The Two-Parameter Extreme-value Distribution

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Suppose the TTF (time to failure), T, of a component has a Weibull distribution with the cdf given by

$$\mathbf{F(t)=1-e}^{-\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta}}, \quad 0 \le t < \infty \qquad (1)$$

where δ is the minimum life, θ is the characteristic life, and β is the Weibull slope.

In equation (1), make the transformation x = $\beta\,\mathbf{ln}\left(\frac{t-\delta}{\theta-\delta}\right)$. Then the cdf of the rv X at point x is given by

$$W(x) = P(X \le x) = [\beta\,\mathbf{ln}\left(\frac{T-\delta}{\theta-\delta}\right) \le x\,] = P[\,T \le \delta+(\theta-\delta)e^{x/\beta}\,] = \mathbf{1-e}^{-\left(\frac{\delta+(\theta-\delta)e^{x/\beta}-\delta}{\theta-\delta}\right)^{\beta}}$$

$$= \mathbf{1-e}^{-(e^{x/\beta})^{\beta}} = \mathbf{1-e}^{-(e^{x})}, \quad -\infty < x < \infty \qquad (2)$$

The cdf, W(x), is the standard distribution function for an extreme-value, where by extreme-value we mean the log base e of a component life-time. To obtain the pdf of the standard extreme-value distribution, differentiate Eq. (2) with respect to x.

$$\text{pdf}(x) = w(x) = \mathbf{e}^{-(e^{x})}\,\mathbf{e}^{x}, \quad -\infty < x < \infty \qquad (3)$$

In order to convert the standard extreme-value density in (3) to the two-parameter case, let x = (y− ξ)/θ and G(y) represent the cdf of Y at y. Then, by definition,

$$G(y) = P(Y \le y) = P[(\theta X + \xi) \le y] = P[X \le (y-\xi)/\theta] = W[(y-\xi)/\theta] =$$

$$= \mathbf{1-e}^{-e^{(y-\xi)/\theta}} = \mathbf{1-e}^{-\mathrm{Exp}[(y-\xi)/\theta]}, \quad -\infty < y < \infty \qquad (4)$$

Equation (4) represents the cdf of the two-parameter extreme-value distribution. To obtain its density function g(y), differentiate equation (4) wrt y.

$$g(y) = \frac{1}{\theta}\mathbf{e}^{-e^{(y-\xi)/\theta}}\,\mathbf{e}^{(y-\xi)/\theta}, \quad -\infty < y < \infty \qquad (5)$$

The parameter ξ is a measure of location and because at y = ξ the value of the cdf in (4) becomes G(ξ) = $\mathbf{1-e}^{-e^{0}} = \mathbf{1-e}^{-1}$ = 0.6321205588, then $-\infty < \xi < \infty$ is the 63.21206 percentile of the extreme-value pdf in (5). The parameter θ > 0 is related to the spread (or standard deviation) of the density function in (5).

To obtain the inverse (or percentile) function, solve y in terms of G from equation (4), which yields

$y = \xi + \theta\times\ln[\ln(\frac{1}{1-G})] \;\rightarrow\; G^{-1}(y) = \xi + \theta\times\ln[\ln(\frac{1}{1-y})]$, $0 \le y \le 1$, i.e., the percentile

function is given by $$y_p = \xi + \theta\times\ln[\ln(\frac{1}{1-p})] \qquad (6)$$

For example, the median of the distribution is obtained by putting p = 0.50 in equation (6), i.e.,

$y_{0.50} = \xi + \theta\times\ln(\ln\frac{1}{0.5}) = \xi + \theta\times\ln(\ln 2) = \xi - 0.366513\theta$. The 25th and 75th percentiles are

$y_{0.25} = \xi + \theta\times\ln\ln(\frac{1}{1-0.25}) = \xi - 1.24590\theta$ and $y_{0.75} =$

$\xi + \theta\times\ln\ln(\frac{1}{1-0.75}) = \xi + 0.32663430\theta \;\rightarrow\;$ IQR = 1.57253360θ.

The mean of Y is given by [after the transformation $u = e^{(y-\xi)/\theta}$]

$$E(Y) = \int_{-\infty}^{\infty} y\frac{1}{\theta}e^{-e^{(y-\xi)/\theta}}e^{(y-\xi)/\theta}dy = \int_{0}^{\infty}(\xi+\theta\ln u)e^{-u}du = \xi + \theta\int_{0}^{\infty}(\ln u)e^{-u}du. \qquad (7)$$

This last integral apparently can be shown to become $\int_{0}^{\infty}(\ln u)e^{-u}du = -\gamma$ so that E(Y) = ξ − $\theta\times\gamma$, where γ = 0.5772157 is the Euler's constant. This proof is very difficult. Further, it can be shown that $V(Y) = \frac{\pi^2\theta^2}{6}$ so that $\sigma_Y = \frac{\pi\theta}{\sqrt{6}}$ = 1.28255 θ and the $CV(y) = \frac{1.282550}{\xi - 0.57721570}$.

Setting the 1st derivative of g(y) in (5) equal to zero yields the modal point of the extreme-value distribution to be MO = ξ. Since the distribution is negatively skewed ($\alpha_3 < 0$), then MO = ξ > $y_{0.50}$ = ξ − 0.3665130 > E(Y) = ξ − 0.5772157 θ. Finally, the hazard function (or failure rate function)

is given by $h(y) = \frac{f(y)}{R(y)} = \frac{\frac{1}{\theta}e^{-e^{(y-\xi)/\theta}}e^{(y-\xi)/\theta}}{e^{-e^{(y-\xi)/\theta}}} = \frac{1}{\theta}e^{(y-\xi)/\theta}$.