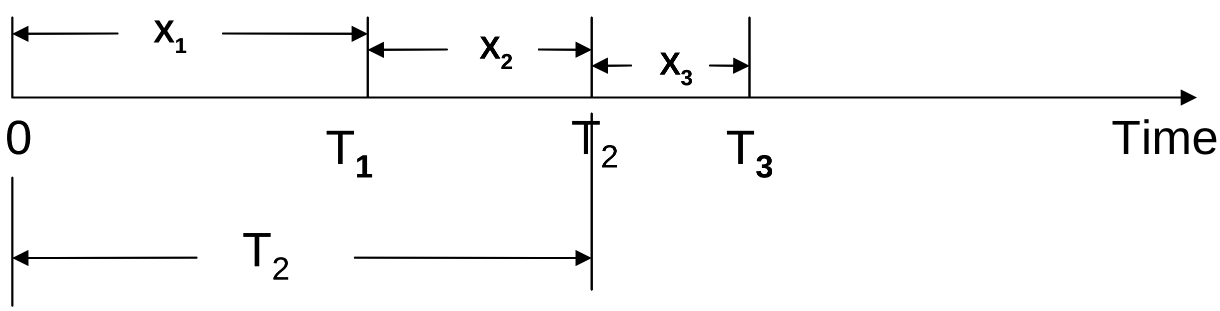
Renewal Processes and Availability Reference: Chapters 9 and 11 of Ebeling

S. Maghsoodloo

As an example, suppose a machine component is replaced ASA it fails. Let N(t) be the number of replacements during the interval [0, t] of length t. Then N(t) is called a renewal counting process, and the expected no. of failures is called the renewal function. The study of renewal processes focuses on the following topics:

(1) The pmf of N(t). (2) The expected number of renewals during [0, t], denoted by E[N(t)], where E[N(t)] = M(t) is called the renewal function. Note that Ebeling uses the notation m(t) for E[N(t)]. (3) The occurrence Pr (mass) or density function of a renewal at specific epochs of time, and (4) The time needed for the occurrence of k events (such as failures) to occur. [For more details See U. N. Bhat (1984), *Elements of Applied Stochastic Processes*, 2nd Ed. , Chapter 8.]

Suppose that failures occur at times T_k (k = 1, 2, 3, 4, …) measured from zero and assuming for the time being that replacement (or restoration time) is negligible relative to operational time, then T_k represents the operating time (measured from zero) until the k^{th} failure, where $T_0 = 0$. Because the pdf of T_1 may be different from the pdf of intervening times $X_2 = (T_2 - T_1)$, $X_3 = (T_3 - T_2)$, $X_4 = (T_4 - T_3)$, ..., we let $f_1(t)$ represent the pdf of the time to the 1^{st} failure and f(t) represent the pdf of intervening times X_2, X_3, X_4, … as depicted below.



Note that X_1, X_2, X_3, … represent intervening times between failures, while T_i represents time to the i^{th} renewal measured from zero. The above figure clearly shows that T_k (Time to the k^{th} renewal from zero) = $\sum_{i=1}^{k} X_k$ = sum of the times to the 1^{st} failure plus the intervening times of 2^{nd} failure until the k^{th} failure. If k > 30, then the central theorem states that the distribution of T_k approaches normality with mean $\mu_1 + (k-1)\mu$, where $\mu = E(X_k)$= mean time between successive renewals, k = 2, 3, 4, … and with variance $\sigma_1^2 + (k-1)\sigma^2$, where $\sigma^2 = V(X_k)$, k = 2, 3, 4, … (see pp. 194-5 of Ebeling). However, if the pdf of X_k is highly skewed and/or k is not sufficiently large, then the pdf of $T_k = \sum_{i=1}^{k} X_k$ is given by the k^{th}-fold convolution given below.

$$f_{T_k}(t) = f_1(t) * f_{(k-1)}(t)$$

where $f_{(k-1)}(t)$ is the (k – 1) convolution of the failure density f(t) with itself.

The most common counting process is the Poisson process where the intervening times are exponentially distributed at the arrival (or failure) rate λ. Because λ is a constant and intervening times are iid, a Poisson process is also referred to as a homogeneous renewal process. Then during the interval [0, t] the renewal function is $E[N(t)] = \lambda t$ and the pmf of N(t) is given by $P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$. For a homogeneous renewal process the pdf of X_1 is given by $f_1(t) = \lambda e^{-\lambda t}$, and because $f(t) = f_1(t)$ for i = 2, 3, 4, …, k then $f_{T_k}(t) = f_{(k)}(t)$, where $f_{(k)}(t)$ is the k-fold convolution of the exponential with itself and hence $f_{(k)}(t) = \frac{\lambda}{\Gamma(k)}(\lambda t)^{k-1} e^{-\lambda t}$ (the Gamma density with parameters λ and k).

The Renewal function M(t) = E[N(t)]

Because the two events $\{N(t) \geq k\}$ and $\{T_k \leq t\}$ are equivalent, i.e., the

$$P[N(t) \geq k] = P(T_k \leq t) = F_{(k)}(t)$$

where $F_{(k)}(t) = F_{(1)}(t) * F_{(k-1)}(t)$ is the k^{th}-fold convolution representing the cdf of $T_k = \sum_{i=1}^{k} X_k$.

Thus, $P[N(t) = k] = P[N(t) \geq k] - P[N(t) \geq k + 1] = F_{(k)}(t) - F_{(k+1)}(t)$. Then the renewal function is obtained as follows:

$M(t) = E[N(t)] = \sum_{n=1}^{\infty} n \times P[N(t)=n] = \sum_{n=1}^{\infty} n \times P_n(t)$, where $P_n(t) = P[N(t) = n]$.

$$M(t) = E[N(t)] = \sum_{n=1}^{\infty} P_n(t) + \sum_{n=2}^{\infty} P_n(t) + \sum_{n=3}^{\infty} P_n(t) + \sum_{n=4}^{\infty} P_n(t) + \ldots$$

$$M(t) = \sum_{n=1}^{\infty} P[N(t) \geq n] = \sum_{n=1}^{\infty} F_{(n)}(t) \quad (130)$$

Note that Ebeling uses m(t) for M(t), where m stands for the mean. You may wish to replace M(t) in Eq. (130) with $\overline{N}(t)$ if this notation $\overline{N}(t)$ would be less confusing.

As an example, if $X_1, X_2, \ldots, X_k$ are ii and exponentially distributed like $f(x) = \lambda e^{-\lambda x}$, then $f_{(n)}(t) = \dfrac{\lambda}{\Gamma(n)}(\lambda t)^{n-1}e^{-\lambda t}$ which is the n-fold convolution of f(t) with itself, and from Eq. (130) we obtain $E[N(t)] = M(t) = \sum_{n=1}^{\infty} F_{(n)}(t) = \sum_{n=1}^{\infty}\int_{x=0}^{t}\dfrac{\lambda}{\Gamma(n)}(\lambda x)^{n-1}e^{-\lambda x}dx = \int_{x=0}^{t}\sum_{n=1}^{\infty}\dfrac{\lambda}{\Gamma(n)}(\lambda x)^{n-1}e^{-\lambda x}dx =$ $\int_{x=0}^{t}\lambda e^{-\lambda x}\sum_{n=1}^{\infty}\dfrac{(\lambda x)^{n-1}}{(n-1)!}dx = \int_{x=0}^{t}\lambda e^{-\lambda x}e^{\lambda x}dx = \int_{x=0}^{t}\lambda dx = \lambda t$ = Mean number of Poisson occurrences (or failures) during an interval of length t, as expected.

Unfortunately, obtaining a closed-form expression for the renewal function is not as simple as the case of exponential interarrival times. For the sake of illustration, consider, Examples 9.5 and 9.6 on pp. 195 and 196 of Ebeling where a cutting tool has a N(5 hours, 1) TTF distribution. The objective is to compute the renewal function E[N(12 hours)] = M(12 hours). Then the P_0= Pr[N_f(12 hours)= 0] = Pr(TTF_1 > 12) = Pr(Z > 7) = $\Phi(-7) = 1.279812543886\times10^{-12}$; $P_1(12)$ = Pr[N_f(12 hours) =1] = Pr[N_f(12 hours) ≥ 1] − Pr[N_f(12 hours)≥2] = Pr(TTF_1 ≤ 12) − Pr(TTF_2 ≤ 12) = $\Phi(7) - \Pr(Z \leq \frac{12-10}{\sqrt{2}})$ = 0.99999999999872 −0.92135039647486 = 0.07864960352386

P_2 = Pr[N_f(12 hours)= 2] = Pr[N_f(12 hours) ≥ 2] − Pr[N_f(12 hours) ≥ 3] = Pr(TTF_2 ≤ 12) − Pr(TTF_3 ≤ 12) = $\Pr(Z \leq \frac{12-10}{\sqrt{2}}) - \Pr(Z \leq \frac{12-15}{\sqrt{3}})$ =0.92135039647486 − 0.04163225833178 = 0.87971813814308; P_3 = Pr[N_f(12 hours)= 3] = Pr[N_f(12 hours) ≥ 3] − Pr[N_f(12 hours) ≥ 4] = Pr(TTF_3 ≤ 12) − Pr(TTF_4 ≤ 12) = $\Pr(Z \leq \frac{12-15}{\sqrt{3}}) - \Pr(Z \leq \frac{12-20}{\sqrt{4}})$ = 0.04163225833178 − 0.00003167124183312 = 0.04160058708995; P_4 = Pr[N_f(12 hours)=4] = Pr[N_f(12 hours) ≥ 4] − Pr[N_f(12 hours) ≥ 5] = Pr(TTF_4 ≤ 12) − Pr(TTF_5 ≤ 12) =

$\Pr(Z \le \frac{12-20}{\sqrt{4}}) - \Pr(Z \le \frac{12-25}{\sqrt{5}}) = 0.00003167124183312 - 3.053943179854770 \times 10^{-9} =$

0.00003166818789; $P_5 = \Pr[N_f(12 \text{ hours})=5] = \Pr[N_f(12 \text{ hours}) \ge 5] - \Pr[N_f(12 \text{ hours}) \ge 6] =$

$\Pr(TTF_5 \le 12) - \Pr(TTF_6 \le 12) = \Pr(Z \le \frac{12-25}{\sqrt{5}}) - \Pr(Z \le \frac{12-30}{\sqrt{6}}) = 3.053943179854770 \times 10^{-9} -$

$1.002448040140151 \times 10^{-13} = 3.053842935050756 \times 10^{-9} \rightarrow E[N_f(12)] = M(12) = \sum_{n=0}^{\infty} n \times P_n(12) =$

1.96301433, which agrees with Ebeling's answer atop page 196 to 2 decimals $\rightarrow$ The approximate value of renewal intensity $\rho(12) \approx M(12)/12 = 1.96301433/12 = 0.16358453$ per hour [see Eqs. (9.11 & 9.12) on page 196 of Ebeling]. In other words, the MTBF $\approx$ 12 hours/M(12) = 12/1.96301433 = 6.1130476 hours, or $M(12) \cong t/MTTF = 12/5 = 2.4$ failures which is not close to 1.963014 because t = 12 hours is too short. If we consider t = 24 hours of operations, then M(24) = 4.311779321 which much closer to 24/5 = 4.8. Note that I am using the same notation $\rho(t) = dM(t)/dt$ as Ebeling for the renewal intensity function, while some authors use m(t), both of which are common notation for renewal intensity. As yet another good example, see Example 9.6 atop page 197 of Ebeling.

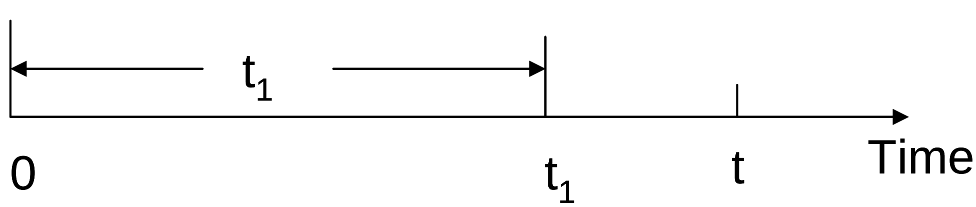
Suppose now that the 1st failure occurs at $T_1 = X_1 = t_1$; then, the renewal function must satisfy the following relationship:

$$M(t) = E[N(t)] = \int_0^{\infty} E[N(t) | X_1 = t_1) f(t_1) dt_1 , \qquad t_1 < t$$

Clearly, if $t_1 > t$, then no failures have occurred by t and thus E[N(t)] = 0, as depicted below. Therefore, $E[N(t) | X_1 = t_1) = 1 + E[N(t - t_1)]$ and hence M(t) = E[N(t)] =

$$\int_0^t \{1 + E[N(t-t_1)]\} f(t_1) dt_1 = F_1(t) + \int_0^t E[N(t-t_1)] f(t_1) dt_1 \rightarrow$$

$$M(t) = E[N(t)] = F_1(t) + \int_0^t M(t-t_1) f(t_1) dt_1 \qquad (131)$$



Eq. (131) is called the Renewal Equation and in simple cases (i.e., in cases when the integrand on the RHS can be made free of t) can be used to obtain the renewal function M(t). For example, suppose the time to the 1st failure of a component is uniformly distributed over [500, 1000 hours], and the succeeding failure have the same pdf; then $F_1(t) = (t-500)/500$ and $f(t_1) = 1/500$, $500 \le t_1 \le 1000$ hours. Upon substitution into (131), we obtain

$M(t) = E[N(t)] = \dfrac{t-500}{500} + \int_0^t M(t-t_1)dt_1 / 500$; letting $t - t_1 = x$ yields

$$M(t) = \frac{t-500}{500} + \frac{1}{500}\int_t^0 M(x)(-dx) = \frac{t-500}{500} + \frac{1}{500}\int_0^t M(x)dx \tag{132}$$

The above Eq. (132) shows that the density of the Renewal Function, ρ(t), is given by

$$\rho(t) = \frac{\partial M(t)}{\partial t} = \frac{1}{500} + \frac{M(t)}{500} \rightarrow \frac{dM(t)}{dt} - \frac{M(t)}{500} = \frac{1}{500} \rightarrow$$

$$\frac{dM(t)}{dt}e^{-t/500} - \frac{M(t)}{500}e^{-t/500} = \frac{1}{500}e^{-t/500} \rightarrow \frac{d}{dt}[M(t)e^{-t/500}] = \frac{1}{500}e^{-t/500} \rightarrow M(t)e^{-t/500}$$

$= \int \frac{1}{500}e^{-t/500}dt + C \rightarrow M(t) = (-e^{-t/500}) \times e^{t/500} + C\,e^{t/500} = C\,e^{t/500} - 1$; applying the boundary condition $M(t = 0) = 0 \rightarrow 0 = C - 1 \rightarrow C = 1 \rightarrow M(t) = e^{t/500} - 1$; for example, the expected number of renewals (or failures) during [600, 900 hours] is given by $M(300) = e^{300/500} - 1 = 1.8221 - 1 = 0.8221$, while the expected number of failures during the mission time [500, 1000 hours] is given by $M(500) = e^1 - 1 = 1.7183$. Note that the renewal intensity $\rho(t) = \frac{1}{500}e^{t/500}$ is not a pdf; in fact its integral over [0, ∞) diverges in this case. The renewal intensity, ρ(t), gives the instantaneous renewal rate at time t, i.e., $\rho(t) = \lim_{\Delta t \to 0}\frac{M(t+\Delta t) - M(t)}{\Delta t}$ so that $\rho(t)\times\Delta t$ gives the Pr element of a renewal during the interval (t, t+Δt). When $M(t) = E[N(t)] = F_1(t) + \int_0^t M(t-t_1)f(t_1)dt_1$ depends on both t and t_1 (where t_1 is the TTFF), then the above procedure applied in the case of the uniform distribution for the time to the 1st failure (TTFF) will not work because the renewal intensity, ρ(t), will still have an integral on the RHS of the equation, and hence we will have to resort to Laplace Transforms described below. In the case of the Poisson process $\rho(t) = \lambda$.

Let f(t), not necessarily a pdf, be any function whose range space is [0, ∞); then the Laplace transform of f(t) is defined as $\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$, $s > 0$. For example, the Laplace transform of f(t) = 1 is given by $\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$; note that some authors use θ in lieu of s, while I may use them interchangeably. Further, $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\,dt = \left[\frac{-te^{-st}}{s}\right]_0^\infty +$

$$\frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s^2};\ \mathcal{L}\{t^n\} = \int_0^\infty e^{-st}t^n\,dt = \left[\frac{-t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt = \ldots = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{-\lambda t}\} = \int_0^\infty e^{-st}e^{-\lambda t}dt = \left[\frac{-e^{-t(s+\lambda)}}{s+\lambda}\right]_0^\infty = \frac{1}{s+\lambda},\ \lambda > 0;\ \mathcal{L}\{df(t)/dt\} = \int_0^\infty e^{-st}f'(t)dt = \left[e^{-st}f(t)\right]_0^\infty$$

$$+ s\int_0^\infty e^{-st}f(t)dt = -f(0) + s\bar{f}(s) == sf^*(s) - f(0);\quad \mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-(s-a)t} f(t)dt = \bar{f}(s-a).$$

$$\mathcal{L}\{\int_o^t f(x)dx\} = \int_0^\infty e^{-st}\int_0^t f(x)dx\,dt = \left[\frac{-e^{-st}}{s}\int_0^t f(x)dx\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f(t)dt = 0 + \bar{f}(s)/s = \bar{f}(s)/s$$

$$\mathcal{L}\{\frac{1}{a-b}(e^{-bt} - e^{-at})\} = \int_0^\infty e^{-st}\frac{1}{a-b}(e^{-bt} - e^{-at})dt = \int_0^\infty \frac{1}{a-b}[e^{-(b+s)t} - e^{-(a+s)t}]dt =$$

$\frac{1}{a-b}[\frac{1}{(b+s)} - \frac{1}{(a+s)}] = \frac{1}{(s+a)(s+b)}$; Table 7 gives a summary of basic Laplace transforms.

Properties of Laplace Transforms

(1) The Laplace transform $\mathcal{L}$ is a linear operator because if C_1 and C_2 are any constants, it can easily be verified that $\mathcal{L}\{C_1f_1(t) + C_2f_2(t)\} = C_1\,\mathcal{L}\{f_1(t)\} + C_2\,\mathcal{L}\{f_2(t)\}$. **(2)** Let $f^*g = \int_0^t f(t-x)g(x)dx$ be the convolution of f(t) with g(t). Then it can be proven that $\mathcal{L}\{f^*g\} = \bar{f}(s)\bar{g}(s)$, where $\bar{g}(s) = \mathcal{L}\{g(t)\}$.

Table 7. A summary of $L\{f(t)\} = \bar{f}(s)$ for different useful f(t).

Function f(t), or $L^{-1}\{\bar{f}(s)\}$	Laplace Transform $L\{f(t)\} = \bar{f}(s)$
λ	λ/s
t	$1/s^2$
t^n	$n!/s^{n+1}$
$e^{-\lambda t}$	$\frac{1}{s+\lambda}$
$\frac{1}{\lambda}(1-e^{-\lambda t})$	$\frac{1}{s(s+\lambda)}$
$\frac{1}{\lambda^2}(e^{-\lambda t}+\lambda t-1)$	$\frac{1}{s^2(s+\lambda)}$
$\frac{1}{a-b}(e^{-bt}-e^{-at})$	$\frac{1}{(s+a)(s+b)}$
$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
$\frac{t^{n-1}e^{-\lambda t}}{(n-1)!}$	$\frac{1}{(s+\lambda)^n}$
$\frac{(n-1)-\lambda t}{(n-1)!}t^{n-2}e^{-\lambda t}$	$\frac{s}{(s+\lambda)^n}$
df/dt	$s\bar{f}(s) - f(0)$
d^2f/dt^2	$s^2\bar{f}(s) - sf(0) - df(0)/dt$
f(at)	$\bar{f}(s/a)/a$ f
$\frac{1}{a^2}(e^{-at}+at-1)$	$1/[s^2(s+a)]$
$\frac{1}{a^2}[(1/a)-t+(at^2/2)-e^{-at}/a]$	$1/[s^3(s+a)]$
$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$
$e^{-\lambda t}f(t)$	$\bar{f}(s+\lambda)$

Now let $f_1(t)$ be the pdf of the time to the 1st failure (TTFF) and $\bar{f}_1(s)$ be its Laplace transform, i.e.,

$\bar{f}_1(s) = L\{f_1(t)\} = \int_0^\infty e^{-st} f_1(t)dt$ = The Laplace transform of the TTFF. Similarly, let f(t) be the lifetime density between the 1st & 2nd, 2nd & 3rd, … renewals (or failures). Then, $\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$. In the special case $f_1(t) = f(t)$, then $\bar{f}_1(s) = \bar{f}(s)$. Similarly, let $L\{M(t)\} = \int_0^\infty e^{-st} M(t)dt = \bar{M}(s)$ and $L\{\rho(t)\} = \int_0^\infty e^{-st}\rho(t)dt = \bar{\rho}(s)$. Thus, $\bar{\rho}(s) =$

$$\int_0^\infty e^{-st}\rho(t)dt = \int_0^\infty e^{-st}[dM(t)/dt]dt = \int_0^\infty e^{-st}\frac{d}{dt}\sum_{n=1}^\infty F_{(n)}(t)\,dt = \sum_{n=1}^\infty \int_0^\infty e^{-st} f_{(n)}(t)dt = \sum_{n=1}^\infty \bar{f}_{(n)}(s) =$$

$$= \sum_{n=1}^\infty E[e^{-(X_1+X_2+...+X_n)s}] = \sum_{n=1}^\infty \bar{f}_1(s)\bar{f}^{\,n-1}(s) = \frac{\bar{f}_1(s)}{1-\bar{f}(s)}.$$

Because $\rho(t) = dM(t)/dt$ and $M(t) = \int_0^t \rho(x)dx$, then its Laplace transform $\bar{\rho}(s) = -M(0) + s\bar{M}(s) = -0 + s\bar{M}(s) = s\bar{M}(s)$, as the expected number of renewals during an interval of length 0 must be zero $\rightarrow \bar{M}(s) = \frac{\bar{\rho}(s)}{s} \rightarrow \bar{M}(s) = \frac{\bar{f}_1(s)}{s[1-\bar{f}(s)]}$.

As an example, suppose $f_1(t) = f(t) = \lambda e^{-\lambda t}$; then $\bar{f}(s) = \int_0^\infty e^{-st}\lambda e^{-\lambda t}dt = \frac{\lambda}{s+\lambda} \rightarrow \bar{M}(s) =$

$$\frac{\lambda/(s+\lambda)}{s[1-\lambda/(s+\lambda)]} = \frac{\lambda}{s[(s+\lambda)-\lambda]} = \frac{\lambda}{s^2} \rightarrow M(t) = E[N(t)] = L^{-1}\{\bar{M}(s)\} = L^{-1}\{\frac{\lambda}{s^2}\} = \lambda L^{-1}\{\frac{1}{s^2}\} =$$

λt (which implies $\rho(t) = \lambda$), as expected. As yet another example, suppose the TTF of a machine component has the gamma pdf at the rate of $\lambda = 0.003$ and n = 2; we wish to obtain the expression for the renewal function M(t) and to compute the expected number of failures during t =1000 hours of operations.

$$\bar{f}(s) = \int_0^\infty e^{-st}\lambda(\lambda t)e^{-\lambda t}dt = \lambda^2\int_0^\infty te^{-(\lambda+s)t}dt = \lambda^2\frac{\partial}{\partial s}\int_0^\infty -e^{-(\lambda+s)t}dt = \lambda^2\frac{\partial}{\partial s}\left[\frac{e^{-(\lambda+s)t}}{\lambda+s}\right]_0^\infty =$$

$$\lambda^2\frac{\partial}{\partial s}[\frac{-1}{\lambda+s}] = \frac{\lambda^2}{(\lambda+s)^2} \rightarrow \bar{M}(s) = \frac{\lambda^2/(s+\lambda)^2}{s[1-\lambda^2/(s+\lambda)^2]} = \frac{\lambda^2}{s[(s+\lambda)^2-\lambda^2]} = \frac{\lambda^2}{s^2[s+2\lambda]}.$$

I will now use the partial fraction techniques to write this last $\bar{M}(s)$ in such a manner that we can recognize its inverse Laplace transform (L^{-1}) a bit easier. $\bar{M}(s) = \frac{\lambda^2}{s^2[s+2\lambda]} = \lambda^2(\frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+2\lambda}) =$

$\lambda^2[\frac{-1/(4\lambda^2)}{s} + \frac{1/(2\lambda)}{s^2} + \frac{1/(4\lambda^2)}{s+2\lambda}] \rightarrow \bar{M}(s) = \frac{-1}{4s} + \frac{\lambda}{2s^2} + \frac{1}{4(s+2\lambda)} \rightarrow M(t) = E[N(t)] =$

$L^{-1}\{\bar{M}(s)\} = L^{-1}\{\frac{-1}{4s} + \frac{\lambda}{2s^2} + \frac{1}{4(s+2\lambda)} = -\frac{1}{4} + \frac{\lambda}{2}t + \frac{1}{4}e^{-2\lambda t}$; thus, during 1000 operating hours we expect $M(1000) = -\frac{1}{4} + \frac{0.003}{2}1000 + \frac{1}{4}e^{-6} = 1.2506$ failures; note that if the above process lifetime were exponential at the rate $\lambda = 0.003$, then $M(1000) = \lambda t = 3$. The renewal intensity for the above gamma lifetime is $\rho(t) = dM/dt = \frac{\lambda}{2} - \frac{\lambda}{2}e^{-2\lambda t}$.

As yet another example, suppose the time to the 1st failure of a new machine is exponential at the rate $\lambda_1 = 0.0005$/hour, but the succeeding renewals occur at the constant rate $\lambda_2 = 0.001$; this is called a modified renewal process because the TTFF(Time To first Failure) has a different distribution from succeeding failures. We wish to compute the expected number of failures during t = 10,000 hours of operations.

$\bar{f}_1(s) = \int_0^{\infty} e^{-st}\lambda_1 e^{-0.0005t}dt = \frac{\lambda_1}{s+\lambda_1}$; similarly, $\bar{f}(s) = \int_0^{\infty} e^{-st}\lambda_2 e^{-\lambda_2 t}dt = \frac{\lambda_2}{s+\lambda_2} \rightarrow \bar{M}(s) =$

$\frac{\lambda_1/(s+\lambda_1)}{s[1-\lambda_2/(s+\lambda_2)]} = \frac{\lambda_1(s+\lambda_2)}{s[(s+\lambda_1)(s+\lambda_2)-\lambda_2(s+\lambda_1)]} = \frac{\lambda_1(s+\lambda_2)}{s^2(s+\lambda_1)} = \frac{\lambda_1-\lambda_2}{\lambda_1 s} + \frac{\lambda_2}{s^2} + \frac{\lambda_2-\lambda_1}{\lambda_1(s+\lambda_1)}$

$\rightarrow M(t) = E[N(t)] = L^{-1}\{\bar{M}(s)\} = L^{-1}\{\frac{\lambda_1-\lambda_2}{\lambda_1 s} + \frac{\lambda_2}{s^2} + \frac{\lambda_2-\lambda_1}{\lambda_1(s+\lambda_1)}\} = \frac{\lambda_1-\lambda_2}{\lambda_1} + \lambda_2 t +$

$\frac{\lambda_2-\lambda_1}{\lambda_1}e^{-\lambda_1 t} = \frac{0.0005-0.001}{0.0005} + 10 + e^{-5} = 9.00674$ failures.

Some Limiting Results For Renewal Processes

The most important limiting result in renewal theory is the fact that

$$\lim_{t \to \infty} \rho(t) = 1/\mu$$

which states in the limit one cannot identify when the renewal process began so that over the long-term the rate at which components are replaced is inversely proportional to the average time, μ, between renewals. Further, because $dM(t)/dt = \rho(t) \cong \frac{M(t+\Delta t) - M(t)}{\Delta t} \rightarrow M(t+\Delta t) - M(t) \cong$ $m(t)\Delta t$ $\rightarrow$ The expected number of renewals during (t, t+Δt) is approximately equal to $\rho(t)\Delta t =$ $\Delta t/\mu$. This implies that the expected number of renewals during an interval of length t is roughly given by $M(t) \cong t/\mu$, this approximation improving as $t \rightarrow \infty$. For the example involving gamma pdf with $\lambda = 0.003$ and n = 2, MTBF = μ = 2/0.003 = 666.6667; thus during 1000 hours of operation, $M(t) \cong t/\mu$ = 1000/666.6667 = 1.5000 (not very close to the exact M(t = 1000) = 1.2506 because 1000 hours is too short). However, at t = 10,000 hours, M(10000) = E[N(10000 hours)] = $-\frac{1}{4} + \frac{0.003}{2}10000 + \frac{1}{4}e^{-2\lambda(10000)}$ = 14.7500 failures, and $M(t) \cong t/\mu$ = 10000/666.6667 = 15 failures (much better approximation). As yet another example, see Example 6.9 on page 197 of Ebeling.

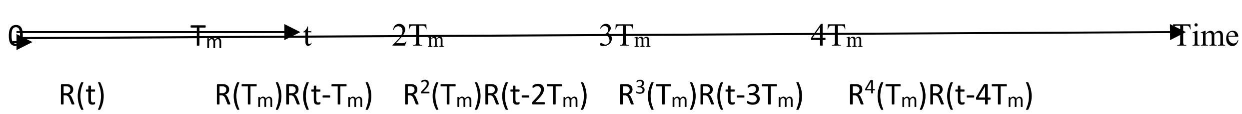
Section 9.5 of Ebeling (The Reliability function under preventive maintenance)

Ebeling (2nd Ed. pp. 237 to 241) has an excellent discussion as how preventive maintenance (PM) can improve system RE iff the hazard function h(t) is increasing. Suppose PM is performed periodically every T_m days. We further assume that once preventive maintenance is performed on a system, then the system is practically as-good-as-new. There are two MUEX (mutually exclusive) possibilities:

(1) The system fails by T_m, (2) the system survives beyond at least one cycle of length T_m. These 2 possibilities lead to the following RE function with PM:

$$R_m(t) = \begin{cases} R(t), & 0 \le t < T_m \\ R(T_m)^n R(t - nT_m), & nT_m \le t < (n+1)T_m \end{cases} . \qquad \text{(9.26 of Ebeling)}$$

Note that Ebeling uses T for T_m in his Eq. (9.26) to represent the length of one PM cycle. The argument for the above RE function is depicted in the following figure.



The MTTF of a PM system is given by (m standing for with maintenance)

$\text{MTTF}_{\text{Pm}} = \int_0^{\infty} R_m(t)dt = \sum_{n=0}^{\infty} \int_{nT_m}^{(n+1)T_m} R(T_m)^n R(t - nT_m)dt = \sum_{n=0}^{\infty} R(T_m)^n \int_{nT_m}^{(n+1)T_m} R(t - nT_m)dt$

In the above integral, put x = t– nT_m. This leads to

$$\text{MTTF}_{\text{Pm}} = \sum_{n=0}^{\infty} R(T_m)^n \int_0^{T_m} R(x)dx = \frac{\int_0^{T_m} R(t)dt}{1 - R(T_m)} \qquad \text{(Eq. 9.27 of Ebeling)}$$

Ebeling provides a good example of an exponential lifetime with PM in his Example 9.17 on p. 238 in which he illustrates that PM does not alter R(t) iff h(t) = λ is a constant. Note that if we compute the MTTF_{Pm} using Eq. (9.27) under constant h(t), we obtain $\text{MTTF}_{\text{PM}} = \frac{\int_0^{T_m} e^{-\lambda t}dt}{1 - e^{-\lambda T_m}} = \frac{\frac{-1}{\lambda}\left[e^{-\lambda t}\right]_0^{T_m}}{1 - e^{-\lambda T_m}} =$

$\frac{\frac{-1}{\lambda}\left[e^{-\lambda T_m} - 1\right]}{1 - e^{-\lambda T_m}} = 1/\lambda$, i.e., for an exponential TTF the MTTF with and without PM are identically equal to $1/\lambda$. For another excellent example, see the Example 9.18 on p. 238 of Ebeling. For this Example, I am changing the value of t_c for the compressors from 100 days to 120 days so that now TTF ~ W(0, θ =120 days, shape= β =2); further, I am changing the maintenance cycle to one month = 30 days (25% of θ). Thus, the RE function with PM is given

by $R_m(t) = \begin{cases} R(t),\ 0 \le t < 30 \text{ days} \\ R(30)^n R(t - 30n),\ 30n \le t < 30(n+1) \end{cases}$, n = 0, 1, 2, 3, 4, …. The value of RE at 160 days from time zero is computed first by recognizing that n = 5 PM cycles and that 165 days lies within the interval (5×30, 6×30), and hence t = 10 days; thus,

$R_m(160) = R(30)^5 R(160 - 150),\ 150 \le t < 180 \text{ days}$. That is, PM improves RE by 329.88%. The

MTTF(W/O PM) = 106.347 days versus MTTF(with PM) $= \text{MTTF}_{\text{Pm}} = \frac{\int_0^{T_m} R(x)dx}{1 - R(T_m)} =$

$\frac{\int_0^{30} e^{-(x/120)^2}dx}{1 - 0.9394131} = \frac{\int_0^{\sqrt{2}/4} e^{-z^2/2}(120dz/\sqrt{2})}{0.0605869}$, where $x/120 = z/\sqrt{2}$. Thus, MTTF_{Pm}

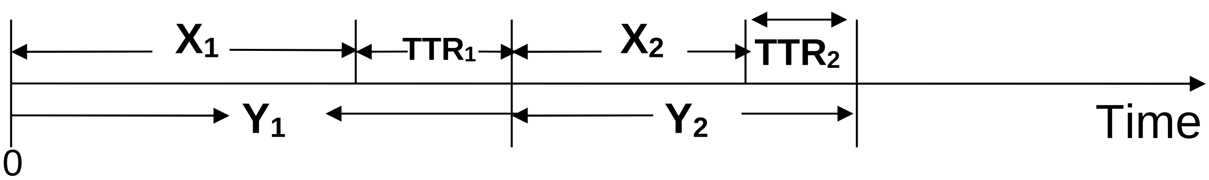
$= \dfrac{120\sqrt{\pi}\int_0^{\sqrt{2}/4} e^{-z^2/2}(dz/\sqrt{2\pi})}{0.0605869} = \dfrac{120\sqrt{\pi}[\Phi(\sqrt{2}/4)-0.50]}{0.0605869} = \dfrac{120\sqrt{\pi}(0.1381632)}{0.0605869}$ = 485.03136,

which is an improvement of 356.083% in MTTF due to PM.

Ebeling also discusses the case when PM induces failure into the system with a Pr, p, and the RE function with PM is given atop page 240 in his Eq. (9.28). See his Example 9.19 on p.

Alternating (or Cyclical) Renewal Processes (see pp. 234-235 of Ebeling)

So far we have assumed that the repair time has been negligible compared to operating times so that E[N(t)] is simply $\sum_{n=1}^{\infty} F_{(n)}(t)$, where $F_{(n)}(t)$ is the cdf of the time to the n^{th} renewal measured from zero. Or, E[N(t)] = M(t) = $\mathcal{L}^{-1}\{\bar{M}(s)\}$ = $\mathcal{L}^{-1}\{\dfrac{\bar{f}_1(s)}{s[1-\bar{f}(s)]}\}$. Suppose now active restoration times TTR_1, TTR_2, TTR_3,… (TTR = Time to Repair) are not negligible but are rvs with identical pdfs $g_r(t)$, r for repair. Then time to the 1^{st} cycle (or renewal) now is given by $Y_1 = X_1 + TTR_1$; time to the 2nd cycle (measured from the last) now is given by $Y_2 = X_2 + TTR_2$; time to the 3^{rd} renewal period is given by $Y_3 = X_3 + TTR_3$, and so on, as depicted below. A network exhibits such a cyclic behavior between up and down states, but generally down states include administrative and logistic times in addition to active repair time.



Assuming that X_i's and TTR_i's are each iid rvs with pdfs f(x) and $g_r(\xi)$, then Y_i's are iid rvs with the cdf W(t) = $P(Y_i \le t)$ and pdf $w_y(t)$. Clearly $w_y(t)$ is the convolution of f(x) with $g_r(\xi)$, i.e., $w_y(t)$ = $f(x)*g_r(\xi) = \int_0^t f(x)g_r(t-x)dx = \int_0^t f(t-\xi)g_r(\xi)d\xi$, where ξ = TTR stands for active repair time. Then the Laplace transform of $w_y(t)$ is given by $\mathcal{L}\{w_y(t)\} = \mathcal{L}\{f(x)\}\times\mathcal{L}\{g_r(\xi)\} = \bar{f}(s)\bar{g}(s)$. Thus the Laplace transform of the renewal function and renewal density that involves maintenance are given by

$$\bar{M}(s) = \frac{\bar{f}(s)\bar{g}_r(s)}{s[1-\bar{f}(s)\bar{g}_r s)]} \quad \text{and} \quad \bar{\rho}(s) = \frac{\bar{f}(s)\bar{g}_r(s)}{1-\bar{f}(s)\bar{g}_r s)} \qquad (133a)$$

As an example, suppose a machine component TTF is exponential at the rate λ and its repair time is also exponential at the rate λ_r, i.e., $g_r(t) = \lambda_r e^{-\lambda_r t}$. Thus, $\bar{f}(s) = \frac{\lambda}{s+\lambda}$ and $\bar{g}_r(s) = \frac{\lambda_r}{s+\lambda_r} \rightarrow$

$$\bar{M}(s) = \frac{\bar{f}(s)\bar{g}_r(s)}{s[1-\bar{f}(s)\bar{g}_r s)]} = \frac{\lambda\lambda_r}{s[(s+\lambda)(s+\lambda_r)-\lambda\lambda_r]} = \frac{\lambda\lambda_r}{s^2[s+(\lambda+\lambda_r)]} = \frac{-\lambda\lambda_r}{s(\lambda+\lambda_r)^2} + \frac{\lambda\lambda_r}{s^2(\lambda+\lambda_r)} +$$

$$\frac{\lambda\lambda_r}{(s+\lambda+\lambda_r)(\lambda+\lambda_r)^2} \rightarrow M(t) = \mathcal{L}^{-1}\{\bar{M}(s)\} = \mathcal{L}^{-1}\{\frac{-\lambda\lambda_r}{s(\lambda+\lambda_r)^2} + \frac{\lambda\lambda_r}{s^2(\lambda+\lambda_r)} +$$

$$\frac{\lambda\lambda_r}{(s+\lambda+\lambda_r)(\lambda+\lambda_r)^2}\} = \frac{-\lambda\lambda_r}{(\lambda+\lambda_r)^2} + \frac{\lambda\lambda_r t}{(\lambda+\lambda_r)} + \frac{\lambda\lambda_r}{(\lambda+\lambda_r)^2} e^{-(\lambda+\lambda_r)t} \qquad (133b)$$

For example, if λ = 0.0005 and λ_r = 0.05 per hour, then the expected number of cycles (or renewals) in 10000 hours from Eq. (133b) is equal to $M(10000) = \frac{-(0.0005)(0.05)}{(0.0505)^2} + \frac{(0.000025)10000}{0.0505}$ $+ \frac{0.25}{(0.0505)^2} e^{-(0.0505)10000} = 4.9406920890$. The instantaneous renewal rate is given by $\rho(t) = \frac{\lambda\lambda_r}{(\lambda+\lambda_r)} - \frac{\lambda\lambda_r}{(\lambda+\lambda_r)} e^{-(\lambda+\lambda_r)t}$, and its limiting behavior is $\lim_{t\to\infty} \rho(t) = \frac{\lambda\lambda_r}{(\lambda+\lambda_r)} = \frac{1}{1/\lambda_r + 1/\lambda} = \frac{1}{MTTR + MTTF}$, where MTTR stands for the component's mean time to repair. Thus, in the limit the mean time between cycles is given by MTBC = $1/\left(\lim_{t\to\infty} \rho(t)\right)$ = MTTF + MTTR; for the above example, MTBC = 1/0.05+1/.0005= 2020 hours, while if the machine component were irreparable (was just merely replaced), then MTBC = MTBF = 1/0.0005 = 2000 hours. Note that Eqs. (133a &b) are valid only for a single component, and in order to use them for a system, then $\bar{f}(s)$ and $\bar{g}_r(s)$ have to represent the Laplace transforms of $f_{Sys}(t)$ and system TTR density $g_{Sys}(t)$. In short, these Eqs. are either for a single component or a single system taken as an ensemble. In general, obtaining the renewal function for a repairable system, i.e., E[no. of cycles during an interval of length t], is much more difficult than a single component or machine and sometimes only an approximation is possible.

Instantaneous (or Point) Availability of a component

By (instantaneous) availability at time t, A(t), we mean the Pr that a component is functioning reliably at time t. Thus, if there is no repair, the availability function is simply A(t) = R(t). However, if the component is repairable, then there are two mutually exclusive possibilities: (1) The component is reliable at t, in which case $A_1(t) = R(t)$, (2) the component fails at time x, $0 < x < t$, gets renewed in the interval $(x, x+\Delta x)$ with Pr element m(x)dx, and then is reliable from time x to t − x (Trivedi, 1982). This second Pr is given by $A_2(t) = \int_0^t \rho(x)dx\, R(t-x)$. Because the above two cases are mutually exclusive, then

$$A(t) = A_1(t) + A_2(t) = R(t) + \int_0^t \rho(x)dx\, R(t-x) \qquad (134a)$$

Taking Laplace transform of Eq. (134a) yields $\bar{A}(s) = \bar{R}(s) + \bar{R}(s)\,\bar{\rho}(s) = \bar{R}(s)\,[1+\bar{\rho}(s)] \rightarrow$

$$\bar{A}(s) = \bar{R}(s)\,[1+\frac{\bar{f}(s)\bar{g}_r(s)}{1-\bar{f}(s)\bar{g}_r(s)}] = \frac{\bar{R}(s)}{1-\bar{f}(s)\,\bar{g}_r(s)}. \qquad (134b)$$

For the above example where TTF has a constant failure rate λ and time to repair is also exponential at the rate $r = \lambda_r$, $\bar{R}(s) = \int_0^\infty e^{-\lambda t}e^{-st}dt = 1/(\lambda+s)$; $\bar{f}(s) = \int_0^\infty \lambda\, e^{-\lambda t}e^{-st}dt = \lambda/(\lambda+s)$; $\bar{g}_r(s) = \int_0^\infty \lambda_r\, e^{-\lambda_r t}e^{-st}dt = \lambda_r/(\lambda_r+s)$. Hence, from (134b) $\bar{A}(s) = \frac{1/(\lambda+s)}{1-[\lambda/(\lambda+s)][\lambda_r/(\lambda_r+s)]} =$

$\frac{\lambda_r+s}{(\lambda+s)(\lambda_r+s)-\lambda\lambda_r} = \frac{\lambda_r+s}{s[s+(\lambda+\lambda_r)]} = \frac{\lambda_r}{s(\lambda+\lambda_r)} + \frac{\lambda/(\lambda+\lambda_r)}{s+\lambda+\lambda_r} \rightarrow A(t) = \mathcal{L}^{-1}\{\bar{A}(s)\} =$

$\mathcal{L}^{-1}\{\frac{\lambda_r}{s(\lambda+\lambda_r)} + \frac{\lambda/(\lambda+\lambda_r)}{s+\lambda+\lambda_r}\} = \frac{\lambda_r}{(\lambda+\lambda_r)} + \frac{\lambda_r}{(\lambda+\lambda_r)}\, e^{-(\lambda+\lambda_r)t}$. For example, given that λ = 0.0005 and $\lambda_r = 0.05$ per hour, then the Pr that the component is available (i.e., not under repair) at t = 1000 hours is given by $A(1000) = \frac{0.05}{0.0505} + \frac{0.0005}{0.0505}\, e^{-0.0505(1000)} = 0.990099009901.$

More Availability Measures

(1) Steady-State (Inherent or Intrinsic) Availability

By intrinsic availability (A_I) we mean the limiting value of A(t) as $t \rightarrow \infty$, i.e., availability over the long haul. Thus $A_I = A(\infty) = \lim_{t \to \infty} A(t)$. For the case of constant failure and repair rates, $A_I = A(\infty)$

$= \lim_{t \to \infty} A(t) = \lim_{t \to \infty} [\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}] = \frac{\lambda_r}{\lambda + \lambda_r} = \frac{1/\mu_r}{1/\mu + 1/\mu_r}$, where $\mu_r = 1/\lambda_r$ represents

mean repair time , or MTTR. Thus, $A_I = \frac{1/\mu_r}{1/\mu + 1/\mu_r} = \frac{\mu}{\mu_r + \mu} = \frac{MTBF}{MTTR + MTBF}$, where

MTTR includes only active repair time.

(2) By interval availability (IA) we mean the proportion of the time within the mission interval (t_1, t_2) the system is expected (on the average) to be in the success mode. Thus, $IA = A(t_1, t_2) = \frac{\int_{t_1}^{t_2} A(t)dt}{t_2 - t_1}$. For the exponential failures and repairs the interval

availability $A(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}]dt = \frac{\lambda_r}{\lambda + \lambda_r} +$

$$\frac{\lambda/(t_2 - t_1)}{(\lambda + \lambda_r)^2}[e^{-(\lambda+\lambda_r)t_1} - e^{-(\lambda+\lambda_r)t_2}]. \qquad (135)$$

For the interval (100, 200 hours) availability from Eq. (135) we obtain A(100, 200) =

$\frac{1}{100} \int_{100}^{200} [\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}]dt = \frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda/(100)}{(\lambda + \lambda_r)^2}[e^{-(\lambda+\lambda_r)100} - e^{-(\lambda+\lambda_r)200}]$; when λ = 0.0005 and the repair rate r = 0.05, then this IA becomes A(100, 200) = 0.99011149545.

A special case of IA(t) is the average availability over the interval [0, T] given by

$$\bar{A}(T) = A(0, T) = \frac{1}{T} \int_0^T A(t)dt$$

For the example with $\lambda = 0.0005$ and $\lambda_r = 0.05$ per hour, the average availability over the interval [0, 1000 hours] is given by $\bar{A}(1000) = \frac{1}{1000} \int_0^{1000} [\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}]dt = 0.99029506911$.

(3) By long-term availability, $\bar{A}$, we mean the proportion of the times a system is available and considers only MTBF and MDT (mean down time) but excludes mean idle time. Thus $\bar{A}$ =

$$\underset{T \to \infty}{\text{Limit}} \ \overline{A}(T) = \frac{MTBF}{MTBF + MDT} \le A_I$$ because MDT is generally larger than MTTR(mean active repair time). Downtime involves both administrative (or logistic) time and active repair time.

Achieved Availability

We mean the proportion of the times a system is available and considers only MTBM and MMDT (mean maintenance down time) but excludes mean idle time. The time horizon is depicted atop the next page in Figure 20. The achieved availability is defined in Eq. (11.6) on page 255 of Ebeling as

$$\overline{\mathbf{A}}_{\mathbf{a}} = \frac{\mathbf{MTBM}}{\mathbf{MTBM + MMDT}} \qquad (11.6Ep255)$$

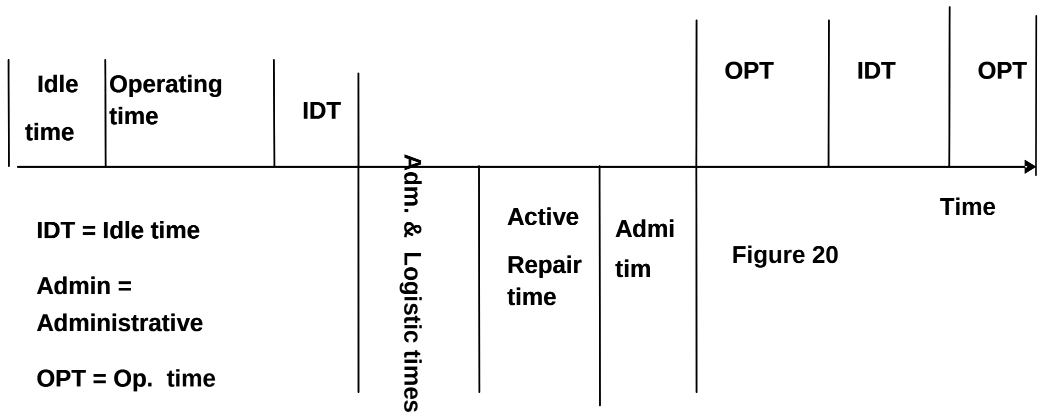


Figure 20

where MMDT = $\overline{M}$ stands for mean maintenance downtime that includes both corrective and preventive maintenance downtimes. For example, suppose a certain maintenance on a car occurs every 5000 miles and the value of MMDT is 8 hours. Then, assuming that the car averages 12000 miles/year, then the MTBM = $\frac{5000 \text{ miles}}{12000/\text{year}}$ = 0.4167 year = 0.4167year×(365×24 hours/year) = 3650 hours → $\overline{A}_a$ = 3650/(3650 +8) = 0.9978130126.

Example 19. Suppose a system of 5 components fails a total of 6 times and goes under repair a total of 6 times in 300 days of operations. The first failure occurs at TTF_1 = 55.3 days and repair on it starts immediately with TTR_1 = 1.3 days; the 2nd failure occurs at TTF_2 = 85 days (measured from zero) with TTR_2 = 0.90 days; the 3rd failure occurs at TTF_3 = 165 days with TTR_3 = 1.4 days; the 4th failure occurs at 205 days with TTR_4 = 1.5 days; the 5th failure occurs at 220 hours with TTR_5 = 1.8, and the 6th failure occurs at 260 hours (from zero) with TTR_6 = 1.6 days. Note that

Our objective is to estimate the MTBF and MTTR.

$$\text{MT}\hat{\text{B}}\text{F} = \frac{[55.3+(85-55.3-1.3)+(165-85-0.9)+(205-165-1.4)+(220-205-1.5)+(260-220-1.8)]}{6} = 42.1833 \text{ days}$$

and $\text{MTT}\hat{\text{R}}$ = (1.3+0.9+1.4+1.5+1.8+1.6)/6 = 1.4167 days. Assuming roughly constant failure and repair rates, then $\hat{\lambda}_f$ = 1/(42.1833*24) = 0.000987752/hour and $\hat{r} = \hat{\lambda}_r$ = 1/(1.4167*24) = 0.029411765/hour, and $A_I = \dfrac{\lambda_r}{\lambda + \lambda_r} \approx$ 0.96751.

To further illustrate the exact difference between R(t) and A(t), consider another system of N = 12 new components that are placed in service at time 0. Table 8 shows their TTFs and TTRs and the corresponding estimates of R(t) and A(t). Note that for a repairable system A(t) ≥ R(t) for all t. Further, in Table 8 $N_s(t)$ stands for number of components surviving at t and $N_A(t)$ stands for number of components available for service at time t, and the last two columns give $\hat{R}(t)$ and $\hat{A}(t)$.

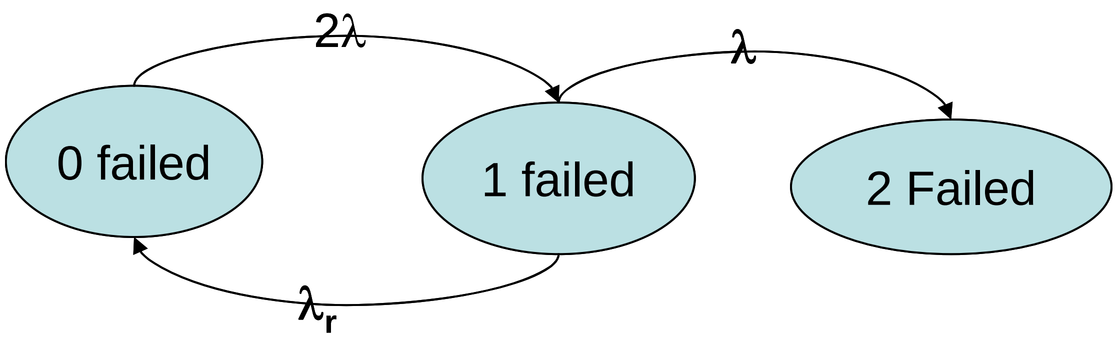
Table 8. [t = time; $\hat{R}(t) = N_s(t)/N$; $\hat{A}(t) = N_A(t)/N$]

t in days	N	TTF_i	TTR_i	$N_s(t)$	$N_A(t)$	$\hat{R}(t)$	$\hat{A}(t)$
0	12			12	12	1	1
200	12	250 days	3 days	12	12	1	1
400	12	598	5	11	12	11/12	12/12
600	11	795	8	10	11	10/12	11/12
800	11	980	4	9	11	9/12	11/12
1000	12	1130	6	8	12	8/12	12/12
1200	12			7	12	7/12	12/12

Section 9.6 of Ebeling

Figure 9.5 on page 207 of Ebeling describes one cycle of a 2-component repairable pure-parallel system where state 3 is absorbing. For convenience I have modified Ebeling's Figure 9.5 as follows, where λ_r represents repair rate and state 2 represents system failure. From the modified Figure 9.5 we can deduce that

$$\begin{cases} P_0(t+\Delta t) = P_0(t)(1-2\lambda\Delta t) + P_1(t)\lambda_r\Delta t \\ P_1(t+\Delta t) = P_0(t)(2\lambda\Delta t) + P_1(t)(1-\lambda_r\Delta t - \lambda\Delta t) \\ P_2(t+\Delta t) = P_1(t)(\lambda\Delta t) \end{cases}$$



After transposing the pertinent pr functions from the RHS to the LHS, dividing by Δt and taking the limit as $\Delta t \to 0$ result in the following system of differential equations:

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + \lambda_r P_1(t) \\ dP_1(t)/dt = 2\lambda P_0(t) - (\lambda_r + \lambda)P_1(t) \\ dP_2(t)/dt = \lambda P_1(t) \end{cases}$$

Because $P_0(t) + P_1(t) + P_2(t) = 1$, then $dP_2(t)/dt = -\, dP_1(t)/dt \,-\, dP_0(t)/dt$ $\to$ The above system of differential equations can be reduced to

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + \lambda_r P_1(t) \\ dP_1(t)/dt = 2\lambda P_0(t) - (\lambda_r + \lambda)P_1(t) \end{cases}$$

Taking Laplace transforms, we obtain

$$\begin{cases} s\overline{P}_0(s) - P_0(0) = -2\lambda\overline{P}_0(s) + \lambda_r\overline{P}_1(s) \\ s\overline{P}_1(s) - P_1(0) = 2\lambda\overline{P}_0(s) - (\lambda_r + \lambda)\overline{P}_1(s) \end{cases}$$

Applying the boundary conditions $P_0(0) = 1$ and $P_1(0) = 0$, we obtain

$$\begin{cases} s\overline{P}_0(s) - 1 = -2\lambda\overline{P}_0(s) + \lambda_r\overline{P}_1(s) \\ s\overline{P}_1(s) = 2\lambda\overline{P}_0(s) - (\lambda_r + \lambda)\overline{P}_1(s) \end{cases} \to \begin{cases} (s+2\lambda)\overline{P}_0(s) - \lambda_r\overline{P}_1(s) = 1 \\ 2\lambda\overline{P}_0(s) - (s+\lambda_r+\lambda)\overline{P}_1(s) = 0 \end{cases} \to$$

$$\overline{P}_0(s) = \frac{\begin{vmatrix} 1 & -\lambda_r \\ 0 & -(s+\lambda_r+\lambda) \end{vmatrix}}{\begin{vmatrix} (s+2\lambda) & -\lambda_r \\ 2\lambda & -(s+\lambda_r+\lambda) \end{vmatrix}} = \frac{(s+\lambda_r+\lambda)}{(s+\lambda_r+\lambda)(s+2\lambda)-2\lambda\lambda_r} = \frac{s+\lambda_r+\lambda}{(s-u_1)(s-u_2)} = \frac{a}{s-u_1} +$$

$\frac{b}{s-u_2}$, where u_1 and u_2 are the roots of the quadratic $s^2+(\lambda_r+3\lambda)s+2\lambda^2=0$,

$$u_i = \frac{-(\lambda_r+3\lambda)\pm\sqrt{(\lambda_r+3\lambda)^2-8\lambda^2}}{2} = \frac{-(\lambda_r+3\lambda)\pm\sqrt{\lambda_r^2+6\lambda\lambda_r+\lambda^2}}{2} \rightarrow$$

$a = \frac{\lambda+\lambda_r+u_1}{\sqrt{\lambda_r^2+6\lambda\lambda_r+\lambda^2}} = \frac{\lambda+\lambda_r+u_1}{u_1-u_2}$; $b = -\frac{\lambda+\lambda_r+u_2}{u_1-u_2}$; note that both u_1 and $u_2 < 0$.

Thus, $P_0(t) = L^{-1}\{\overline{P}_0(s)\} = L^{-1}\{\frac{a}{s-u_1} + \frac{b}{s-u_2}\} = a e^{u_1 t} + b e^{u_2 t} \rightarrow$

$dP_0(t)/dt = a u_1 e^{u_1 t} + b u_2 e^{u_2 t}$; because $dP_0(t)/dt + 2\lambda P_0(t) = \lambda_r P_1(t) \rightarrow$

$a u_1 e^{u_1 t} + b u_2 e^{u_2 t} + 2\lambda(a e^{u_1 t} + b e^{u_2 t}) = \lambda_r P_1(t) \rightarrow a(u_1+2\lambda)e^{u_1 t} + b(u_2+2\lambda)e^{u_2 t} = \lambda_r P_1(t)$; but a

$= \frac{(-\lambda+\lambda_r+\sqrt{\lambda_r^2+6\lambda\lambda_r+\lambda^2})/2}{u_1-u_2}$ and $u_1+2\lambda = \frac{(\lambda-\lambda_r+\sqrt{\lambda_r^2+6\lambda\lambda_r+\lambda^2})/2}{u_1-u_2} \rightarrow$

$a(u_1+2\lambda) = \frac{2\lambda\lambda_r}{u_1-u_2}$; similarly, $b(u_2+2\lambda) = \frac{2\lambda\lambda_r}{u_2-u_1} \rightarrow$

$P_1(t) = \frac{2\lambda}{u_1-u_2}e^{u_1 t} - \frac{2\lambda}{u_1-u_2}e^{u_2 t} \rightarrow R(t) = P_0(t) + P_1(t) = \frac{u_2}{u_2-u_1}e^{u_1 t} + \frac{u_1}{u_1-u_2}e^{u_2 t}$; Note that the intrinsic availability $A_I = A(\infty) = 0$ because state 2 (i.e., 2 units failed) is absorbing and the system eventually has to transition to state 2 plus the fact that once we are in state 2 no repair is done. The system MTTF can be obtained by integrating the RE function from 0 to ∞, which Ebeling does near the bottom of page 208 in his Eq. (9.33) as $MTTF_{Sys} = \frac{3\lambda+\lambda_r}{2\lambda^2}$. If $\lambda = 0.0005$ and $\lambda_r = 0.05$ per hour,

$MTTF_{Sys} = \frac{0.0015+0.05}{2(0.0005)^2} = 103000$ hours so that the system's effective failure rate is $\lambda_{Sys} = 1/MTTF_{Sys}$ $= 0.00000970874$ per hour. It is clear that as a system becomes more complex, obtaining its transient

RE function is very difficult and time-consuming because nearly always one has to solve a system of differential equations simultaneously, as was done above. Due to the fact that A_I, $MTTF_{Sys}$ and effective failure rate λ_{Sys} provide nearly all the information about a system that is needed, we may obtain $MTTF_{Sys}$ by using Markovian analysis as in Chapter 6. To compute the $MTTF_{Sys}$ for a 2-unit repairable system, we have to assume that the state 2 is absorbing so that we may obtain the fundamental matrix N. Figure 9.5E of Ebeling shows that the transition pr $P_{01} = (2\lambda)\,\Delta t$ and thus $P_{00} = 1 - (2\lambda)\,\Delta t$. Similarly, $P_{10} = \lambda_r\,(\Delta t)$, $P_{12} = \lambda\,(\Delta t)$, and hence $P_{11} = 1 - \lambda_r\,(\Delta t) - \lambda\,(\Delta t) = 1 - (\lambda_r + \lambda)\Delta t$. If we take Δt equal to 1 unit of time, then our one-step transitional pr matrix for a duration Δt is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1-2\lambda & 2\lambda & 0 \\ \lambda_r & 1-\lambda_r-\lambda & \lambda \\ 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Making state 2 absorbing, this last matrix reduces to

$$\tilde{P} = \begin{array}{c} 2 \\ 0 \\ 1 \end{array}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1-2\lambda & 2\lambda \\ \lambda & \lambda_r & 1-\lambda-\lambda_r \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ R & Q \end{bmatrix} \rightarrow Q = \begin{bmatrix} 1-2\lambda & 2\lambda \\ \lambda_r & 1-\lambda-\lambda_r \end{bmatrix} \rightarrow$$

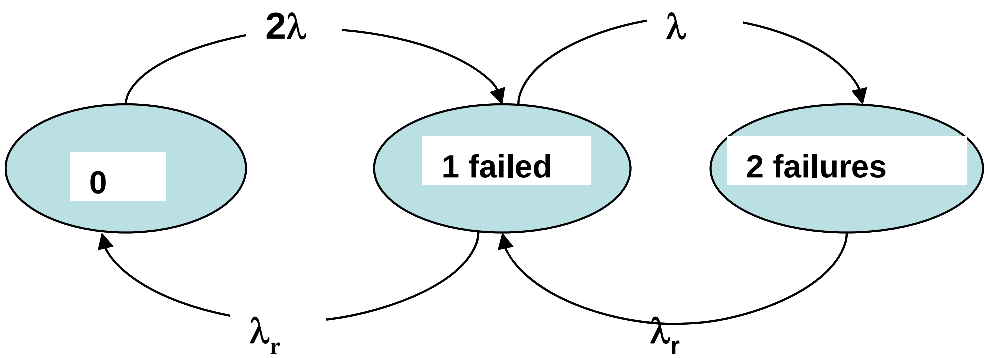
Thus, the matrix N $=(I_2 - Q)^{-1} = \begin{bmatrix} 2\lambda & -2\lambda \\ -\lambda_r & \lambda+\lambda_r \end{bmatrix}^{-1} = \dfrac{1}{2\lambda^2}\begin{bmatrix} \lambda+\lambda_r & 2\lambda \\ \lambda_r & 2\lambda \end{bmatrix}$.

Given that the system starts in state 0, then the mean time to failure is given by $MTTF_0 =$ $\dfrac{(\lambda+\lambda_r)+2\lambda}{2\lambda^2} = \dfrac{3\lambda+\lambda_r}{2\lambda^2} = \dfrac{3\times 0.0005+0.05}{2(0.0005)^2}$ = 103000 hours, and $MTTF_1 = \dfrac{2\lambda+\lambda_r}{2\lambda^2}$ = 102000 hours.

The material on pages 104-105 of my notes may be used to compute the variance of TTF. However, the process can be made a renewal one by either adding on-line or off-line restoration as depicted below. The equilibrium (i.e., as $t \to \infty$) TRS equations for the on-line restoration case are $2\lambda\pi_0 = \lambda_r \pi_1 \to \pi_1 = (2\lambda/\lambda_r)\,\pi_0$; $\lambda\pi_1 = \lambda_r\,\pi_2 \to \pi_2 = \lambda\pi_1/\lambda_r = 2(\lambda/\lambda_r)^2\,\pi_0$; because $\pi_0 + \pi_1 + \pi_2 = 1$, then $\pi_0 + (2\lambda/\lambda_r)\,\pi_0 + 2(\lambda/\lambda_r)^2\,\pi_0 = 1 \to \pi_0 = 1/[1+(2\lambda/\lambda_r) + 2(\lambda/\lambda_r)^2]$ and $\to \pi_1 =$ $(2\lambda/\lambda_r)/[1+(2\lambda/\lambda_r) + 2(\lambda/\lambda_r)^2]$. Thus the intrinsic availability is given by $A_I = \pi_0 + \pi_1 =$

$\dfrac{1+(2\lambda/\lambda_r)}{1+(2\lambda/\lambda_r)+2(\lambda/\lambda_r)^2} = \dfrac{\lambda_r^2+2\lambda\lambda_r}{\lambda_r^2+2\lambda\lambda_r+2\lambda^2}$. For example, if $\lambda = 0.0005$ and $\lambda_r = 0.05$, then $A_I =$

$\dfrac{\lambda_r^2+2\lambda\lambda_r}{\lambda_r^2+2\lambda\lambda_r+2\lambda^2} = \dfrac{(0.05)^2+2(0.000025)}{0.0025+0.00005+2(0.0005)^2} = 0.99980396001 \rightarrow$ this implies that the



The State-TRD for a Two-unit Parallel System with On-line one-unit-at-a-time Restoration

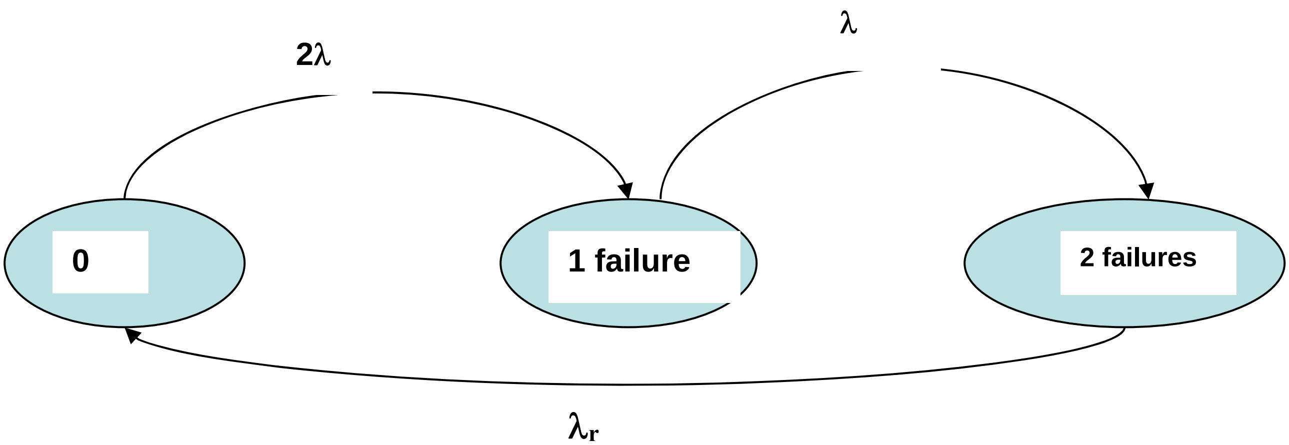
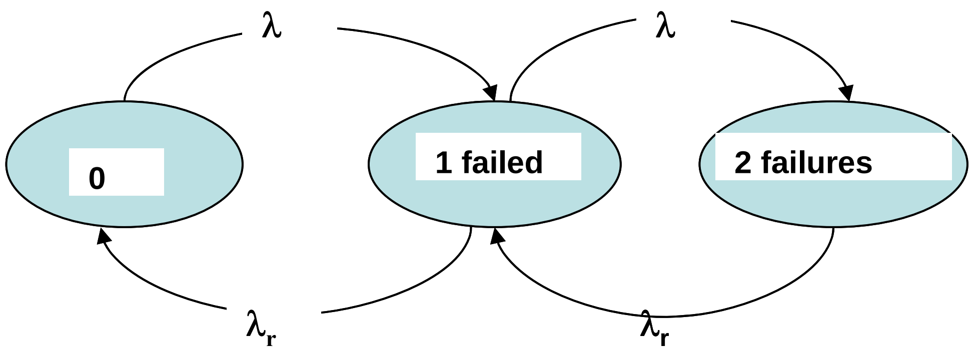


Figure 21. Two-unit Parallel System with Off-line Restoration

system intrinsic unavailability in the long run is roughly $U_I \cong 0.0002 = 0.02\%$, i.e., the system spends roughly 0.02% of the time in state 2 under repair.

For the case of off-line system restoration (where there is appreciable system downtime), the same procedure will show that $\pi_0 = \dfrac{\lambda_r}{3\lambda_r+2\lambda}$, $\pi_1 = \dfrac{2\lambda_r}{3\lambda_r+2\lambda}$ and hence, system availability is given by

$A_I = \pi_0 + \pi_1 = \dfrac{3\lambda_r}{3\lambda_r + 2\lambda}$. When $\lambda = 0.0005$ and $\lambda_r = 0.05$, $A_I = \dfrac{0.15}{0.15 + 0.001} = 0.9933775 \rightarrow$ On the average the system will be down 0.6623% of the time with both units under repair. Ebeling obtains the transient solution for the 2-unit standby system (with perfect sensor and switching) using TRD 9.6 and 9.7 on pp. 209 and 210 and quiescent failure rate of roughly zero. Since the time-dependent solutions again require solving a system of differential equations, I will first provide the equilibrium solution using on-line and off-line restoration as depicted in the following figures, and then I will obtain the transient solution. Ebeling's Figure 9.7 atop his page 210 is somewhat vague. Thus, I have modified it a bit given atop the next page. Proceeding as above we obtain

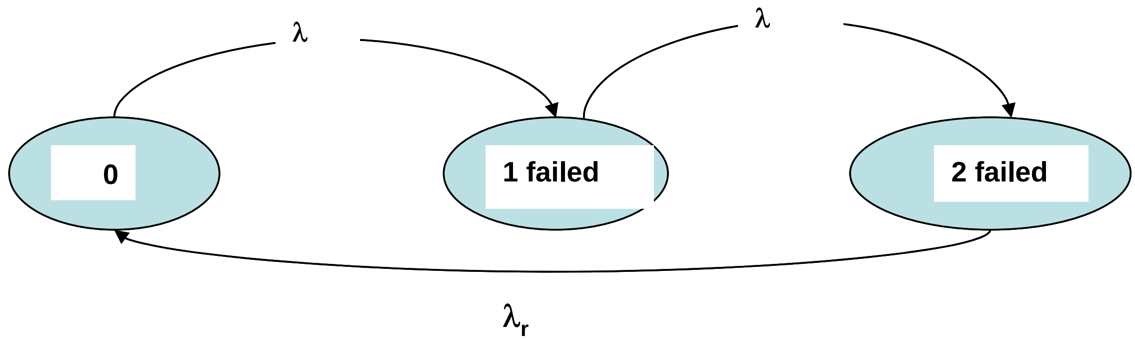


The TRD for a 2-unit standby system with on-line one-at-a-time repair

$\pi_0 = 1/[1+(\lambda/\lambda_r) + (\lambda/\lambda_r)^2]$, $(\lambda/\lambda_r)/[1+(\lambda/\lambda_r)+(\lambda/\lambda_r)^2]$ and $A_I = \dfrac{1+(\lambda/\lambda_r)}{1+(\lambda/\lambda_r)+(\lambda/\lambda_r)^2} =$

$\dfrac{\lambda_r^2 + \lambda\lambda_r}{\lambda_r^2 + \lambda\lambda_r + \lambda^2}$; When $\lambda = 0.0005$ and $\lambda_r = 0.05$, $A_I = \dfrac{(0.05)^2 + (0.000025)}{0.0025 + 0.000025 + (0.0005)^2} = 0.999901$

$\rightarrow$ Thus, on the average the system is unavailable 0.01% of the time when both units are down and only one is under repair. For the 2-unit standby system with off-line restoration, the figure below shows that $\lambda\pi_0 = \lambda_r \pi_2 \rightarrow \pi_2 = (\lambda/\lambda_r)\pi_0$; $\lambda\pi_0 = \lambda\pi_1 \rightarrow \pi_1 = \pi_0$; because $\pi_0 + \pi_1 + \pi_2 = 1$, then $\pi_0 + \pi_0 + (\lambda/\lambda_r)\pi_0 =$



The TRD for a 2-unit Standby with off-line system repair rate λ_r

$1 \rightarrow \pi_0 = 1/[2 + (\lambda/\lambda_r)]$ and $\rightarrow \pi_1 = 1/[2 + (\lambda/\lambda_r)]$ $\rightarrow A_I = 2/[2 + (\lambda/\lambda_r)]$. When $\lambda = 0.0005$ and $\lambda_r = 0$ $\lambda\pi_0 = \lambda_r\,\pi_2 \rightarrow \pi_2 = (\lambda/\lambda_r)\,\pi_0$; $\lambda\pi_0 = \lambda\pi_1 \rightarrow \pi_1 = \pi_0$; because $\pi_0 + \pi_1 + \pi_2 = 1$, then $\pi_0 + \pi_0 + (\lambda/\lambda_r)\pi_0 =$ $1 \rightarrow \pi_0 = 1/[2 + (\lambda/\lambda_r)]$ and $\rightarrow \pi_1 = 1/[2 + (\lambda/\lambda_r)]$ $\rightarrow A_I = 2/[2 + (\lambda/\lambda_r)]$. When $\lambda = 0.0005$ and $\lambda_r = 0$ 0.05, $A_I = 2/[2 + 0.0005/0.05] = 0.995025 \rightarrow$ The system will down an average of roughly 0.50% of the time when both units are under repair. To obtain the $MTBF_{Sys}$, we need the TRM (transition-rate matrix) for the above 2-unit standby system which is given by

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1-\lambda & \lambda & 0 \\ 1 & 0 & 1-\lambda & \lambda \\ 2 & \lambda_r & 0 & 1-\lambda_r \end{array} \quad \rightarrow \quad \tilde{P} = \begin{array}{c|ccc} & 2 & 0 & 1 \\ \hline 2 & 1 & 0 & 0 \\ 0 & 0 & 1-\lambda & \lambda \\ 1 & \lambda & 0 & 1-\lambda \end{array}$$

$$\rightarrow Q = \begin{bmatrix} 1-\lambda & \lambda \\ 0 & 1-\lambda \end{bmatrix} \rightarrow N = (I_2 - Q)^{-1} = N = \begin{bmatrix} \lambda & -\lambda \\ 0 & \lambda \end{bmatrix}^{-1} = \frac{1}{\lambda^2}\begin{bmatrix} \lambda & \lambda \\ 0 & \lambda \end{bmatrix} \rightarrow$$

$MTBF_0 = 2/\lambda$ and $MTBF_1 = 1/\lambda$.

The Transient Solution for a Single Unit

By instantaneous (or point) component availability at time t, A(t), we mean the Pr that a component is functioning reliably at time t. Thus, if there is no repair, the point availability function is simply A(t) = R(t). However, if the component is repairable, then there are two mutually exclusive possibilities: (1) The component is reliable at t, in which case $A_1(t) = R(t)$, (2) the component fails at time x, $0 < x < t$, gets renewed in the interval $(x, x+\Delta x)$ with pr element $m(x)dx$, and then is reliable from time x to $t - x$ (Trivedi, 1982). This second pr is given by $A_2(t) = \int_0^t \rho(x)dx\, R(t-x)$. Because

the above two cases are mutually exclusive, then a component's availability is

$$A(t) = A_1(t) + A_2(t) = R(t) + \int_0^t \rho(x)dx\, R(t-x) \tag{136a}$$

Taking Laplace transform of the above Eq. (136a) yields $\bar{A}(s) = \bar{R}(s) + \bar{R}(s)\,\bar{\rho}(s) = \bar{R}(s)\,[1+\bar{\rho}(s)]$

$$\rightarrow \qquad \bar{A}(s) = \bar{R}(s)\,[1 + \frac{\bar{f}(s)\bar{g}_r(s)}{1-\bar{f}(s)\bar{g}_r(s)}] = \frac{\bar{R}(s)}{\mathbf{1} - \bar{f}(s)\,\bar{g}_r(s)}. \tag{136b}$$

If the component's TTF has a constant failure rate λ and time to repair is also exponential at the rate λ_r, then $\bar{A}(s) = \int_0^\infty e^{-\lambda t}e^{-st}dt = 1/(\lambda+s)$; $\bar{f}(s) = \int_0^\infty \lambda\, e^{-\lambda t}e^{-st}dt = \lambda/(\lambda+s)$; $\bar{g}_r(s) = \int_0^\infty \lambda_r\, e^{-\lambda_r t}e^{-st}dt =$ $\lambda_r/(\lambda_r+s)$. Hence, the $L\{A(t)\} = \bar{A}(s) = \dfrac{1/(\lambda+s)}{1-[\lambda/(\lambda+s)][\lambda_r/(\lambda_r+s)]} = \dfrac{\lambda_r+s}{(\lambda+s)(\lambda_r+s)-\lambda\lambda_r}$

$$= \frac{\lambda_r+s}{s[s+(\lambda+\lambda_r)]} = \frac{\lambda_r}{s(\lambda+\lambda_r)} + \frac{\lambda/(\lambda+\lambda_r)}{s+\lambda+\lambda_r} \rightarrow A(t) = L^{-1}\{\bar{A}(s)\} = L^{-1}\{\frac{\lambda_r}{s(\lambda+\lambda_r)} + \frac{\lambda/(\lambda+\lambda_r)}{s+\lambda+\lambda_r}\}$$

$$\rightarrow \quad A(t) = \frac{\lambda_r}{(\lambda+\lambda_r)} + \frac{\lambda}{\lambda+\lambda_r}\, e^{-(\lambda+\lambda_r)t} \tag{137}$$

Eq. (137) gives the simplest availability function for M/M/1 queuing system where both arrival and repair rates are exponential and is identical to Eq. (11.10) on page 257 of Ebeling. For example, given that a component has a constant failure rate $\lambda_f = 0.0005$ and repair rate $\lambda_r = r = 0.05$ per hour, then the pr that the component is available (i.e., not under repair) at t = 1000 hours is given by

$A(1000) = \dfrac{0.05}{0.0505} + \dfrac{0.0005}{0.0505}\, e^{-0.0505(1000)} = 0.990099009901$, while $R(1000) =$ $e^{-0.5} = 0.60653066 < A(1000)$.

System Availability (Section 11.3 of Ebeling)

Because A(t) can stand for availability of a system, then the laws of Pr will prevail. For example, for a serial system consisting of n units each with availability of $A_i(t)$ and its own server, the system availability is given by

$$A_{Sys}(t) = \prod_{i=1}^{n} A_i(t) \le \text{Minimum}[A_i(t)] \qquad (11.15E\ p.\ 259)$$

Similarly, for a pure redundant parallel system, the system will be unavailable iff all n units are under repair and hence $U_{Sys}(t) = \prod_{i=1}^{n} U_i(t) = \prod_{i=1}^{n}[1 - A_i(t)]$. Therefore,

$$A_{Sys}(t) = 1 - U_{Sys}(t) = 1 - \prod_{i=1}^{n}[1 - A_i(t)] \ge \text{Maximum}[A_i(t)] \qquad (11.16E,\ p.\ 259)$$

where the above Eq. (11.1E) is valid only if there are n servers. If there is a single server, then $A_{Sys}(t) < 1 - \prod_{i=1}^{n}[1 - A_i(t)]$, as will be shown later (See Example 11.3 on page 259 of Ebeling).

Note that we can easily use Markov analysis to obtain the long-term inherent availability of both a series and parallel system. For the example 11.3 of Ebeling, the TRD for the 2-unit series system is given in Figure 22. Figure 22 clearly shows that $2\lambda\, \pi_0 = \lambda_r\, \pi_1 \rightarrow \pi_1 = (2\lambda/\lambda_r)\pi_0$; $\lambda\, \pi_1 = 2\lambda_r\, \pi_2$

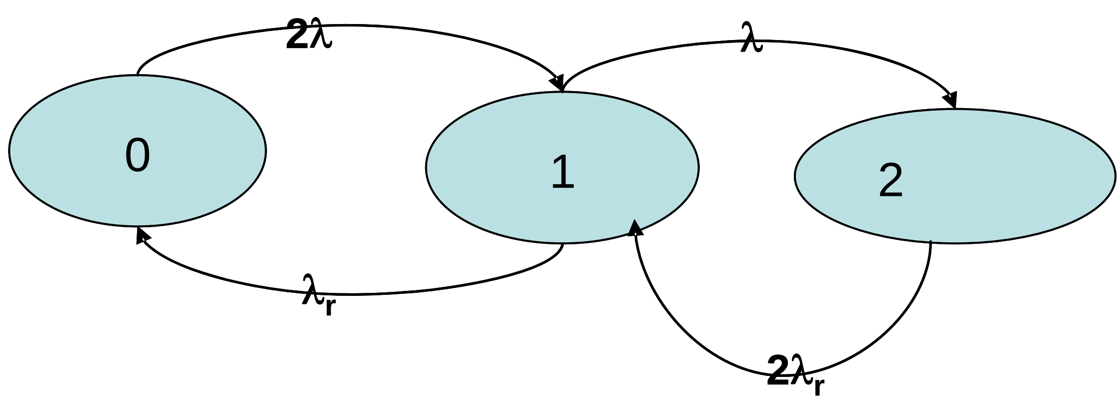


Figure 22. The TRD for a 2-unit Serial system with 2 servers

$\pi_2 = \dfrac{\lambda\pi_1}{2\lambda_r} \rightarrow \pi_2 = (\lambda/\lambda_r)^2\pi_0 \rightarrow \pi_0 + (2\lambda/\lambda_r)\pi_0 + (\lambda/\lambda_r)^2\pi_0 = 1 \rightarrow \pi_0 = \dfrac{1}{1+(2\lambda/\lambda_r)+(\lambda/\lambda_r)^2} =$

$\dfrac{\lambda_r^2}{\lambda_r^2 + 2\lambda\lambda_r + \lambda^2} = \dfrac{\lambda_r^2}{(\lambda+\lambda_r)^2} = [\lambda_r/(\lambda_r+\lambda)]^2$. Once π_0 is computed, the other two steady-state prs can easily be obtained. For the example 11.3 of Ebeling, $\lambda = 0.10$ and $\lambda_r = 0.20$ per hour, then $\pi_0 = (0.2/0.3)^2 = 0.444\overline{4}$, which agrees with Ebeling's answer to 3 decimals. Then for a series system the intrinsic system availability is $A_I = \pi_0 = 0.444\overline{4}$, and $U = 0.555\overline{5}$. However, if the system is a pure parallel one, then $A_I = \pi_0 + \pi_1 = 0.444\overline{4} + (0.2/0.2)\pi_0 = 0.888\overline{8}$ and $U_I = 0.111\overline{1}$, which again aggress with Ebeling's answer on page 259 to 3 decimals.

The transient solution for availability of a 2-identical-unit serial system can be obtained also thru solving the following system of differential equations.

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t) \\ dP_1(t)/dt = 2\lambda P_0(t) - (r+\lambda)P_1(t) + 2rP_2(t) \end{cases}$$

Because $P_2(t) = 1 - P_0(t) - P_1(t)$, the above system reduces to

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t) \\ dP_1(t)/dt = (2\lambda - 2r)P_0(t) - (3r+\lambda)P_1(t) + 2r \end{cases}$$

. Taking the Laplace transform of this system results in

$$\begin{cases} s\overline{P}_0(s) - 1 = -2\lambda\overline{P}_0(s) + r\overline{P}_1(s) \\ s\overline{P}_1(s) - 0 = (2\lambda - 2r)\overline{P}_0(s) - (3r+\lambda)\overline{P}_1(s) + 2r/s \end{cases} \rightarrow$$

$$\begin{cases} (s+2\lambda)\overline{P}_0(s) - r\overline{P}_1(s) = 1 \\ (2r-2\lambda)\overline{P}_0(s) + (s+3r+\lambda)\overline{P}_1(s) = 2r/s \end{cases} \rightarrow$$

$$\overline{P}_0(s) = \frac{\begin{vmatrix} 1 & -r \\ 2r/s & (s+3r+\lambda) \end{vmatrix}}{\begin{vmatrix} (s+2\lambda) & -r \\ 2r-2\lambda & (s+3r+\lambda) \end{vmatrix}} = \frac{(s+3r+\lambda) + 2r^2/s}{(s+2\lambda)(s+3r+\lambda) + r(2r-2\lambda)} =$$

$$\frac{s(s+3r+\lambda) + 2r^2}{s[s^2 + (3r+3\lambda)s + (2\lambda^2 + 2r^2 + 4\lambda r)]} = \frac{s(s+3r+\lambda) + 2r^2}{s[(s-u_1)(s-u_2)]}$$

where

$$u_i = \frac{-(3\lambda+3r) \pm \sqrt{(3\lambda+3r)^2 - 4(2\lambda^2 + 2r^2 + 4\lambda r)}}{2} = \frac{-(3\lambda+3r) \pm \sqrt{\lambda^2 + r^2 + 2\lambda r}}{2}$$

$$= \frac{-(3\lambda+3r) \pm (\lambda+r)}{2} \rightarrow u_1 = -(\lambda+r) \text{ and } u_2 = -2(\lambda+r) \rightarrow$$

$$\overline{P}_0(s) = \frac{s+3r+\lambda}{(s+\lambda+r)(s+2\lambda+2r)} + \frac{2r^2}{s(s+\lambda+r)(s+2\lambda+2r)} \rightarrow$$

$$\overline{P}_0(s) = \frac{s}{(s+\lambda+r)(s+2\lambda+2r)} + \frac{3r+\lambda}{(s+\lambda+r)(s+2\lambda+2r)} + \frac{2r^2}{s(s+\lambda+r)(s+2\lambda+2r)} \quad \rightarrow$$

$$P_0(t) = L^{-1}\{\overline{P}_0(s)\} = \frac{1}{\lambda+r}[2(\lambda+r)e^{-2(\lambda+r)t} - (\lambda+r)e^{-(\lambda+r)t}] + \frac{3r+\lambda}{\lambda+r}[e^{-(\lambda+r)t} - e^{-2(\lambda+r)t}] +$$

$$\frac{2r^2}{2(\lambda+r)^2}\left\{1+\frac{1}{\lambda+r}[(\lambda+r)e^{-2(\lambda+r)t} - 2(\lambda+r)e^{-(\lambda+r)t}]\right\}$$

$$\rightarrow A_{Sys}(t) = P_0(t) = \frac{r^2}{(\lambda+r)^2} + \frac{2\lambda r}{(\lambda+r)^2}e^{-(\lambda+r)t} + \frac{\lambda^2}{(\lambda+r)^2}e^{-2(\lambda+r)t} \qquad (138)$$

Eq. (138) gives $A_{Sys}(10 \text{ hours}) = 0.467674$, which is in agreement with that of Ebeling's on his page 259 for $A_{Sys}(10) = (0.684)^2 = 0.468$ to 3 decimals. Further, Eq. (138) shows that the availability of a serial system is simply the product of the individual availabilities, i.e., $A_{Sys}(t) = \prod_{i=1}^{n} A_i(t)$ (see Eqs. (11.13) and (11.15) on pp. 258-259 of Ebeling.

System Transient Availability For a 2-identical-unit Parallel System with a Single Server

It can easily be verified that the system of differential equations is given by

$$\begin{cases} \mathbf{dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t)} \\ \mathbf{dP_1(t)/dt = 2\lambda P_0(t) - (r+\lambda)P_1(t) + rP_2(t)} \end{cases}$$

Because $P_2(t) = 1 - P_0(t) - P_1(t)$, the above system reduces to

$$\begin{cases} \mathbf{dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t)} \\ \mathbf{dP_1(t)/dt = (2\lambda - r)P_0(t) - (2r+\lambda)P_1(t) + r} \end{cases}.$$

Taking the Laplace transform of this system results in

$$\begin{cases} \mathbf{s\overline{P}_0(s) - 1 = -2\lambda\overline{P}_0(s) + r\overline{P}_1(s)} \\ \mathbf{s\overline{P}_1(s) - 0 = (2\lambda - r)\overline{P}_0(s) - (2r+\lambda)\overline{P}_1(s) + r/s} \end{cases} \quad \rightarrow$$

$$\begin{cases} \mathbf{(s+2\lambda)\overline{P}_0(s) - \qquad r\,\overline{P}_1(s) \;= 1} \\ \mathbf{(r-2\lambda)\overline{P}_0(s) + (s+2r+\lambda)\overline{P}_1(s) = r/s} \end{cases} \rightarrow \overline{P}_0(s) = \frac{\begin{vmatrix} 1 & -r \\ r/s & (s+2r+\lambda) \end{vmatrix}}{\begin{vmatrix} (s+2\lambda) & -r \\ r-2\lambda & (s+2r+\lambda) \end{vmatrix}} =$$

$$\frac{(s+2r+\lambda)\ + r^2/s}{(s+2\lambda)(s+2r+\lambda)+r(r-2\lambda)} = \frac{s(s+2r+\lambda)+r^2}{s[s^2+(3\lambda+2r)s+(2\lambda^2+r^2+2\lambda r)]} \rightarrow \overline{P}_0(s) =$$

$$\frac{s(s+2r+\lambda)+r^2}{s[(s-u_1)(s-u_2)]}, \quad \text{where } u_i = \frac{-(3\lambda+2r) \pm \sqrt{(3\lambda+2r)^2 - 4(2\lambda^2+r^2+2\lambda r)}}{2} =$$

$$\frac{-(3\lambda+2r) \pm \sqrt{\lambda^2+4\lambda r}}{2}, \; u_1 - u_2 = \sqrt{\lambda^2+4\lambda r}\ , \quad \text{and} \quad u_1u_2 = 2\lambda^2 + 2\lambda r + r^2 \rightarrow$$

$$\overline{P}_0(s) = \sqrt{b^2-4ac} = \frac{s+2r+\lambda}{(s-u_1)(s-u_2)} + \frac{r^2}{s(s-u_1)(s-u_2)} \rightarrow \overline{P}_0(s) = \frac{s}{(s-u_1)(s-u_2)}$$

$$+\frac{2r+\lambda}{(s-u_1)(s-u_2)} + \frac{r^2}{s(s-u_1)(s-u_2)} \qquad \rightarrow P_0(t) = \mathcal{L}^{-1}\{\overline{P}_0(s)\} = \frac{1}{u_1-u_2}(u_1e^{u_1t} - u_2e^{u_2t}) +$$

$$\frac{\lambda+2r}{u_1-u_2}(e^{u_1t} - e^{u_2t}) + \frac{r^2}{u_1u_2}\left[1 + \frac{1}{u_1-u_2}(u_2e^{u_1t} - u_1e^{u_2t})\right] = \frac{r^2}{r^2+2\lambda^2+2\lambda r} +$$

$$\frac{u_1^2+u_1(\lambda+2r)+r^2}{u_1(u_1-u_2)}e^{u_1t} + \frac{u_2^2+u_2(\lambda+2r)+r^2}{u_2(u_2-u_1)}e^{u_2t} \rightarrow P_0(t) = \frac{r^2}{r^2+2\lambda^2+2\lambda r} +$$

$$\frac{u_1^2+u_1(\lambda+2r)+r^2}{u_1(u_1-u_2)}e^{u_1t} + \frac{u_2^2+u_2(\lambda+2r)+r^2}{u_2(u_2-u_1)}e^{u_2t} \tag{139a}$$

Similarly, $\overline{P}_1(s) = \dfrac{2r\lambda}{s(s-u_1)(s-u_2)} + \dfrac{2\lambda}{(s-u_1)(s-u_2)}$, and

$$P_1(t) = \frac{2\lambda r}{r^2+2\lambda^2+2\lambda r} + \frac{2\lambda u_1 + 2\lambda r}{u_1(u_1-u_2)}e^{u_1t} + \frac{2\lambda r + 2\lambda u_2}{u_2(u_2-u_1)}e^{u_2t} \tag{139b}$$

Combining Eqs. (139 a &b) yields

$$A(t) = P_0(t) + P_1(t) = \frac{r^2 + 2\lambda r}{r^2 + 2\lambda^2 + 2\lambda r} + \frac{u_1^2 + u_1(3\lambda + 2r) + (r^2 + 2\lambda r)}{u_1(u_1 - u_2)} e^{u_1 t} +$$

$$\frac{u_2^2 + u_2(3\lambda + 2r) + (r^2 + 2\lambda r)}{u_2(u_2 - u_1)} e^{u_2 t}, \ u_2 < u_1 < 0. \ \rightarrow$$

$$A(t) = \frac{r^2 + 2\lambda r}{r^2 + 2\lambda^2 + 2\lambda r} + \frac{2\lambda^2}{u_1(u_2 - u_1)} e^{u_1 t} + \frac{2\lambda^2}{u_2(u_1 - u_2)} e^{u_2 t} = \frac{r^2 + 2\lambda r}{r^2 + 2\lambda^2 + 2\lambda r} +$$

$$\frac{4\lambda^2 e^{u_1 t}}{(3\lambda + 2r)\sqrt{\lambda^2 + 4\lambda r} - (\lambda^2 + 4\lambda r)} - \frac{4\lambda^2 e^{u_2 t}}{\lambda^2 + 4\lambda r + (3\lambda + 2r)\sqrt{\lambda^2 + 4\lambda r}} \tag{140}$$

At $\lambda = \lambda_f = 0.10$ and $\lambda_r = r = 0.20$, the value of A(10 hours) from Eq. (140) is A(10) = 0.844213368. This availability is less than $A_s(10) = 0.900$, as expected, given by Ebeling on his page 259 for the case of 2-identical parallel units with 2 servers because the system availability given in Eq. (11.16) on page 259 of Ebeling is valid only for the case of n servers, while availability function of Eq. (140) developed above is valid for a 2-uint parallel system with only one server.

System Transient Availability For a 2-unit Standby System

Figure 11.4 on page 259 of Ebeling describes one cycle of a 2-component repairable Standby system where the quiescent failure rate of the standby unit λ_f^- is negligible relative to its active failure rate λ_2. For convenience I have modified his Figure 11.4 as follows, where $\lambda_r = r$ represents repair rate and state 2 represents system failure. From the modified Figure 11.4 we can deduce that

$$\begin{cases} dP_0(t)/dt = -\lambda_1 P_0(t) + rP_1(t) \\ dP_1(t)/dt = \lambda_1 P_0(t) - (r + \lambda_2)P_1(t) + rP_2(t) \end{cases}$$

Because $P_0(t) + P_1(t) + P_2(t) = 1$, then $P_2(t) = 1 - P_1(t) - P_0(t)$ $\rightarrow$ the above system of differential equations reduces to

$$\begin{cases} dP_0(t)/dt = -\lambda_1 P_0(t) + rP_1(t) \\ dP_1(t)/dt = (\lambda_1 - r)P_0(t) - (2r + \lambda_2)P_1(t) + r \end{cases}$$

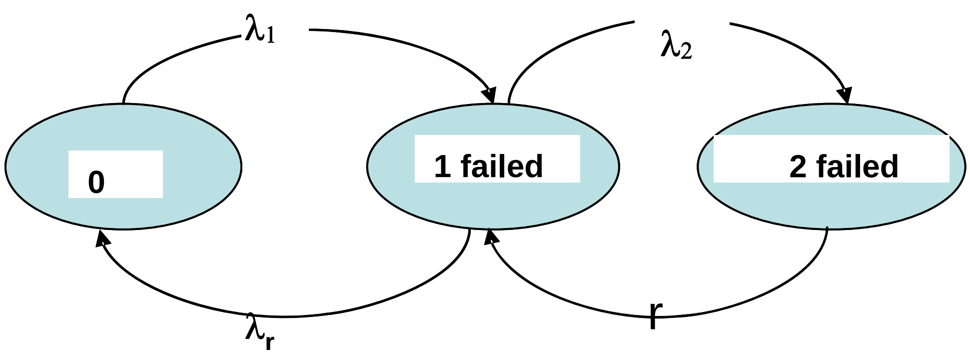
Taking Laplace transforms, we obtain $\begin{cases} s\overline{P}_0(s) - P_0(0) = -\lambda_1 \overline{P}_0(s) + r\overline{P}_1(s) \\ s\overline{P}_1(s) - P_1(0) = (\lambda_1 - r)\overline{P}_0(s) - (2r + \lambda_2)\overline{P}_1(s) + r/s \end{cases}$.

Applying the boundary conditions $P_0(0) = 1$ and $P_1(0) = 0$, we obtain

$$\begin{cases} s\bar{P}_0(s) - 1 = -\lambda_1\bar{P}_0(s) + r\bar{P}_1(s) \\ s\bar{P}_1(s) = (\lambda_1 - r)\bar{P}_0(s) - (2r+\lambda_2)\bar{P}_1(s) + r/s \end{cases} \rightarrow \begin{cases} (s+\lambda_1)\bar{P}_0(s) - \quad r\bar{P}_1(s) = 1 \\ (r-\lambda_1)\bar{P}_0(s) + (s+2r+\lambda_2)\bar{P}_1(s) = r/s \end{cases} \rightarrow$$

$$\bar{P}_0(s) = \frac{\begin{vmatrix} 1 & -r \\ r/s & (s+2r+\lambda_2) \end{vmatrix}}{\begin{vmatrix} (s+\lambda_1) & -r \\ r-\lambda_1 & (s+2r+\lambda_2) \end{vmatrix}} = \frac{(s+2r+\lambda_2) + r^2/s}{(s+2r+\lambda_2)(s+\lambda_1) + r(r-\lambda_1)} =$$

$$\frac{s(s+2r+\lambda_2)+r^2}{s[(s+2r+\lambda_2)(s+\lambda_1)+r(r-\lambda_1)]} \rightarrow$$



A 2-unit standby system with on-Line Repair

$$\bar{P}_0(s) = \frac{s(s+2r+\lambda_2)+r^2}{s[s^2+(\lambda_1+2r+\lambda_2)s+(r^2+r\lambda_1+\lambda_1\lambda_2)]} = \frac{s(s+2r+\lambda_2)+r^2}{s(s-u_1)(s-u_2)},$$

where u_1 and u_2 are the roots of the quadratic $s^2+(\lambda_1+2r+\lambda_2)s+(r^2+r\lambda_1+\lambda_1\lambda_2)=0$, i.e.,

$u_i = \dfrac{-(\lambda_1+2r+\lambda_2) \pm \sqrt{(\lambda_2-\lambda_1)^2+4r\lambda_2}}{2}$, and $u_1u_2 = \lambda_1\lambda_2 + r^2 + r\lambda_1$.

Thus, $\bar{P}_0(s) = \dfrac{s+2r+\lambda_2}{(s-u_1)(s-u_2)} + \dfrac{r^2}{s(s-u_1)(s-u_2)} = \dfrac{s}{(s-u_1)(s-u_2)} + \dfrac{2r+\lambda_2}{(s-u_1)(s-u_2)} +$

$\dfrac{r^2}{s(s-u_1)(s-u_2)}$; inverting this Laplace transform yields

$P_0(t) = = L^{-1}\{\overline{P}_0(s)\} = \dfrac{1}{u_1 - u_2}(u_1 e^{u_1 t} - u_2 e^{u_2 t}) + \dfrac{\lambda_2 + 2r}{u_1 - u_2}(e^{u_1 t} - e^{u_2 t}) +$

$$\frac{r^2}{u_1 u_2}[1 + \frac{1}{u_1 - u_2}(u_2 e^{u_1 t} - u_1 e^{u_2 t})] = \frac{r^2}{r^2 + \lambda_1\lambda_2 + \lambda_1 r} + \frac{u_1^2 + u_1(\lambda_2 + 2r) + r^2}{u_1(u_1 - u_2)} e^{u_1 t} +$$

$$\frac{u_2^2 + u_2(\lambda_2 + 2r) + r^2}{u_2(u_2 - u_1)} e^{u_2 t}. \qquad (141)$$

Similarly, $\overline{P}_1(s) = \dfrac{\begin{vmatrix} (s+\lambda_1) & 1 \\ (r-\lambda_1) & r/s \end{vmatrix}}{\begin{vmatrix} (s+\lambda_1) & -r \\ r-\lambda_1 & (s+2r+\lambda_2) \end{vmatrix}} = \dfrac{(s+\lambda_1)r/s - (r-\lambda_1)}{(s+2r+\lambda_2)(s+\lambda_1) + r(r-\lambda_1)} =$

$$= \frac{r(s+\lambda_1) - (r-\lambda_1)s}{s[(s+2r+\lambda_2)(s+\lambda_1) + r(r-\lambda_1)]} = \frac{r\lambda_1 + s\lambda_1}{s[s^2 + (\lambda_1 + 2r + \lambda_2)s + (r^2 + r\lambda_1 + \lambda_1\lambda_2)]} =$$

$$\frac{r\lambda_1 + \lambda_1 s}{s(s-u_1)(s-u_2)} = \frac{r\lambda_1}{s(s-u_1)(s-u_2)} + \frac{\lambda_1}{(s-u_1)(s-u_2)} \rightarrow$$

$P_1(t) = L^{-1}\{P_1^*(s)\} = \dfrac{r\lambda_1}{u_1 u_2}[1 + \dfrac{1}{u_1 - u_2}(u_2 e^{u_1 t} - u_1 e^{u_2 t})] + \dfrac{\lambda_1}{u_1 - u_2}(e^{u_1 t} - e^{u_2 t}) \rightarrow$

$$P_1(t) = \frac{r\lambda_1}{r^2 + \lambda_1\lambda_2 + \lambda_1 r} + \frac{r\lambda_1 + \lambda_1 u_1}{u_1(u_1 - u_2)} e^{u_1 t} + \frac{r\lambda_1 + \lambda_1 u_2}{u_2(u_2 - u_1)} e^{u_2 t} \qquad (142)$$

Combining Eqs. (141) and (142), the point (or instantaneous) availability for a 2-unit standby system is given by $A(t) = P_0(t) + P_1(t) = \dfrac{r^2 + r\lambda_1}{r^2 + \lambda_1\lambda_2 + \lambda_1 r} + \dfrac{u_1^2 + u_1(\lambda_2 + 2r + \lambda_1) + r^2 + r\lambda_1}{u_1(u_1 - u_2)} e^{u_1 t} +$

$$\frac{u_2^2 + u_2(\lambda_2 + 2r + \lambda_1) + r^2 + r\lambda_1}{u_2(u_2 - u_1)} e^{u_2 t}. \qquad (143a)$$

Substituting $u_i = \dfrac{-(\lambda_1 + 2r + \lambda_2) \pm \sqrt{(\lambda_2 - \lambda_1)^2 + 4r\lambda_2}}{2}$ into Eq. (143a) reduces the above availability function to

$$A(t) = \frac{r^2 + r\lambda_1}{r^2 + \lambda_1\lambda_2 + \lambda_1 r} + \frac{\lambda_2\lambda_1}{u_1(u_2 - u_1)} e^{u_1 t} + \frac{\lambda_2\lambda_1}{u_2(u_1 - u_2)} e^{u_2 t} \qquad (143b)$$

The intrinsic availability is given by $A_I = \underset{t \to \infty}{\text{Limit}}[P_0(t) + P_1(t)] = \pi_0 + \pi_1 \to$

$$A_I = \frac{r(r+\lambda_1)}{r^2 + r\lambda_1 + \lambda_1\lambda_2} \tag{144}$$

Eq. (144) clearly shows that for an irreparable 2-unit standby system, i.e. r = 0, the value of $A_I = 0$ while for the same system the RE function from Eq. (143b) is given by

$$R(t) = \frac{\lambda_2\lambda_1}{u_1(u_2 - u_1)} e^{u_1 t} + \frac{\lambda_2\lambda_1}{u_2(u_1 - u_2)} e^{u_2 t} \tag{145a}$$

However, when r = 0, $u_1 = -\lambda_1$ and $u_2 = -\lambda_2$ so that the Eq. (145a) reduces to

$$R(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 t} \tag{145b}$$

Eq. (145b) is identical to the RE function for a 2-unit standby system given in Eq. (6.27) on page 113 of Ebeling where we are assuming that the quiescent failure rate of the standby unit $\lambda_2^- = 0$. For the Example 11.4 on page 260 of Ebeling, $\lambda_1 = 0.002$, $\lambda_2 = 0.001$, and r = 0.01 per hour. Substituting these into Eq. (144) results in $A_I = 0.9836065574$, which agrees with that of Ebeling's to 3 decimals near the bottom of his page 260. Ebeling's failure rates in this example are unrealistic because the primary unit in a 2-unit standby should never (or hardly ever) have a failure rate twice that of the standby unit. Therefore, the more realistic failure rates are $\lambda_1 = 0.001$, $\lambda_2 = 0.002$, and r = 0.01 per hour. Substituting these into Eq. (144) results in $A_I = 0.98214286$, which should be a bit smaller than the case of $\lambda_1 > \lambda_2$. The value of RE at t = 500, $\lambda_1 = 0.001$, $\lambda_2 = 0.002$, and r = 0.01 from Eq. (145) is given by R(500) $= \frac{0.002}{0.002 - 0.001} e^{-0.5} + \frac{0.001}{0.001 - 0.002} e^{-1} = 1.21306131942527 - 0.36787944117144 =$ 0.845181878254. While the value of the point availability A(t) at t = 500 hours from Eq. (143b) is A(500) = 0.9830968, where $u_1 = -0.007000$ and $u_2 = -0.01600000$.

Summary

1. Exponential Failures and Repairs of a Single unit (or Component)

(a) For the exponential failures and no repairs (MTTR = $\mu_r \cong 0$), the RE function is R(t) = $e^{-\lambda t}$.

The renewal function is M(t) = E[N_f(t)] = λt.

(b) For the exponential failures with exponential repairs (MTTR = μ_r > 0), then

$$A(t) = \frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}$$

$$A(t_1,\ t_2) = \frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda / (t_2 - t_1)}{(\lambda + \lambda_r)^2} [e^{-(\lambda+\lambda_r)t_1} - e^{-(\lambda+\lambda_r)t_2}]$$

$$A_I = \frac{\lambda_r}{\lambda + \lambda_r}$$; The renewal function is

$$M(t) = E[N_C(t)] = \frac{-\lambda\lambda_r}{(\lambda + \lambda_r)^2} + \frac{\lambda\lambda_r t}{(\lambda + \lambda_r)} + \frac{\lambda\lambda_r}{(\lambda + \lambda_r)^2} e^{-(\lambda+\lambda_r)t} .$$

Note that the above expected number of cycles when r = $\lambda_r = \infty$ (i.e., MTTR = $\mu_r \cong 0$), its value does become equal to the mean number of failures M(t) = λt for the case of W/O repair, as expected. E. A. Elsayed provides the same expression as above near the bottom of his page 426 W/O detailed proof.

2. Two-Identical-Unit Series System

(a) No Repairs (r = λ_r = 0): R(t) = $e^{-2\lambda t}$

The renewal function is M(t) = E[N_f(t)] = 2λt

(b) r > 0; $A(t) = [\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}]^2$ $A_I = (\frac{r}{\lambda + r})^2$

3. Two-Identical-Unit Active Redundant System

(a) No Repairs (MTTR = $\mu_r \cong 0$): R(t) = $2e^{-\lambda t} - e^{-2\lambda t}$

In order to obtain the exact renewal function $E[N_f(t)] = M(t)$, one must first derive the Laplace transform of the renewal function $\bar{M}(s) = \dfrac{\bar{f}(s)}{s[1-\bar{f}(s)]}$ iff $f(t) = f_1(t)$, and then $M(t) = \mathcal{L}^{-1}\{\bar{M}(s)\}$. The approximate value of $M(t) = E[N_f(t)] \cong t/MTTF$; $MTTF = \dfrac{2}{\lambda} - \dfrac{1}{2\lambda} = 1.5/\lambda \rightarrow$

$$M(t) = E[N_f(t)] \cong \lambda t/1.5$$

Because $f(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}$, then $\bar{f}(s) = \dfrac{2\lambda}{s+\lambda} - \dfrac{2\lambda}{s+2\lambda}$ resulting in $\bar{M}(s) =$

$$\left(\frac{2\lambda}{s+\lambda} - \frac{2\lambda}{s+2\lambda}\right)/\{s[1 - \frac{2\lambda}{s+\lambda} + \frac{2\lambda}{s+2\lambda}]\} = \frac{2\lambda(s+2\lambda) - 2\lambda(s+\lambda)}{s\left[(s+\lambda)(s+2\lambda) - 2\lambda(s+2\lambda) + 2\lambda(s+\lambda)\right]}$$

$$= \frac{2\lambda^2}{s\left[(s+\lambda)(s+2\lambda) - 2\lambda^2\right]} = \frac{2\lambda^2}{s^2(s+3\lambda)} \rightarrow M(t) = \frac{2\lambda^2}{(3\lambda)^2}(e^{-3\lambda t} + 3\lambda t - 1) = 2(e^{-3\lambda t} + 3\lambda t - 1)/9 .$$

To check the validity of this last expected number of failures for a length of time t, we note that $E[N_f(t=0)] = 2(e^0 - 1)/9 = 2(1-1)/9 = 0$, as expected. Secondly, the value renewal density is $\rho(t) = dM(t)/dt = 2(-3\lambda e^{-3\lambda t} + 3\lambda - 0)/9 = -2/3\lambda e^{-3\lambda t} + 2\lambda/3$. Because $\mu = 1.5/\lambda$, then the $\lim_{t\to\infty} \rho(t) = 1/\mu = 2\lambda/3$.

(b) The case of $MTTR = \mu_r > 0$. $A(t) = 1 - [1 - \dfrac{\lambda_r}{\lambda+\lambda_r} - \dfrac{\lambda}{\lambda+\lambda_r} e^{-(\lambda+\lambda_r)t}]^2$

assuming two servers, i.e., each unit has its own server.

$$A_I = \frac{2\lambda_r}{\lambda+\lambda_r} - \left(\frac{\lambda_r}{\lambda+\lambda_r}\right)^2 \text{ (assuming two servers)}$$

For one server, $A(t) = \dfrac{r^2 + 2\lambda r}{r^2 + 2\lambda^2 + 2\lambda r} + \dfrac{2\lambda^2}{u_1(u_2 - u_1)} e^{u_1 t} + \dfrac{2\lambda^2}{u_2(u_1 - u_2)} e^{u_2 t}$

where $u_i = \dfrac{-(3\lambda + 2r) \pm \sqrt{\lambda^2 + 4\lambda r}}{2}$ and $A_I = \dfrac{r^2 + 2\lambda r}{r^2 + 2\lambda r + 2\lambda^2}$, In order to obtain M(t) = E[N_C(t)], one must obtain $\bar{M}(s) = \dfrac{\bar{f}(s)\bar{g}_r(s)}{s[1-\bar{f}(s)\bar{g}_r(s)]}$, and then

E[N_C(t)] = $\mathcal{L}^{-1}\{\bar{M}(s)\}$. This is probably too difficult to accomplish because $\mathbf{\bar{f}(s)}$ **and** $\mathbf{\bar{g}(s)}$ now must represent the system Laplace transforms of TTF and TTR. The system TTF distribution can easily be obtained, but the system TTR distribution, $g_{Sys}(t)$, is difficult to obtain. The approximate value of M(t) = E[N_C(t)] is given by t/MTBC, where MTBC represents mean time between cycles.

4. Two-Unit Standby Redundant System

(a) No Repair ($\mu_r \cong 0$, $\lambda_1 = \lambda_2 = \lambda$).

$$R(t) = (1+\lambda t)\, e^{-\lambda t}$$

$$M(t) \cong t/(2/\lambda) = \lambda t/2$$

(b) $\mu_r > 0$

$$A(t) = \frac{r^2 + r\lambda_1}{r^2 + \lambda_1\lambda_2 + \lambda_1 r} + \frac{\lambda_2\lambda_1}{u_1(u_2 - u_1)} e^{u_1 t} + \frac{\lambda_2\lambda_1}{u_2(u_1 - u_2)} e^{u_2 t}$$

where $u_i = \dfrac{-(\lambda_1 + 2r + \lambda_2) \pm \sqrt{(\lambda_2 - \lambda_1)^2 + 4r\lambda_2}}{2}$.

$$A_I = \frac{r(r+\lambda_1)}{r^2 + r\lambda_1 + \lambda_1\lambda_2}$$