INSY 7380 Summer 2014 S. Maghsoodloo

The pdf of the Sum of Two Continuous Random Variables

Suppose X_1 and X_2 are irvs (independent random variables) with pfds $f_1(x_1)$ and $f_2(x_2)$ so that the joint pdf of of the vector $[X_1 \quad X_2]'$ is given by $f(x_1, x_2) = f_1(x_1)\, f_2(x_2)$. The objective is to find the pdf of $Y = X_1 + X_2$ given that $f_1(x_1)$ and $f_2(x_2)$ are known. First, let g(y) denote the pdf of Y; then g(y), which is the function we are seeking to find, is called the convolution of $f_1(x_1)$ with $f_2(x_2)$. In statistical literature this is written as $g(y) = f_1(x_1)^* f_2(x_2)$.

To this end, make the transformations $Y = X_1 + X_2$ and $W = X_2$; thus, the inverse transformations are $X_1 = Y - W$ and $X_2 = W$. Letting h(y, w) denote the joint pdf of Y and W, then

$$\iint_{R} \mathbf{f(x_1, x_2)\, dx_1\, dx_2} = \int\int_{R^*} \mathbf{f(y-w, w)|J|\, dy\, dw} = \int\int_{R^*} \mathbf{h(y, w)\, dy\, dw}$$

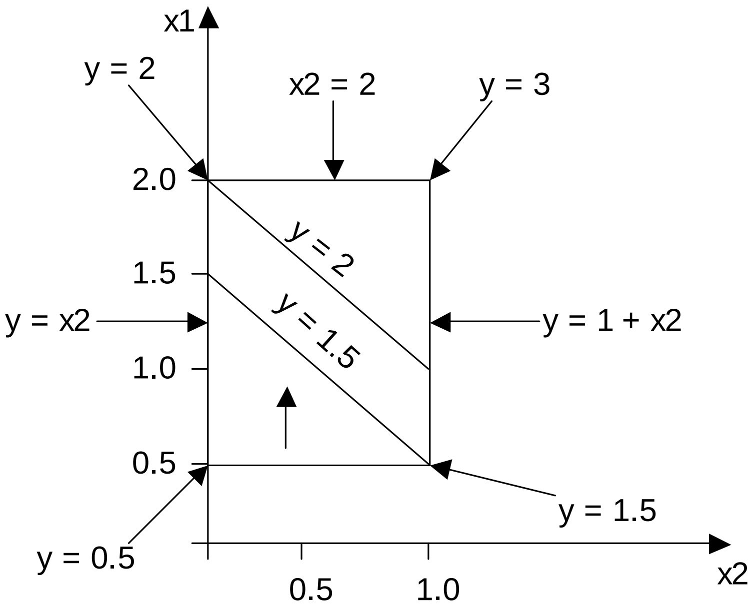
The reason behind the above equality is the fact that in general the differential of area dA = $dx_1\, dx_2$ in $x_1 - x_2$ plane does not transform to $dA^* = dy\, dw$ in the y–w coordinate system, but $dx_1\, dx_2 \rightarrow |J|\, dy\, dw$, where

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial w} \end{vmatrix}$$

is called the Jacobian determinant of the transformation. For our example, $|J| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$, and hence the $\mathbf{h(y,w)} = \mathbf{f(x_1, x_2)}\,|J| = \mathbf{f(y-w, w)|J|} = f(y - w, w)$.

Therefore, the marginal density of y is given by $g(y) = \int_{-\infty}^{\infty} \mathbf{h(y, w)\, dw} =$

$$\int_{-\infty}^{\infty} \mathbf{f(y-w, w)\, dw} = \int_{-\infty}^{\infty} \mathbf{f_1(y - x_2) f_2(x_2)\, dx_2} = \int_{-\infty}^{\infty} \mathbf{f_1(x_1) f_2(y - x_1)\, dx_1}$$



The above Figure was drawn by Luke Miller, a Ph.D. Student in ISE Department of AU, where $y = x_1 + x_2$.

<u>Example1.</u> Suppose $X_1 \sim U(0, 1)$ and $X_2 \sim U(0.50, 2)$; then $f_1(x_1) = 1$, $0 \le x_1 \le 1$ and $f_2(x_2) = 2/3$, $0.50 \le x_2 \le 2.0$. The objective is to find the convolution of $f_1(x_1)$ with $f_2(x_2)$ denoted by g(y), where $0.50 \le y \le 3$. I have shown, by drawing a figure similar to the above, that the pdf of y will be given by

$$g(y) = \int_{R_2} (2/3)\,dx_2 = \begin{cases} (2y-1)/3, & 0.50 \le y \le 1.5, \\ 2/3, & 1.50 \le y \le 2.0, \\ 2(1 - y/3), & 2.0 \le y \le 3.0, \\ 0, & \text{Elsewhere} \end{cases} = f_1(x_1)^* f_2(x_2),$$

where the integration is carried out over the range of x_2. The analytic geometry involved in developing the above convolution is extremely difficult and will be discussed in class, if necessary, for the interested reader. I do believe that integration in the x_1 direction is feasible but more difficult in order to obtain the marginal pdf of y from the joint pdf h(y,

x_1). For $0.50 \le y \le 1.5$, x_1 will range from 0 to $x_1 = y - 0.50$; for $1.50 \le y \le 2.0$, x_1 will range from 0 to 1, and for $2.0 \le y \le 3$, x_1 will range from $y - 2$ to 1.

Example2. Suppose X_1 and X_2 are irvs (independent rvs) which are Exponentially distributed at rates $\lambda_2 \ge \lambda_1$. As before, let $Y = X_1 + X_2$ and $g(y) = (\lambda_1 e^{-\lambda_1 x_1})^*(\lambda_2 e^{-\lambda_2 x_2})$. Then $g(y) = \int_0^y f_2(y - x_1) f_1(x_1)\,dx_1 =$

$$\int_0^y \lambda_2 e^{-\lambda_2 (y - x_1)} (\lambda_1 e^{-\lambda_1 x_1})\,dx_1 =$$

$$\lambda_1 \lambda_2 e^{-\lambda_2 y} \int_0^y e^{-(\lambda_1 - \lambda_2) x_1}\,dx_1 = \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)} \left[e^{-\lambda_1 y} - e^{-\lambda_2 y} \right], \; 0 \le y < \infty$$

Exercise. Obtain the above convolution again by integrating wrt x_2 and prove that the above convolution reduces to the Gamma pdf at the rate λ and $n = 2$ as $\lambda_1 \to \lambda_2 = \lambda$.