hat{\beta}^2 / \mathbf{n} \end{bmatrix}. \quad (5.37 \text{ of E. A. Elsayed})$$

For example, if a sample is 70% completed to failure, or 30% censored (i.e., $p = r/n = 0.70$), then

Table 5.11 of Elsayed shows that $v(\hat{\theta}) \cong 1.4473 \times \frac{(\hat{\theta} / \hat{\beta})^2}{\mathbf{n}}$, $v(\hat{\beta}) \cong \mathbf{c}_{22} \times \hat{\beta}^2 / \mathbf{n} =$

$1.1225 \times \hat{\beta}^2 / \mathbf{n}$, and $\text{cov}(\hat{\theta}, \hat{\beta}) \cong -0.1448 \hat{\theta} / \mathbf{n}$. The above \mathbf{I}^{-1} clearly shows that the two Weibull estimators $\hat{\theta}$ and $\hat{\beta}$ are not independent, i.e., they are correlated.

However, Minitab first obtains the Local-Fisher's Info-matrix as follows:

$$\mathbf{F} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{21} & \mathbf{I}_{11} \end{bmatrix} = \begin{bmatrix} \mathbf{I}(\theta) & \mathbf{I}(\theta, \beta) \\ \mathbf{I}(\theta, \beta) & \mathbf{I}(\beta) \end{bmatrix} = \begin{bmatrix} -\mathbf{E}(\partial^2 \mathbf{L} / \partial \theta^2) & -\mathbf{E}(\partial^2 \mathbf{L} / \partial \theta \partial \beta) \\ -\mathbf{E}(\partial^2 \mathbf{L} / \partial \theta \partial \beta) & -\mathbf{E}(\partial^2 \mathbf{L} / \partial \beta^2) \end{bmatrix}$$

where $-\mathbf{E}(\partial^2 \mathbf{L} / \partial \theta \partial \beta) \cong \mathbf{n}(\hat{\beta} / \hat{\theta}) \ln(\hat{\theta}) - (\hat{\beta} / \hat{\theta}) \hat{\theta}^{-\hat{\beta}} \sum_{i=1}^{\mathbf{n}} [\mathbf{t}_i^{\hat{\beta}} \times \ln(\mathbf{t}_i)] = \hat{\mathbf{I}}_{12}$. Minitab uses the

inverse of the above approximated Fisher's matrix to obtain the *se*'s of $\hat{\theta}$ and $\hat{\beta}$, which takes the correlation between the 2 Weibull estimators into account. I will now do so for the Example 5.17 of E. A. Elsayed.

$$\hat{\mathbf{I}}_{11} \cong -[\partial^2 \mathbf{L}(\theta, \beta) / \partial \theta^2] = \mathbf{n}(\hat{\beta} / \hat{\theta})^2 = 0.06135283, \hat{\mathbf{I}}_{12} = -0.0480405433 = \hat{\mathbf{I}}_{21} \text{ and } \hat{\mathbf{I}}_{22} \cong \frac{\mathbf{n}}{\hat{\beta}^2}$$

$$+ \sum_{i=1}^{\mathbf{n}} \left[(\mathbf{t}_i / \hat{\theta})^{\hat{\beta}} \times [\ln(\mathbf{t}_i / \hat{\theta})]^2 \right] = \frac{\mathbf{n}}{\hat{\beta}^2} + \hat{\theta}^{-\hat{\beta}} \sum_{i=1}^{\mathbf{n}} \left[\mathbf{t}_i^{\hat{\beta}} \times [\ln(\mathbf{t}_i / \hat{\theta})]^2 \right] = 0.484697408; \hat{\mathbf{F}} =$$

$$\begin{bmatrix} \mathbf{0.000000135283} & -\mathbf{0.000080405433} \\ \hat{\mathbf{I}}(\theta, \beta) & \mathbf{0.484697408} \end{bmatrix}. \text{ Upon inverting this last Local-Fisher's matrix,}$$

$$\text{we obtain } \hat{\mathbf{F}}^{-1} = \begin{bmatrix} \mathbf{8200408.0038111} & \mathbf{1360.348435} \\ \mathbf{1360.348435} & \mathbf{2.288808208} \end{bmatrix}. \text{ Thus, the } se(\hat{\theta}) =$$

$\sqrt{8200408.0038111} = 2863.6354523$ and the $se(\hat{\beta}) = \sqrt{2.288808208} = 1.512881$, which are identical to those of Minitab's to 5 decimals.

The Example 5.18 on page 309 of Elsayed. This example provides a complete sample with times to failure of $n = 10$ identical units. Since the unit of TTF is not specified, I will assume the times TF were measured in hours. The failure data are 20, 22, 24, 25, 26, 27, 30, 35, 42, 52 hours, which give $mttf = 30.3000$, $S = 10.0117$, and $cv_t = 33.04\% \ll 1$. The Minitab output on page 233-234 of my notes yields $\hat{\beta} = -0.270 + 1.20/0.3304 = 3.3620$ as the initial value of $\hat{\beta}$. Next, using the MS-solver and solving (111a) and (111b) simultaneously yields $\hat{\theta} = 33.7585$ $\hat{\beta} = 3.27550$. These MLEs are consistent with those of Elsayed's listed in the middle of his page 309. Using $v(\hat{\beta}) \cong 0.6079(\hat{\beta})^2/n$ yields $v(\hat{\beta}) = 0.65232$ (Elsayed's answer is 0.6521) and the $se(\hat{\beta}) = 0.80766$. Thus, the asymptotic 95% CIs are provided in the Excel file that has already been emailed to you. Further, Elsayed uses the more exact SMD of $(c \times r)(\beta / \hat{\beta})^{(1+p^2)}$ which is $\chi_{c(r-1)}^2$, where $p =$ proportion of the sample that has failed and $(1 - p)$ represents the censored proportion of the sample), and $c = 2[(1+p^2)^2 pc_{22}]^{-1}$. Table 5.11 on page 308 of Elsayed gives the values of c_{ij} for $p = 0.10(0.10)1$. The 95% lower confidence limit for the characteristic life is given by $\hat{\theta} - 1.645se(\hat{\theta})$, where $se(\hat{\theta}) = \sqrt{c_{11} \frac{(\hat{\theta} / \hat{\beta})^2}{n}} = \sqrt{1.1087 \times \frac{(33.7585 / 3.2755032)^2}{10}} = 3.4317 \rightarrow \theta_L = 33.7585 - 1.645 \times 3.4317 = 28.1133 \rightarrow 28.1133 \leq \theta < \infty$ at the 95% confidence level.

Obtaining More Exact 95% Lower One-Sided CI for the Weibull

W(0, θ , β) Characteristic Life and Reliability Function

Since the characteristic life, $t_c = \theta$, is an LTB type population parameter, then there is absolutely no concern on the high side and hence, as before, we will obtain only a lower 95% CI for θ of the form $\theta_L \leq \theta < \infty$. As always, before experimentation θ_L is a rv (random variable), but after data have been gathered and a numerical value computed for θ_L , then the deterministic interval $\theta_L \leq \theta < \infty$ is no longer random and its pr of containing the true value of θ

reduces to 0 or 1. I will discuss the CI estimation for the uncensored sample case first. This estimation is easier than CI estimation for a censored data. We will first obtain the lower one-sided CI for θ , followed by a lower 1-sided CI for $R(t)$.

Since there does not exist a closed-form solution for the ML estimator $\hat{\theta}$, then it will always be impossible to use mathematical statistics to obtain the precise SMD of the statistic $\hat{\theta} / \theta$. However, Bain, L. J. and Engelhardt, M. (1991), 2nd Edition, New York, Marcel Dekker, used simulation to obtain the approximate percentiles of the SMD of the statistic $U = \hat{\beta} \sqrt{n} \ln(\hat{\theta} / \theta)$, which they tabulate on their page 230 for different sample sizes and confidence levels $\gamma = 1 - \alpha = 0.02, 0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95$, and 0.98 . I do not know why the tabulation was not done for $1 - \alpha = 0.005, 0.99$, and 0.995 . Since, we are interested only in developing a 95% lower one-sided CI for θ , then our interest centers only at the value of $\gamma = 1 - \alpha = 0.95$. I took Bain and Engelhardt's 95 percentiles of U , for $n = 5$ to $n = 120$, given in their Table 4A on their page 230 and used Minitab to obtain the following models for the percentiles $u_{0.95}$ and $u_{0.975}$.

$$\hat{u}_{0.95}(n) = 1.7093 + 3.8522/n - 11.3533/n^2 + 56.3650/n^3 \quad (118a)$$

$$\hat{u}_{0.975}(n) = 2.1043 + 6.3308/n - 30.918/n^2 + 153.95/n^3 \quad (118b)$$

Minitab reported an $R_{\text{Model}}^2 \geq 99.6\%$ for the above models and I tried to improve them to obtain nearly a 100% value of R_{Model}^2 by adding $1/n^4$, but the models would not improve any further. I tested the values of $\hat{u}_{0.95}(n)$ from equation (118a) against those given in equation (5.48) of Elsayed on page 322 as $U_{0.95} = 1.720 + 3.163/n + 18.25 e^{-n}$, both Elsayed's and mine agree (to 3 decimals) with those listed in Table 4A of Bain and Engelhardt (on their page 230) for $n = 5$ to $n = 120$.

The Example 5.18 on page 309 of Elsayed provides a complete sample for $n = 10$ with times to failure as 20, 22, 24, 25, 26, 27, 30, 35, 42, 52 hours (note that units are not provided in this example and I am guessing that it is hours to failure). Inserting $n = 10$ in equation (109a) gives $\hat{u}_{0.95}(10) = 2.03733$ (B&E report $U_{0.95} = 2.037$ in their Table 4A on page 230 for $n = 10$).

Therefore, the $P[\hat{\beta} \sqrt{n} \ln(\hat{\theta} / \theta) \leq 2.03733] \cong 0.95$. Rearranging the inequality inside this last brackets yields

$$\theta_L(0.95) = \hat{\theta} e^{-u_{0.95}/(\hat{\beta}\sqrt{n})} \quad (119)$$

Inserting the values of $\hat{\theta} = 33.7585$, $\hat{\beta} = 3.2755032$, and $\hat{u}_{0.95}(10) = 2.03733$ into equation (119) yields $\theta_L(0.95) = 27.7307$. Thus, we are 95% confident that the true value of θ lies in the interval $27.63201 \text{ hours} \leq \theta < \infty$. This CI is not as conservative as that of Elsayed's $26.20 \text{ hours} \leq \theta < \infty$, given on his page 313, because of the fact that Elsayed used the unbiased estimates of β and θ , which he computed to be $\hat{\beta} = 2.81$ and $\hat{\theta}_1 = 33.01$. Since his 95% lower bound for θ is more conservative, then you may wish to also use the unbiased estimates of β and θ , as he did. Using equation (119), $\theta_L(0.975) = \hat{\theta} e^{-u_{0.975}/(\hat{\beta}\sqrt{n})} = 26.30984$.

A point MLE of $R(t)$ is given by $\hat{R}(t) = e^{-(t/33.7585)^{3.2755}}$. For example, $\hat{R}(25 \text{ hours}) = 0.6881$. Due to the fact that the RE function of a Weibull is not a monotonically increasing function of the slope β (recall that the Weibull RE increases with increasing β up to the characteristic life θ and then decreases with increasing β). Hence, $R_L(t)$ will not equal to $e^{-(t/\theta_L)^{\beta_L}}$ and as a result in order to obtain a lower 95% CI for $R(25 \text{ hours})$, we need to have some idea about the SMD of $\hat{R}(25 \text{ hours})$. As stated before, the SMD of all MLEs approach normality (as $n \rightarrow \infty$) with mean equal to the corresponding parameter, namely $R(25)$, and a variance whose asymptotic value is equal to $1/I[R(t)]$. I surmise that $I[R(t)] = -E[\partial^2 L(\theta, \beta)/\partial R^2]$ cannot be computed directly. Thus, Bain and Engelhardt (p. 217) provide an empirical formula to compute the asymptotic variance of $\hat{R}(t)$, but their formula does not provide a conservative glb for $R(t)$ as shown in their Table 7A (pp. 236-243). Thus, I took the liberty, through trial/error, to provide a revised version of their formula, given below, so that the normal approximation will provide a more conservative value of $R_L(t)$ for almost all n .

$$v[\hat{R}(t)] \cong \hat{R}^2(t) \times [\ln(1/\hat{R}(t))]^2 \times \{1.70 - 0.70 \ln(\ln(1/\hat{R}(t))) + 0.70 [\ln(\ln(1/\hat{R}(t)))]^2\} / n \quad (120)$$

Inserting the value of $\hat{R}(25 \text{ hours}) = 0.6881$ into equation (120) gives $v[\hat{R}(25)] = 0.0203$, and the $se[\hat{R}(25)] = 0.142449$ and $1.645 \times se(\hat{R}(25)) = 0.23433$, and hence $R_L(25 \text{ hours}) \cong \hat{R}(25) -$

$Z_{0.05 \times se}[\hat{R}(25)] = 0.45373 \rightarrow 0.45373 \leq R(25) < 1$. Thus, we are 95% confident that the reliability function at $t = 25$ hours exceeds 0.45373.

Estimation of Weibull Parameters using Weibull Graph Paper (WGP) and Least-Squares.

Recall that because $R(t) = e^{-\left(\frac{t-\delta}{\theta-\delta}\right)^\beta}$, $t > \delta$, then $\ln[R(t)] = -\left(\frac{t-\delta}{\theta-\delta}\right)^\beta \rightarrow \ln[1/R(t)] = \left(\frac{t-\delta}{\theta-\delta}\right)^\beta \rightarrow \ln\{\ln[1/R(t)]\} = \beta \ln\left(\frac{t-\delta}{\theta-\delta}\right) \rightarrow \ln\{\ln[1/R(t)]\} = \beta[\ln(t-\delta) - \ln(\theta-\delta)] = \beta \ln(t-\delta) - \beta \ln(\theta-\delta) = \beta \ln(t-\delta) + C$, where $C = -\beta \ln(\theta-\delta)$ is a constant. Letting $y = \ln[\ln(R^{-1})]$, $x = \ln(t-\delta)$ yields $y = \beta x + C$ which represents a line with slope β and Y-intercept C . This is why the shape parameter β is also called the slope of the Weibull distribution. For the case of zero minimum life, $x = \ln(t)$ and the y-intercept is $C = -\beta \ln(\theta) = \beta \ln(1/\theta)$. Because all Weibull distributions have a RE value of e^{-1} at their characteristic life $t_c = \theta$, the Weibull graph paper (WGP) has two ordinates, where the right scale gives $\ln[\ln(R^{-1})]$ and the opposing left scale gives $F(t)$. Thus, $y = \ln[\ln(R^{-1})]$ is the right coordinate scale and its value at $t = \theta$ is equal to $y(\theta) = \ln[\ln(e^{-1})^{-1}] = \ln[\ln(e)] = \ln(1) = 0$ while the corresponding left ordinate is obtained from $0 = \ln[\ln(1/R)] = \ln\left[\ln\left(\frac{1}{1-F}\right)\right] \rightarrow \ln\left(\frac{1}{1-F}\right) = e^0 = 1 \rightarrow \frac{1}{1-F} = e \rightarrow 1-F = e^{-1} \cong 0.3679 \rightarrow F = 0.6321 \rightarrow$ The 63.21% failure point on the left scale corresponds to the characteristic life θ on the abscissa because $\ln[\ln(1/R(\theta))] = 0$ [see Figure 11.2 of Kapur and Laberson (1977) on their p. 296] reproduced on the next page. Yet as another example, $-1 = \ln[\ln(1/R)]$ on the right scale of figure 11.2 implies that $\ln\left(\frac{1}{1-F}\right) = e^{-1} = 0.3679 \rightarrow \frac{1}{1-F} = 1.4447 \rightarrow 1-F = 0.6922 \rightarrow F = 0.3078 \rightarrow$ the WGP left scale will correspond to roughly 30.78% cumulative failure, etc. Finally, the RE function of the Weibull for the same values of δ and θ is an increasing function of the slope β up to the characteristic life θ and then becomes a decreasing function of β for t values beyond θ . As an example of using the WGP to

estimate the parameters of a $W(0, \theta, \beta)$, see the Example 15.5 on pages 366-7 of Ebeling and my Excel file on my website that obtains the LSE of β followed by θ . The 95% glb on RE at 40 hours for the example 15.5 of Ebeling is roughly given by $1-0.52 = 0.48$, while we are 90% confident that 30% of the units will fail within the interval (20, 87 hours). The 5 failure-times of Ebeling's Example 5.15 are 32 hours, 51, 74, 90, and 120.

Chapter 15 Summary

1. In order to obtain MLE of Weibull β and θ , solve the two Eqs.

$$\hat{\theta} = \left[\frac{1}{n} \sum_{i=1}^n t_i^{\hat{\beta}} \right]^{1/\hat{\beta}} \quad \text{and} \quad \partial L(\theta, \beta) / \partial \beta = \frac{n}{\beta} - n \ln(\theta) + \sum_{i=1}^n \ln(t_i) - \left[\sum_{i=1}^n \frac{\partial}{\partial \beta} (t_i / \theta)^{\beta} \right]$$

$$= \frac{n}{\beta} - n \ln(\theta) + \sum_{i=1}^n \ln(t_i) - \sum_{i=1}^n \left[(t_i / \theta)^{\beta} \times \ln(t_i / \theta) \right] \xrightarrow{\text{Set to}} 0$$

simultaneously in order to obtain $\hat{\beta}$ and $\hat{\theta}$. Then, the point estimate of RE at time t

is given by $\hat{R}(t) = e^{-[t/\hat{\theta}]^{\hat{\beta}}}$. In general, the elements of Fisher's Info-matrix are $\hat{I}_{11} \cong$

$$\hat{\beta} \hat{\theta}^{-2} [(\hat{\beta} + 1) \hat{\theta}^{-\hat{\beta}} \sum_{i=1}^n t_i^{\hat{\beta}} - n], \quad \hat{I}_{12} = n / \hat{\theta} + \hat{\theta}^{-(\hat{\beta}+1)} [\hat{\beta} \ln(\hat{\theta}) - 1] \sum_{i=1}^n t_i^{\hat{\beta}} - \hat{\beta} \hat{\theta}^{-(\hat{\beta}+1)} \sum_{i=1}^n \left[(t_i^{\hat{\beta}} \times \ln(t_i)) \right],$$

$$\text{and } \hat{I}_{22} \cong \frac{n}{\hat{\beta}^2} + \sum_{i=1}^n \left[(t_i / \hat{\theta})^{\hat{\beta}} \times [\ln(t_i / \hat{\theta})]^2 \right] = \frac{n}{\hat{\beta}^2} + \hat{\theta}^{-\hat{\beta}} \sum_{i=1}^n \left[t_i^{\hat{\beta}} \times [\ln(t_i / \hat{\theta})]^2 \right]. \text{ These will}$$

reduce further for MLE, but have to be used as is for LSQ estimation.

2. Better Approximation for CI limits on β and θ when $n < 20$.

Bain and Engelhardt state that a better approximation for censored and complete

samples is that the SMD of $c \times r (\beta / \hat{\beta})^{1+p^2}$ approximately follows a χ^2 with $c(r-1)$ df,

where the constant $c = \frac{2}{p(1+p^2)^2 \times c_{22}}$. Table 5.11 on page 308 of Elsayed (page

223 of my notes) provides the values of c_{22} for different n and censoring proportion

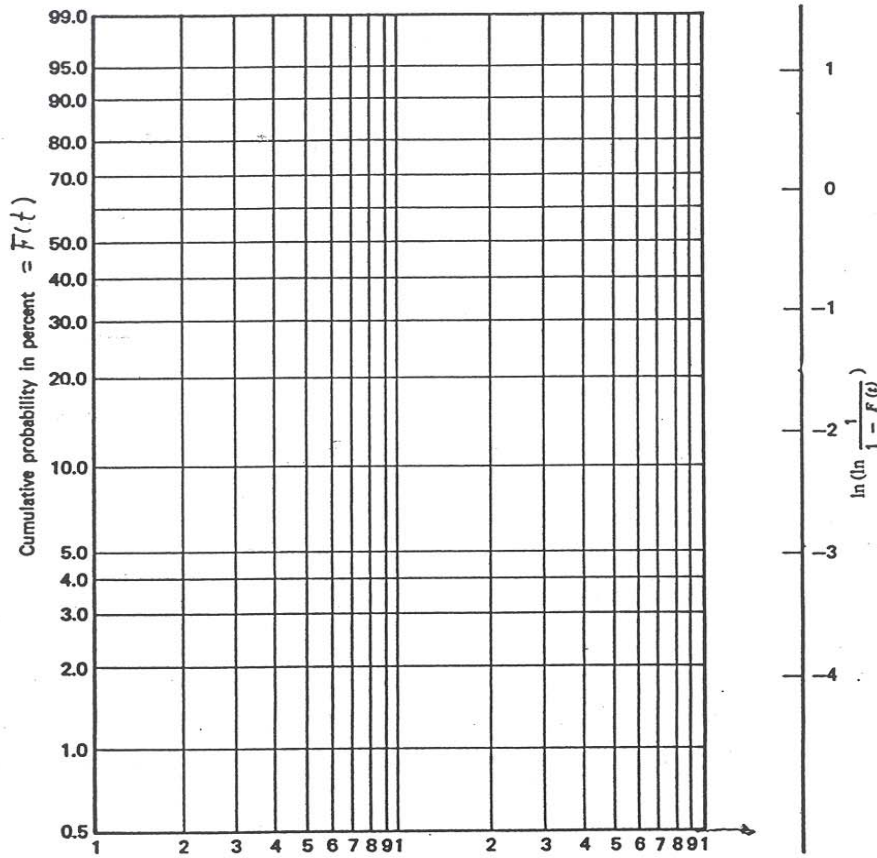


Figure 11.2 Weibull probability paper.

Figure 11.2 of Kapur and Lamberson (1977) from their page 296

Example 15.5 data : 32 hrs, 51, 74, 90, 120

296 reliability estimation: weibull distribution

page 366 of Ebeling

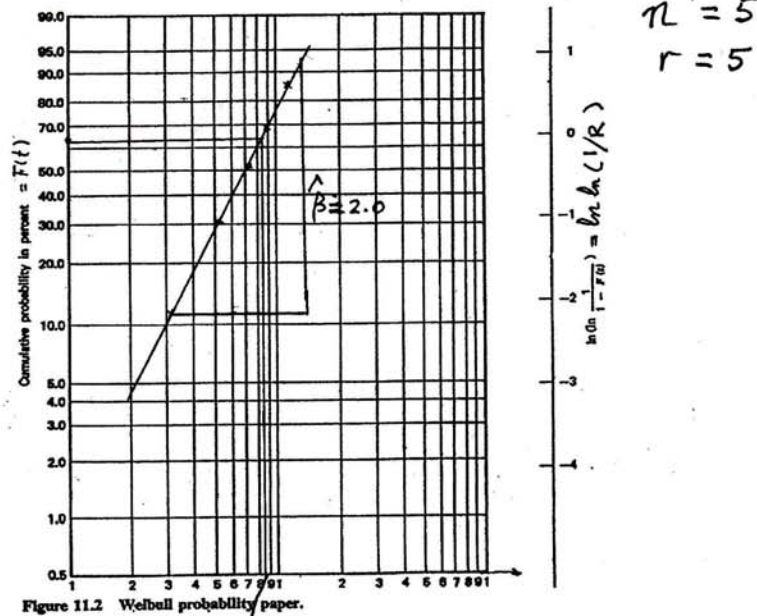


Figure 11.2 Weibull probability paper.

$\hat{\theta} \approx 85.0$ hrs

Figure 11.2 of Kapur and Lamberson (1977) from their page 296

A point estimate of RE at 80 hrs $\approx 1 - 0.60 = 0.40$, while the 90% CI for $R(80)$ is given by $0.15 \leq R(80) \leq 0.72$

$$w_{\alpha; j} = \frac{K}{K + F_{\alpha}(2n - 2j + 2, 2j)}, \quad K = j / (n - j + 1)$$

296 reliability estimation: weibull distribution Data: 32, 51, 74, 90, 120 hr

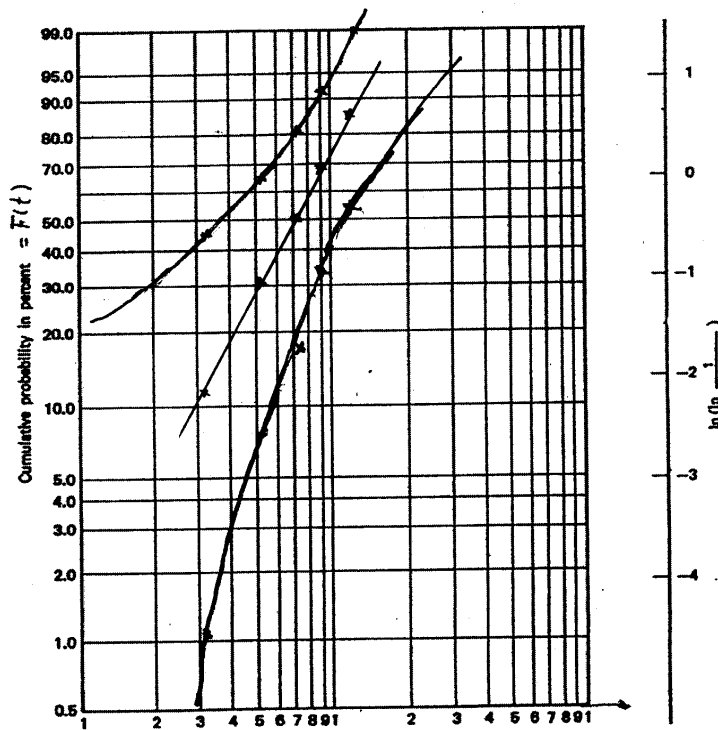


Figure 11.2 Weibull probability paper.

Example 15.5 on page 366 of Ebeling ; $n = 5$

Figure 11.2 of Kapur and Lamberson (1977) from their page 296

$$\text{At } j=1, w_{.05; 1} = \frac{0.20}{0.20 + F_{.05; 10, 2}} = \frac{0.20}{0.20 + 19.3959} = 0.0102$$

$$\text{At } j=4, w_{.95; 4} = \frac{2}{2 + F_{.95; 4, 8}} = \frac{2}{2 + 0.1655} = 0.9236$$

1 - p. For the Example 15.5 of Ebeling, n=5, t_i : 32, 51, 74, 90, 120 hr, p =1, c =

$$\frac{2}{1(1+1^2)^2 \times 0.60793} = 0.822463 \rightarrow df = 0.822463(5 - 1) = 3.289852 \rightarrow$$

$$4.112316(\beta / \hat{\beta})^2 \sim \chi_{3.3}^2 \rightarrow \chi_{0.975;3.3}^2 \approx 0.2964 \text{ and } \chi_{0.025;3.3}^2 \approx 9.8869 \rightarrow P[0.2964 \leq$$

$$4.112316(\beta / \hat{\beta})^2 \leq 9.8869] \approx 0.95 \rightarrow \beta_L \cong \hat{\beta} \sqrt{\frac{0.2964}{4.11232}} = 2.64551 \times 0.26846198 =$$

$$0.710219 \text{ and } \beta_u \cong \hat{\beta} \sqrt{\frac{9.8869}{4.11232}} = 4.101998. \hat{u}_{0.95}(n) = 1.7093 + 3.8522/n -$$

$$11.3533/n^2 + 56.3650/n^3 \rightarrow \hat{u}_{0.95}(5) = 2.476528, \theta_L(0.95) \cong \hat{\theta} e^{-u_{0.95}/(\hat{\beta}\sqrt{n})} =$$

$$82.9054 e^{-2.47653/(2.64551\sqrt{5})} = 54.54644. \text{ Note that the above } \beta_L = 0.71022 \text{ is}$$

troublesome because the baseline distribution is IFR Weibull and $\beta < 1$ implies DFR.

Minitab gives the 95% CI for β as $1.3093 \leq \beta \leq 5.34540$. From their help menu, I got the formulas

$$\beta_L \cong \hat{\beta} e^{-z_{0.025} \times cv(\hat{\beta})} \quad \text{and} \quad \beta_u \cong \hat{\beta} e^{z_{0.025} \times cv(\hat{\beta})}$$

I inverted Minitab's formulas to ascertain that the SMD of $\hat{\beta} \times \ln(\beta / \hat{\beta})$ must be approximately normal with mean zero and stdev of roughly equal to $se(\hat{\beta})$, as

depicted below. The Figure shows that the $P[-1.96 \times se(\hat{\beta}) \leq \hat{\beta} \ln(\beta / \hat{\beta}) \leq$

$1.96 \times se(\hat{\beta})] \cong 0.95$

$$P[-1.96 \times cv(\hat{\beta}) \leq \hat{\beta} \ln(\beta / \hat{\beta}) \leq 1.96 \times cv(\hat{\beta})] \cong 0.95$$

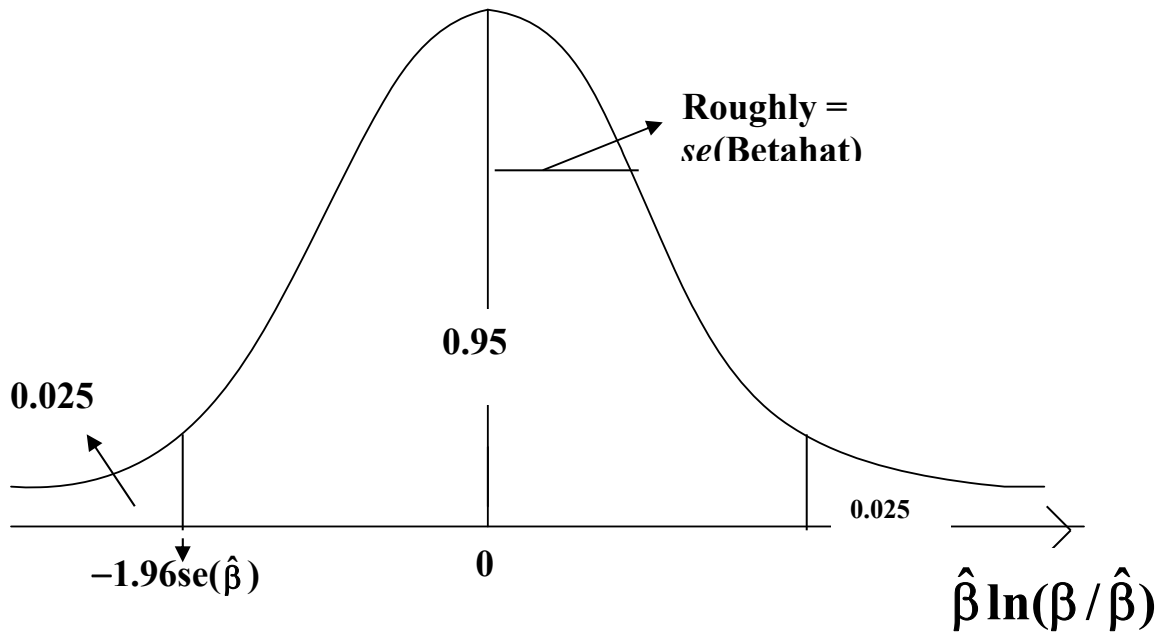
$$P[\hat{\beta} e^{-1.96 \times cv(\hat{\beta})} \leq \beta \leq \hat{\beta} e^{1.96 \times cv(\hat{\beta})}] \cong 0.95 \rightarrow \text{For the data of}$$

Example 15.5 of Ebeling, $\hat{\beta} = 2.64551$, $se(\hat{\beta}) = 0.9223 \rightarrow cv(\hat{\beta}) =$

$$0.348712 \rightarrow \beta_L \cong 2.64551 e^{-1.96 \times 0.34871} = 1.3356 \text{ and } \beta_u \cong \hat{\beta} e^{1.96 \times cv(\hat{\beta})}$$

$$\beta_u \cong 2.64551 \times e^{0.68348} = 5.2401 \rightarrow 1.3356 \leq \beta \leq 5.2401$$

Minitab's answers: $1.3093 \leq \beta \leq 5.3454$.



To obtain the 95% glb on θ , use the SMD of $U = \hat{\beta} \sqrt{n} \ln(\hat{\theta}/\theta)$ with the approximate percentiles $\hat{u}_{0.95}(n) = 1.7093 + 3.8522/n - 11.3533/n^2 + 56.3650/n^3$ and

$$\hat{u}_{0.975}(n) = 2.1043 + 6.3308/n - 30.918/n^2 + 153.95/n^3$$

For $n = 5$, $\hat{u}_{0.95}(5) = 2.47653 \rightarrow P[\hat{\beta} \sqrt{n} \ln(\hat{\theta}/\theta) \leq 2.47653] \cong 0.95 \rightarrow$

$P[\ln(\hat{\theta}/\theta) \leq 2.4765/5.9155402] \cong 0.95 ; \hat{\theta} = 82.9054 \rightarrow$

$P[(\hat{\theta}/\theta) \leq e^{0.4186431}] \cong 0.95 \rightarrow P[\hat{\theta} e^{-0.4186431} \leq \theta < \infty] = 0.95 \rightarrow$

$$\theta_L \cong \hat{\theta} e^{-0.4186431} = 54.5464 \text{ hours}$$

It turns out the 2-sided 95% CI on θ can also be obtained from a similar formula as in the case of the shape parameter β . That is,

$$\theta_L \cong \hat{\theta} e^{-z_{0.025} \times cv(\hat{\theta})} \quad \text{and} \quad \theta_u \cong \hat{\theta} e^{z_{0.025} \times cv(\hat{\theta})}$$

$$\theta_L \cong 82.9054e^{-1.96 \times 0.1780} = 58.4855 \quad \text{and} \quad \theta_u \cong 117.5167$$

To obtain the glb on $R(40 \text{ hours})$, we use the fact that for $n > 20$

$$v[\hat{R}(t)] \cong \hat{R}^2(t) \times [\ln(1/\hat{R}(t))]^2 \times \{1.70 - 0.70 \ln(\ln(1/\hat{R}(t))) + 0.70 [\ln(\ln(1/\hat{R}(t)))]^2\} / n$$

$$\hat{R}(40) \cong e^{-[(40/82.9054)^{2.64551}]} = 0.864656 \rightarrow v[\hat{R}(40)] \cong 0.031379141 \rightarrow$$

$$se[\hat{R}(40)] = 0.1771416 \rightarrow R_L(40) = 0.864656 - 1.645 \times 0.1771416 = 0.5733.$$

Table 5.11 of E. A. Elsayed (on his page 308, $1-p$ = proportion of the sample that is censored; p = failed proportion)

p	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
c ₁₁	1.1087	1.1517	1.2526	1.4473	1.8120	2.5102	3.9330	7.1904	16.4788	60.5171
c ₂₂	0.6079	0.7670	0.9282	1.1225	1.3728	1.7162	2.22474	3.0655	4.7388	9.7447
c ₁₂	0.2570	0.1764	0.0493	-1.1448	-0.4466	-0.9358	-1.7855	-3.4386	-7.3753	-22.1872

Finally, it will be best to compute the se's of the 2 Weibull estimators by inverting the estimated Fisher's Information-Matrix $\hat{I} =$

$$\begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{11} \end{bmatrix}, \text{ i.e., } \hat{I}^{-1} = \begin{bmatrix} v(\hat{\theta}) & \text{cov}(\hat{\theta}, \hat{\beta}) \\ \text{cov}(\hat{\theta}, \hat{\beta}) & v(\hat{\beta}) \end{bmatrix}; \text{ this procedure takes the}$$

correlation between $\hat{\theta}$ and $\hat{\beta}$ into account. The needed formulas for \hat{I}_{ij} are given on p. 243 of my notes. The formulas for the normally

$$\text{approximated CIs are } \beta_L \cong \hat{\beta} e^{-Z_{0.025} \times \text{cv}(\hat{\beta})}, \quad \beta_u \cong \hat{\beta} e^{Z_{0.025} \times \text{cv}(\hat{\beta})},$$

and similarly for the parameter θ .