There are three measures of reliability: (1) The reliability function R(t), (2) The hazard (rate) function h(t) = λ(t), and (3) the mean time to failure (MTTF). If either function (1) or (2) is known, then all 3 measures can be uniquely determined, but the knowledge of (3), i.e., μ = MTTF, is not sufficient to obtain specific functions R(t) and h(t). The chapters 2, 3 & 4 of Charles E. Ebeling (Waveland Press, Inc., ISBN13: 978-1-57766-625-7) deal with reliability at the component level (except sections 3.2, 3.4, 3.6, 4.1.4, 4.1.6) while his Chapters 5 and 6 cover system reliability.

The Dynamic Reliability (or Survivor) Function R(t)

<u>Definition</u>. The reliability of a component is the probability (Pr) that the device will perform without failure during the mission time, t, under specified stress conditions. Note that the terminology survivor function is generally used for non-repairable items, such as light bulbs, transistors, or rocket-motor of an unmanned spacecraft. For example,

R(of a new passenger tire for t = 500 interstate miles) ≅ 100% = 1

However, the reliability (RE) of the same passenger tire under racing conditions at Indianapolis 500 would be almost zero. Let T = the random variable lifetime, or time to failure (TTF), with Pr (probability) density function f(t), acronymed as pdf. Then the RE function at time t, or the survival Pr for a mission of length t, is given by (the Pr that a component Lifetime exceeds t)

$$R(t) = \Pr(T > t) = \int_t^{\infty} f(x)\,dx = 1 - \Pr(T \le t) = 1 - F_T(t) = 1 - Q_T(t) \qquad (1)$$

where F(t) = the cdf of T at the specified time t = $Q_T(t)$ = the unreliability function at time t, or the cumulative failure Pr by time t. The pdf (Pr density function), f(t), is also referred to as the failure (or mortality) density function. Some authors use the notation *S*(t) for the RE (reliability or survivor) function at t to imply survival Pr beyond the interval [0, t], which is of length t; however, the notation R(t) is a bit more prevalent. We now obtain the relationship between R(t) and f(t) by differentiating equation (1) with respect to (wrt) t. Recall that f(t) = $dF_T(t)/dt$ = $dQ_T(t)/dt$ because $F_T(t)$ = $\int_{-\infty}^{t} f(T)\,dT$. Note that time cannot be negative, and thus the lower limit in this last integral must be zero instead of $-\infty$, i.e., $F_T(t) = \int_0^t f(T)\,dT = \int_0^t f(x)\,dx$ = Failure Probability by time t. For the dummy variable of integration Ebeling uses t′, while I am using T and will also sometimes use x as the variable under the integral sign.

$$\frac{dR(t)}{dt} = \frac{d}{dt}[1 - F(t)] = -\frac{dF(t)}{dt} = -f(t) \rightarrow dR(t) = -dF(t), \text{ and}$$

$$f(t) = -\frac{dR(t)}{dt} \quad \rightarrow f(t)\times dt = -dR(t) \tag{2}$$

We are now in a position to obtain the relationship between R(t) and the mean time to failure (MTTF = μ), which is defined as the mathematical expectation of T. Henceforth, the symbol E will represent the Expected-Value operator, V represents the variance operator, and the reader must be cognizant of the fact that anytime the operator E or V is applied to any rv, the end-result will always be a population parameter. The Mean Time to Failure (MTTF) is given by

$$\text{MTTF} = E(T) = \int_0^\infty t f(t)\,dt = \int_0^\infty t(-dR) \rightarrow \text{MTTF} = -tR(t)\Big]_0^\infty + \int_0^\infty R(t)\,dt = 0 + \int_0^\infty R(t)\,dt = \int_0^\infty R(t)\,dt$$

$$\rightarrow \qquad \text{MTTF} = \int_0^\infty R(t)\,dt \tag{3}$$

Eq. (3) clearly shows that the unconditional mean-life, E(T) = MTTF starting at age zero, of any device or system is given by the integral of its RE function evaluated always from 0 to infinity so that the MTTF is the total area under between R(t) and the abscissa t (from 0 to ∞). Note that if a component or system is repairable (or renewable, i.e., failed units are immediately replaced), then E(T) is called the mean time between failures (MTBF). Further, the lower limit of the integral must always be zero even if the minimum life $t_0 = \delta > 0$. Note that Eq. (3) does not in general hold for any continuous rv (random variable), i.e., in general the population mean μ is not equal to $\int_{-\infty}^{\infty} [1 - F(x)]\,dx$.

<u>Example 1</u>. The time to failure, T, of an electronic component is exponentially distributed with MTTF equal to θ and minimum life $t_0 = 0$. Before we obtain its RE function, it is essential for engineers to know when and where the exponential distribution has applications.

The exponential density is generally used to describe the distribution of time to a random (or catastrophic, or chance) failure, and therefore, the exponential distribution is usually used to describe the TTF of a mechanical or electrical component during its useful life cycle (i.e., the 2^{nd} cycle where failures are not due to aging or burn-in but are random or catastrophic; see Figure 2.3 on p. 31 of Ebeling). The exponential pdf has widespread and enormous applications in reliability engineering (or life testing) and also in Markov chains and queuing theory (or general stochastic processes). Majority of its applications occur because of its relation to the Poisson distribution as described below.

Consider a Poisson event (such as a failure) that occurs at a constant rate of $\lambda > 0$ per unit of time. Then the average number of occurrences during an interval of length t is λt (later we will learn that λt is also referred to as the renewal function for a Poisson process). Let X(t) represent the number of Poisson events occurring during an interval of length t, i.e., the range of X(t) = R_x = {0, 1, 2, 3, 4, ...}. Then from the Poisson pmf (Pr mass function)

$$\Pr[X(t) = 0] = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} .$$

Next, define a continuous rv, T, as T = the time between the occurrences of 2 successive Poisson events (or the intervening time of 2 successive failures). Then the following two events are equivalent: $[X(t) = 0] \longleftrightarrow [T > t]$, i.e., no failures during t $\leftrightarrow$ Survival beyond an interval of length t. Thus, $R(t) = \Pr(T > t) = \Pr[X(t) = 0] = 1 - \Pr(T \le t) = e^{-\lambda t} = 1 - F(t) = 1 - Q(t)$ $\rightarrow$ The Unreliability function = $Q(t) = F(t) = 1 - e^{-\lambda t}$ $\rightarrow$ $R(t) = e^{-\lambda t}$ and $f(t) = dF(t)/dt = \lambda e^{-\lambda t}$. The above developments show that if the number of occurrences, X, of an event is Poisson distributed, then the occurrence time, T, of the next Poisson event (such as component or system failure) measured from the last occurrence is exponentially distributed. The RE (or survivor) function of the exponential distribution with MTTF, $\theta = 1/\lambda$, and minimum life zero is then given by

$$R(t) = \Pr(T > t) = \int_t^{\infty} \lambda e^{-\lambda x}\, dx = e^{-\lambda t} = e^{-t/\theta} = e^{-t/\mathrm{MTTF}}$$

Note that for the exponential density, $\mathrm{MTTF} = \int_0^{\infty} R(t)\,dt = \int_0^{\infty} e^{-\lambda t}\,dt = 1/\lambda = \int_0^{\infty} e^{-t/\theta}\,dt =$

$\int_0^\infty t f(t)\,dt = \int_0^\infty t\lambda e^{-\lambda t} dt = \int_0^\infty \frac{t}{\theta} e^{-t/\theta} dt = \theta$. You will see later that the Exponential is the only lifetime density with constant hazard rate $h(t) = \lambda$, and this in turn will lead to a memory-less property for the exponential.

Exercise 1. (a) Show that the MTTF of an exponentially distributed rv is given by $\theta = E(T) = 1/\lambda$. (b) Show that the k^{th} origin moment, μ'_k, of an exponential distribution is given by $\mu'_k = k!/(\lambda^k)$, for k = 1, 2, 3, and 4. Then use this last result to show that the 1^{st} four central moments of the exponential are given by $\mu_k = E[(X - \mu'_1)^k] = 0,\ 1/\lambda^2,\ 2/\lambda^3$, and $9/\lambda^4$, respectively. Therefore, the standard deviation of the exponential is $\sigma = 1/\lambda$ so that the coefficient of variation of the exponential is equal to CV = 100%; the skewness of the exponential is given by $\alpha_3 = 2.00$ and its 4^{th} central moment is given by $\alpha_4 = 9.00$, so that its kurtosis is equal to $\beta_4 = 6$. Hint: Make use of the fact that the 3^{rd} central moment $\mu_3 = \mu'_3 - 3\times\mu'_2\,\mu'_1 + 2(\mu'_1)^3$, and $\mu_4 = \mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$. (c) Further, if 200 components with identical failure rates $\lambda = 0.00025$ per hour and zero minimum life are placed on life test at time zero, then show that the expected number of failures during a mission of length t = 1000 hours is equal to 44.2398434 units.

Example 2. The number of downtimes of a network is Poisson distributed with an average constant rate of 0.20 downtime/week.

(a) Compute the Pr of exactly 2 failures during the next week.

$$p(2;\ 0.20) = \frac{(0.20)^2}{2!} \times e^{-0.20} = 0.0164$$

(b) Compute the Pr that the time between 2 future successive downtimes exceeds one week. $\Pr(T > 1 \text{ week}) = \Pr[X(1) = 0 \text{ downtime}] = e^{-0.20} = 0.81873 = R(1 \text{ week})$.

(c) Compute the Pr that exactly 2 failures occur in the next 3 weeks.

Y = number of failures/3weeks $\rightarrow E(Y) = \lambda t = (0.20)\times 3 = 0.60$ failures. ⟶

Pr(Y = 2 failures) = $\frac{(0.60)^2}{2!} e^{-0.60}$ = 0.09879.

(d) Compute the Pr that the time between 2 successive failures exceeds 3 weeks. Pr(T > 3 weeks) = P[X(3 weeks) = Y = 0 failures] = $e^{-0.60}$ = 0.548812 = R(at 3 weeks).

(e) Compute the Pr that the next downtime (or failure measured from the last) occurs within the interval [2, 4 weeks].

$$\Pr(2 < TTF < 4 \text{ weeks}) = \int_2^4 \lambda e^{-\lambda t}\, dt = F(4) - F(2) = [1 - R(4)] - [1 - R(2)] = R(2) - R(4) =$$

$e^{-0.4} - e^{-0.8}$ = 0.2209911. Note that this last answer can also be obtained by using the relationship between the Poisson and the exponential as follows:

Pr(2 < The next TTF from the last < 4 weeks) = Pr[X(2 weeks) = 0]×Pr[at least one failure occurs during the interval (2, 4 weeks)].

Exercise 2. (a) For the above computer network (with λ = 0.20 per week), compute the Pr that the time between next 2 failures will be shorter than 2 weeks. (b) Compute the Pr that there will be exactly 4 failures in the next 6 weeks. At most 6 failures in the next 6 weeks. (c) Compute the Pr that the time (since the last failure) to the next downtime will be less than 4 weeks. Further, compute the Pr that the next failure occurs 3 to 5 weeks after the last failure. ANS: (a) 0.32968, (b) 0.02602, 0.99975, (c) 0.550671, 0.18093.

The Weibull, $W(t_0 = \delta, \theta, \beta)$, as the Underlying Lifetime Distribution

It is well known that the underlying distribution of almost any manufactured dimension can be approximately modeled by a normal (or Laplace-Gaussian) pdf. The Weibull pdf plays the same important role for the underlying distribution of TTF (or lifetime) of most mechanical or electrical components or systems. To arrive at a Weibull pdf, consider an exponential pdf at the constant rate λ = 1 failure per unit of time. Then, for convenience letting $t_0 = \delta$, we observe that

$$\int_0^\infty e^{-x}\, dx = 1. \qquad (4)$$

In Eq. (4), we make the transformation x = $\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta}$, $\beta > 0$. Then dx/dt = $\beta\,(\frac{t-\delta}{\theta-\delta})^{\beta-1}\,(\frac{1}{\theta-\delta})$, and substitution into (4) yields

$$\int_{\delta}^{\infty} e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}\,\beta(\frac{t-\delta}{\theta-\delta})^{\beta-1}\,\frac{dt}{\theta-\delta} = 1. \qquad (5)$$

Since the value of the integral in Eq. (5) is equal to 1 (or 100%), the integrand $(\frac{\beta}{\theta-\delta})\times(\frac{t-\delta}{\theta-\delta})^{\beta-1}$ $e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$ must be a Pr density function (pdf) over the range $[\delta, \infty)$. The pdf f(t) = $(\frac{\beta}{\theta-\delta})\times$ $(\frac{t-\delta}{\theta-\delta})^{\beta-1}e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$, $t \geq \delta = t_0$, and f(t) = 0 for $0 \leq t < \delta = t_0$, is called the Weibull model, denoted W($\delta = t_0, \theta, \beta$), with minimum (or guaranteed) life $t_0 = \delta$ (the location parameter), the characteristic life θ, and slope (or the shape parameter) β; $(\theta - \delta)$ is called the scaling parameter. Different authors tend to use different symbols for the three parameters of a Weibull pdf, but I believe that Ebeling & I are using symbols that are common, specially β is the most common notation for the slope (or shape) of the Weibull, and θ is the most common notations for the characteristic life t_c.

Special Cases of the Weibull Model

(i) When $\beta = 1$, the Weibull becomes the exponential pdf with minimum life $t_0 = \delta$ and mean-life (or characteristic life t_c) equal to θ. The Weibull pdf with slope $\beta = 1$ can be used to model the TTF of a component during its useful life (constant failure rate = CFR).

(ii) When $\beta = 2$ and $\delta = 0$, the Weibull becomes the Rayleigh density function f(t) = $(2/\theta)(t/\theta)\,e^{-(t/\theta)^2} = \lambda t e^{-\lambda t^2/2}$, where $\theta = t_c = \sqrt{2/\lambda}$.

(iii) When $0 < \beta < 1$, it will be shown that the hazard function of the Weibull is a decreasing function of time (DFR = decreasing failure rate) so that the Weibull may be used to model the TTF during the burn-in (or debugging, or infant-mortality) period of a component (see pp. 31-32 of Ebeling).

(iv) When $\beta > 1$, the hazard function is increasing (i.e., increasing failure rate = IFR) and the Weibull density can be used to model the TTF during the wear-out period of a component.

(v) When $1 < \beta < 2$, h(t) is an IFR concave function, and for $\beta > 2$, h(t) is an IFR convex function because $d^2h(t)/dt^2 > 0$ when $\beta > 2$.

Example 3. The kilometers to failure of a car's water pump has a Weibull distribution with minimum life $t_0 = \delta = 30{,}000$ km, slope $\beta = 4.0$, and characteristic life $\theta = 60{,}000$ km. (a) Obtain the survivor function at time t and compute its RE at 50,000 km. (b) Compute the mean and median lives. (c) Compute the values of σ and the IQR interquartile range). (a) R(t) = Pr(T > t) =

$$\int_t^{\infty} \frac{\beta}{\theta-\delta} e^{-(\frac{x-\delta}{\theta-\delta})^{\beta}} (\frac{x-\delta}{\theta\ -\delta})^{\beta-1} dx = \int_{(\frac{t-\delta}{\theta-\delta})^{\beta}}^{\infty} e^{-u} du \quad \rightarrow \quad R(t) = \begin{cases} 1, & 0 \le t \le \delta \\ e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}, & \delta \le t < \infty \end{cases}, \qquad (6)$$

where we have made use of the transformation $(\frac{x-\delta}{\theta-\delta})^{\beta} = u$ and $du = \beta\,(\frac{x-\delta}{\theta-\delta})^{\beta-1} \times (\frac{dx}{\theta-\delta})$. Therefore, the survivor function for the water pump is R(t) = 1 for $0 \le t \le 30{,}000$ km, and

$R(t) = e^{-(\frac{t-30000}{30000})^4}$, for $t \ge 30000$ km; as a result $R(50000) = e^{-0.19753} = 0.820755$ (see the graph of R(t) on p. 35 of these notes).

Due to the fact that all Weibull rvs in the universe have a characteristic reliability $R(\theta = t_c) = e^{-1} = 0.36788$, the parameter θ is called the characteristic life because this is only the characteristic of all Weibull distributions (no other lifetime density has this property). Further, the Weibull RE function increases as the slope β increases up to the characteristic life $\theta = t_c$ and then decreases with respect to β for values of t beyond θ. To compute the MTTF, we make use of Eq. (3).

(b) $E(T) = \int_0^{\infty} R(t)\,dt = \int_0^{\delta} dt + \int_{\delta}^{\infty} e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}} dt$; letting $x = (\frac{t-\delta}{\theta-\delta})^{\beta}$ in the 2nd integral results in E(T) =

$\delta + \frac{\theta-\delta}{\beta} \int_0^{\infty} e^{-x} (x^{1/\beta})^{1-\beta} dx = \delta + \frac{\theta-\delta}{\beta} \times \int_0^{\infty} x^{(1/\beta)-1} e^{-x} dx$. Since by definition, $\Gamma(\alpha)$ =

$\int_0^\infty x^{\alpha-1}e^{-x}dx$, $\alpha\,\Gamma(\alpha) = \Gamma(\alpha+1)$, $\Gamma(1/2) = \sqrt{\pi}$, and for a positive integer $\Gamma(n) = (n-1)!$, we obtain

$$\text{MTTF} = E(T) = \delta + (\theta - \delta)\,\frac{\Gamma(1/\beta)}{\beta} = \delta + (\theta - \delta)\,\Gamma[(1/\beta) + 1] \qquad (7)$$

The above MTTF $\le \theta$ iff $\beta \ge 1$ (i.e., only for the CFR = constant failure rate, and IFR = increasing failure rate cases).

Note that when the shape $\beta = 1$, the Weibull reduces to the exponential and the MTTF becomes equal to θ. Using equation (7) the MTTF of the water pump is equal to MTTF = 30,000 + 30000 $\Gamma(1.25)$ = 30000 + 30000 $\times$ 0.9064025 = 57192.074312 km, where I have used the Matlab function gamma(1.25) = 0.9064025; equation (7) shows that as $\beta \to \infty$, the value of $E(T) \to \theta$.

Weibull Percentiles (or the p^th Quantiles)

Let t_p be the $100\times p^{th}$ percentile (or p^{th} fractile, or p^{th} quantile) of a Weibull pdf, i.e., p is the fraction of the entire population failing by the time t_p. Therefore, $R(t_p) = 1 - p \to e^{-(\frac{t_p-\delta}{\theta-\delta})^{\beta}} = 1 - p$ $\to$ The p^{th} quantile (or the inverse function) is

$$t_p = \delta + (\theta - \delta)\times[\ln(\frac{1}{1-p})]^{1/\beta} \qquad (8a)$$

Eq. (8a) shows that the Weibull inverse function is given by

$$F^{-1}(p) = \delta + (\theta - \delta)\times[\ln(1-p)^{-1}]^{1/\beta}, \quad 0 \le p \le 1 \qquad (8b)$$

Eq. (8b) paves the way to simulate the Weibull distribution by first generating a U(0, 1) random number, Rand, and then setting p = Rand in Eq. (8a) in order to obtain one simulated Weibull distributed deviate t_W. [It can be proven that all continuous cdfs that are (strictly) increasing have the same pdf, namely U(0,1), see my Appendix]. To compute the median life (or the 50^{th} percentile denoted by $t_{0.50} = t_{med}$), we insert p = 0.50 in (8a), which yields

$$t_{0.50} = \delta + (\theta - \delta)(\ln 2)^{1/\beta} \qquad (8c)$$

For our water pump, the median life is equal to $t_{0.50} = 30000 + 30000(\ln 2)^{0.25} = 57373.329174$. Since the median exceeds the mean, and both are less than the mode (= 57918.14577306299), then the water pump's TTF distribution should be negatively skewed. The 25^{th} percentile, or the 1^{st} quartile, of a Weibull pdf from equation (8a) is given by $t_{0.25} = \delta + (\theta - \delta)\times[\ln(4/3)]^{1/\beta}$ and its 3^{rd} quartile (or 75^{th}

percentile) is given by $t_{0.75} = \delta + (\theta - \delta)\times[\ln(4)]^{1/\beta}$, and hence the interquartile range

$$IQR = t_{0.75} - t_{0.25} = (\theta - \delta)\times[(1.3862944)^{1/\beta} - (0.28768207)^{1/\beta}] \tag{8d}$$

For our water pump, these values are $t_{0.25}$ = 51970.98129744, $t_{0.75}$ = 62552.5578146, and IQR = 10581.57651703 km. Note that almost all family of distributions in the field of statistics have an IQR that exceeds the corresponding standard deviation σ. Nearly all statistical family of distributions have an IQR satisfying $0.20\sigma < IQR \le 1.70\sigma$. Like σ, the IQR is a measure of variability. In fact for a normal universe, $\sigma = IQR/1.3489795$. It can easily be shown that the modal point of the $W(\delta, \theta, \beta)$ is given by

$$MO = \delta + (\theta - \delta)[(\beta - 1)/\beta]^{1/\beta} \tag{9}$$

iff $\beta \ge 1$. The modal point for $0 < \beta < 1$ does not exist. Since the Weibull pdf for water pump is negatively skewed, then for certain MTTF < $t_{0.50}$ < MO. For the water pump, the value of the mode from Eq. (9) is MO = 57918.14577306299 > $t_{0.50}$ > MTTF = 57192.074312 km, as expected.

<u>Exercise 3</u>. Show that the variance of a Weibull rv, T, is given by

$$\sigma_T^2 = V(T) = (\theta - \delta)^2 \left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right]. \tag{10}$$

Thus, the relative IQR is equal to

$$IQR/\sigma_T = \frac{(1.3862944)^{1/\beta} - (0.28768207)^{1/\beta}}{\sqrt{\Gamma(1 + 2/\beta) - \Gamma^2(1 + 1/\beta)}}$$

Verify that the (coefficients of) skewness and the standardized 4th central moment are given by

$$\alpha_3 = E\left[\left(\frac{T - \mu_1'}{\sigma}\right)^3\right] = \frac{\Gamma(1+\frac{3}{\beta}) - 3\Gamma(1+\frac{2}{\beta})\Gamma(1+\frac{1}{\beta}) + 2\Gamma^3(1+\frac{1}{\beta})}{\left[\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})\right]^{3/2}},$$

and

$$\alpha_4 = E\left[\left(\frac{T - \mu_1'}{\sigma}\right)^4\right] = \frac{\Gamma(1+\frac{4}{\beta}) - 4\Gamma(1+\frac{3}{\beta})\Gamma(1+\frac{1}{\beta}) + 6\Gamma(1+\frac{2}{\beta})\,\Gamma^2(1+\frac{1}{\beta}) - 3\Gamma^4(1+\frac{1}{\beta})}{\left[\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})\right]^2}.$$

You may verify that the value of Weibull skewness becomes zero at β_0 = 3.60234942571, i.e.,

α_3(3.602349426) = 0 and its α_4 at β = 5.77278121423 becomes equal to 3. It can be verified that Weibull skewness is positive iff $\beta < \beta_0$. Recall that the skewness of any normal distribution is zero and the corresponding normal α_4 is equal to 3. I have verified using Matlab that the value of β at which the mean, the median and the mode of the Weibull become almost equal is given by β = 3.43954061942, and hence when $\beta \cong 3.44$, the Weibull Prs can closely be approximated by a normal distribution with mean given by equation (7) and with a variance from equation (10). Further, it can be verified that V(T) is a decreasing function of β.

For the water pump, since the value of the shape parameter β = 4 (with α_3 = – 0.087237and α_4 = 2.747830 which are close to those of a Gaussian distribution of 0 and 3), we should be able to approximate its RE at 50000 km fairly closely with a normal pdf with mean MTTF = 57192.074312 and standard deviation σ = 7628.58621 km, which was calculated by taking the square root of Eq. (10). That is, R(50000) = Pr(T > 50000) $\cong$ Pr(Z > $\frac{50000-57192.074312}{7628.58621}$) = Pr(Z > – 0.94278) = Φ(0.94278) = 0.827103, which is fairly close to the exact RE of 0.820755 computed directly from the Weibull RE function. Further, the value of IQR from the Eq. (8d) is 1.3871σ.

Unlike the exponential (with β = 1) with zero minimum life for which the coefficient of variation CV = σ/μ = 100% (i.e., the mean and standard deviations are equal), for a general Weibull with $\delta > 0$, the CV is always less than 100% for all $\beta > 1$. Table 1 gives the relationship between the CV of a Weibull with $\delta = 0$ for various values of the slope β and any characteristic life θ (note that when δ = 0, then θ is called the scale parameter, while β is always the shape parameter). Almost all statistical software report kurtosis of a distribution relative to that of the normal α_4 = 3, i.e., kurtosis =

Table 1. (The Coefficient of Variations for Different Weibull Slopes and δ = 0)

β	0.20	0.40	0.50	0.60	0.80	1	1.5	2.0	2.5	3.0	3.5
CV	15.843	3.1409	2.2361	1.7581	1.2605	100%	0.6790	0.5227	0.4279	0.3635	0.3165
IQR/σ	0.0027	0.2125	0.4112	0.6042	0.9057	1.0986	1.3174	1.3838	1.4014	1.4016	1.3954
β	4.0	4.5	5.0	10	15	20	25	30	40	50	60
CV	0.2806	0.2521	0.2291	0.1203	0.0818	0.0620	0.0499	0.0418	0.0315	0.0253	0.0211
IQR/σ	1.3871	1.3783	1.3697	1.3135	1.2880	1.2739	1.265	1.2589	1.2510	1.2462	1.2430

At β = 100, CV = 0.0127 and Rel-IQR = 1.2363. CV and IQR/$\sigma \to 0$ as $\beta \to \infty$.

$\beta_4 = \alpha_4 - 3$. Further, the values of CV (coefficient of variation) and relative IQR in Table 1 are valid only for the case of zero minimum life.

The Hazard (or The Failure Rate) Function h(t)

By definition the failure rate (FR) of any device is defined as the failure rate during (t, t +Δt) given that the device age is t, i.e.,

$$FR(t) = \frac{P(t \le T \le t+\Delta t \mid T > t)}{\Delta t} = \frac{Pr(t \le T \le t+\Delta t)}{Pr(T > t)\Delta t} = \frac{R(t) - R(t+\Delta t)}{R(t)\Delta t} \qquad (11)$$

Equation (11) implies that the failure rate of a component at time t is the Pr that it will fail in the interval (t, t + Δt) given that its life has exceeded time t (i.e., given that the age of the component is t). Put differently, the failure rate of a population of identical items at time t is the proportion of the units failing per unit of time in the interval (t, t + Δt) amongst all the survivors at time t. The hazard function (HZF), h(t), is simply the instantaneous FR, i.e.,

$$h(t) = \lim_{\Delta t \to 0} [-\frac{1}{R(t)}][\frac{R(t+\Delta t) - R(t)}{\Delta t}] = -\frac{dR(t)/dt}{R(t)} = \frac{f(t)}{R(t)} \qquad (12)$$

Put differently, if we have a system with N_0 identical items on test at time 0, $N_s(t)$ survivors at time t and $N_f(t)$ failed items by time t, then by the above definition h(t) is the rate of failure, $dN_f(t)/dt$ (this derivative is not quite appropriate because $N_f(t)$ is discrete), only amongst the survivors $N_s(t)$ beyond time t, i.e., $h(t) = \frac{dN_f/dt}{N_s(t)} = \frac{d[N_0 - N_s(t)]/dt}{N_s(t)} = \frac{-dN_s(t)/dt}{N_s(t)} = \frac{-d[N_s(t)/N_0]/dt}{N_s(t)/N_0} =$

$\frac{-dR(t)/dt}{R(t)} = \frac{f(t)}{R(t)}$, as before. The quantity h(t)dt = – dR/R = dF(t)/R(t) = $P(t \le T \le t + dt)/R(t) =$ $Pr(t \le T \le t + dt \mid T > t)$ gives the proportion of items that will fail within (t, t + dt) amongst those that are still functioning at t. From the cumulative of the hazard function, CHZF = H(t), we can obtain the relationship amongst the RE measures f(t), R(t), and h(t) as shown below (note that Ebeling uses L(t) for the CHZF on his page 30)

$$H(t) = \int_0^t h(x)dx = \int_0^t -dR/R = -\ln\left[R(x)\right]_0^t = -\ln[R(t)] \rightarrow R(t) = e^{-H(t)}$$

$$R(t) = e^{-\int_0^t h(x)dx} \rightarrow f(t) = h(t)\times R(t) = h(t)e^{-H(t)} \ \& \ MTTF = E(T) = \int_0^{\infty} R(t)\,dt \qquad (13)$$

Exercise 4. Use equations (12) and (13) to prove that the exponential is the only family of density functions in the universe with a constant hazard function (or failure rate) λ. (b) Obtain the HZF of a Weibull pdf.

Exercise 5. Given the hazard function $h(t) = \begin{cases} 0, & 0 \le t \le 50 \text{ hours} \\ 2\times 10^{-5}, & t \ge 50 \end{cases}$, obtain the RE measures R(t), f(t), and E(T). (b) Compute the RE at 1000 hours. ANS: R(1000) = 0.98118. (c) Given that the component's age is T_0 = 100 hrs, compute the Pr that it will survive another 500 hours. (d) Show that the HZF for any Weibull pdf is given by $h(t) = \begin{cases} 0, & t < \delta \\ \frac{\beta}{\theta-\delta}\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta-1}, & t \ge \delta \end{cases}$, and therefore, the HZF of a Weibull pdf with zero minimum life is given by $h(t) = \frac{\beta}{\theta}(t/\theta)^{\beta-1}$, $t \ge 0$. Note that the $W(\delta = t_0, \theta, \beta > 1)$ provide a family of lifetime densities with IFRs, while the $W(\delta, \theta, \beta < 1)$ provide a family of mortality densities with DFRs, and at $\beta = 1$, $h(t) = \frac{1}{\theta}(t/\theta)^0 = \frac{1}{\theta} = \lambda$ represents the family of CFR. Further, when the minimum life $\delta = t_0 = 0$, some authors use $\lambda = 1/\theta$ so that the Weibull, $W(0, \theta = 1/\lambda, \beta)$, hazard rate function becomes $h(t) = \beta\lambda(\lambda t)^{\beta-1}$, $t \ge 0$. Note that when the guaranteed-life, $\delta > 0$, most authors in Reliability Engineering use α in lieu of $\theta-\delta$, where now $\alpha = \theta-\delta$ is the scale of the 3-parameter Weibull. Ebeling illustrates on pp. 53-54 that $R(t) \le e^{(-t/MTTF)}$ iff the process is a DFR. See the Example 3.6 at the bottom of his p. 53.

Hazard Models and Product Life Cycles (See Figure 2.3, p. 31 of Ebeling)

There are three types of failures: (1) early (or burn-in, or debugging) period with DFR, (2) useful life or chance failures with CFR, and (3) wear-out failures with IFR.

(1) Early Life (or Infant-Mortality) Failures. These are caused by sub-standard components (due to manufacturing defects and/or design) and may be eliminated by: (i) Better QC, (ii) Pre-

operational burn-in, or screening. Only during break-in period the hazard model is a decreasing function (see Table 2.1 atop p. 32 of Ebeling). Note that Ebeling uses the notation $\lambda(t)$ for the hazard function which is nearly as common as h(t), while I am using λ to denote the case of only constant hazard (or failure) rate. Any other notation but $h(t) = \lambda(t)$ is not as common in reliability engineering literature.

Example 4. The Weibull $h(t) = \frac{1}{2\theta}(t/\theta)^{-1/2}$ is a hazard function, i.e., T is $W(\delta = 0, \theta, \beta = 1/2)$, $0 \le t \le T_e = t_1$ (see Figure 2.3 on page 31 of Ebeling), and with DFR. Thus, it can be used to model early life (or burn-in period, or break-in period).

(2) Useful Life or Chance Failures. These are due to unexpectedly high stress conditions or random accidents. The impact of such failures may be reduced by (i) improved design, (ii) reduced stress levels. Only during this period the HZF is a constant, i.e., $h(t) = 1/\theta = \lambda$, $T_e \le t \le T_w = t_2$.

(3) Wear-out failures can be reduced by: (i) systematic replacement, (ii) better design, or (iii) reduced operational stress. The hazard function is always increasing over the interval $(t_2 = T_w, \infty)$. As an example $h(t) = \frac{1.5}{\theta}(t/\theta)^{1/2}$, $T_w \le t < \infty$, which is the $W(\delta = 0, \theta, \beta = 3/2)$ hazard function with IFR. The interval (T_e, T_w) is generally referred to as the mission time (or useful life).

Conditional Reliability (Section 2.5, pp. 32-34 of Ebeling)

C. E. Ebeling (2010) defines Conditional Reliability atop his page 33, where T_0 is the end of burn-in or warranty period and it is known that the component has already survived beyond time T_0. However, his definition is somewhat misleading because the 1st statement must be changed according to either of the following 2 modifications:

(1) $$R(T_0 + t \mid T_0) = \Pr(T > T_0 + t \mid T > T_0) = \frac{\Pr(T > T_0 + t)}{\Pr(T > T_0)} = \frac{R(T_0 + t)}{R(T_0)} = \frac{e^{-\int_0^{T_0+t} h(x)\,dx}}{e^{-\int_0^{T_0} h(x)\,dx}} =$$

$$e^{-\int_0^{T_0+t} h(x)\,dx + \int_0^{T_0} h(x)\,dx} = e^{-\int_0^{T_0+t} h(x)\,dx - \int_{T_0}^{0} h(x)\,dx} = e^{-\int_{T_0}^{T_0+t} h(x)dx} \quad \text{(2.17E, p.33)}$$

Here, T_0 is the in-plant burn-in period, while t is the service life.

(2) R(t | age T_0) = Pr(T > t | T > T_0) = $\frac{\Pr(T > t)}{\Pr(T > T_0)} = \frac{R(t)}{R(T_0)} = e^{-\int_{T_0}^{t} h(x)dx}$.

Note that this last conditional RE is different from that of Eq. (2.17) atop page 33 of Ebeling. For the family of Weibull with zero minimum life the above result of part (2) reduces to R(t |T_0) = $e^{-\int_{T_0}^{t}(\lambda\beta)(\lambda x)^{\beta-1}dx} = e^{(\lambda T_0)^{\beta}-(\lambda t)^{\beta}}$, t > T_0. Note that in this case T_0 is the in-plant burn-in period, while t is the age of the component measured from time zero (that includes the burn-in duration T_0).

The Example 2.8 on page 33 of Ebeling

The hazard function in years is given by h(t) = $0.5(0.001)(0.001t)^{-0.5}$, which is the Weibull HZF with $\delta = 0$, $t_c = \theta = 1/\lambda = 1/0.001 = 1000$ years, and slope $\beta = 0.5$ (a DFR lifetime). To obtain the design life at which the unconditional RE function is equal to 0.90 (i.e., the 10% failure point), we let R(t) = $e^{-(\lambda t)^{\beta}} = e^{-(0.001t)^{0.5}} = 0.90 \rightarrow t_{0.10}$ from Eq. (8b) $\rightarrow F^{-1}(0.10) = \theta\times[\ln(1-0.1)^{-1}]^{1/\beta} = 1000$ $[\ln(1/0.90)]^{1/0.5} = 11.10084$ years, where I have made use of Eqs. (8a, b). The 10% failure point is also referred to the B_{10}-life, a term that is a carryover from ball bearing testing. This implies that in order to maintain a survivor Pr of 0.90, the component must be replaced roughly every 11.101 years.

Ebeling now assumes a conditional in-plant burn-in period of 0.50 year and the component still functioning perfectly after six month, and as a result from part (2) above R($t_{0.1}$| T_0 = 0.5) = $e^{(\lambda T_0)^{\beta}-(\lambda t)^{\beta}} = e^{(0.0005)^{0.50}-(0.001t_{0.10})^{0.5}} = 0.90 \rightarrow (0.0005)^{0.50}-(0.001t_{0.10})^{0.5} = \ln(0.90) = -0.10536051566 \rightarrow 0.02236068 + 0.10536051566 = (0.001t_{0.10})^{0.5} \rightarrow 0.001t_{0.10} = (0.1277212)^2$ $\rightarrow t_{0.10}$ = B-10 life = 0.016312704/0.001 = 16.312704 years, which is different from 15.8 years given by Ebeling near the bottom of page 33. However, if we subtract the 0.50 year of in-plant burn-in period from $t_{0.10}$ = 16.312704, we would obtain 15.812704 years which is the length of service life that excludes the burn-in period. Note that Ebeling uses the uncommon notation of $t_{0.90}$ for the 10% life (or the 10^{th} percentile). If we use Ebeling's Eq. (2.17) but include of my revision, we obtain: R(t + T_0| T_0) = $e^{-\int_{T_0}^{T_0+t} h(x)dx} = e^{(\lambda T_0)^{\beta}-[\lambda(T_0+t)]^{\beta}} = 0.90 \rightarrow [0.001(0.5+t_{0.10})]^{0.5} = 0.1277212 \rightarrow$

$0.001(0.5 + t_{0.10})$ = 0.016312704 → $t_{0.10}$ = (0.016312704/0.001) – 0.50 = 15.812704 service life, as before. Note that only when h(t) is a DFR function, then in plant burb-in will improve service life. This is due to the fact that the proportion $F(T_0)$ that fail during burn-in period must either get repaired (i.e., renewed to their perfect operating condition) and then sold to customer, or they must be discarded if manufactured components are irreparable.

An Example of a Normal Density Function TTF (used mainly during the useful and wear-out cycles)

Suppose a car's fan belt has a normal failure density with MTTF = 45,000 miles and standard deviation σ = 5000 miles, i.e., its TTF ~ N(45000, 25×10^6). Compute the RE and hazard functions at t = 30,000 miles. The TTF distribution is depicted in Fig. 1. R(30,000) = Pr(T > 30,000) = P(Z > –3) = $\Phi(3)$ = 1 – $\Phi(-3)$ = 1 – 0.00135 = 0.99865. Thus among 100,000 cars with such fan belts, we expect roughly 100000(1–0.99865) = 135 failures by 30,000 miles. In order to compute the hazard rate at 30,000 miles, we need to compute the value of f(30,000) because h(t) = f(t)/R(t). Clearly, f(30,000)

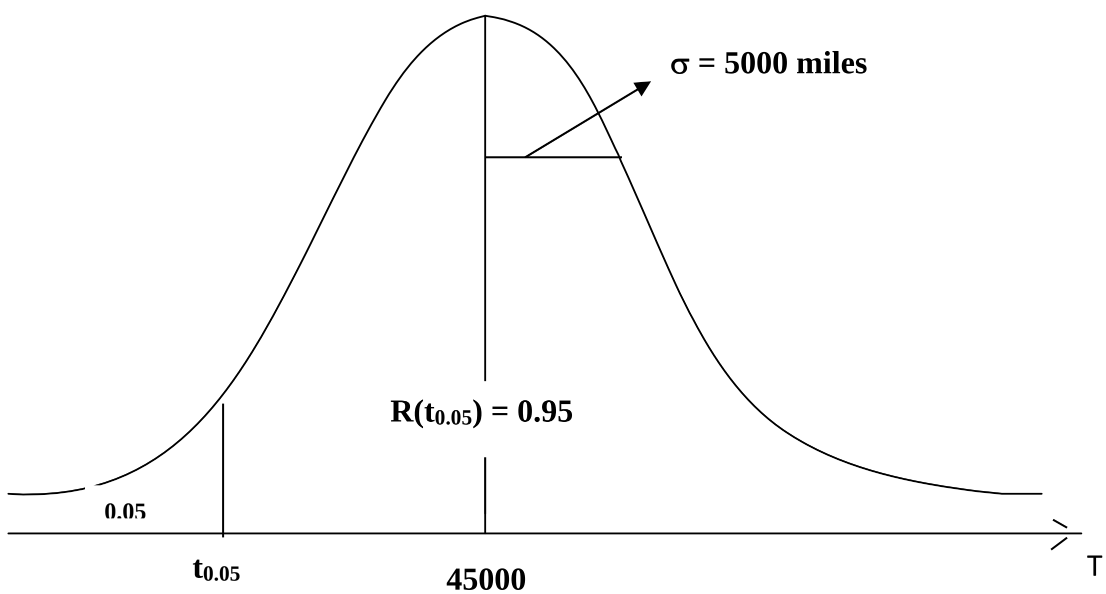

Figure 1. The TTF Distribution of Fan Belts

$= \frac{1}{5000\sqrt{2\pi}} e^{-(-3)^2/2}$ = 0.8863697×10^{-6} which yields h(30000) = $0.8875781\times10^{-6} \cong$ (1/1,000,000) failures per mile. Therefore, if we consider one million such fan belts each of age 30000 miles, then we expect roughly one failure in the next mile amongst the one million belts. Figure 1 shows that if we wish only a 5% chance of field failure, then we must replace the fan belts at the age $t_{0.05} = x_{0.05}$= B-5 life = 45000 – 1.645×σ = 45000 – 8225 = 36775 miles. Note that $F(t_{0.05}) = 0.05 = Q(t_{0.05})$, or $R(t_{0.05})$ = 0.95; further, I have seen the notation U(t) for unreliability instead of Q(t) by some authors.

Summary of the Three RE Measures

1. If f(t) is known or given, then R(t) = Pr(T > t) = 1 – F(t) = $\int_t^\infty f(x)\,dx$.

2. E(T) = $\int_0^\infty R(t)\,dt$, and h(t) = f(t)/R(t), where F(t) gives the proportion of population failing by time t.

3. If R(t) is known or given, then f(t) = –dR/dt, E(T) = $\int_0^\infty R(t)\,dt$, and h(t) = f(t)/R(t).

4. If h(t) is known or specified, then H(t) = $\int_0^t h(x)\,dx$ = – ln[R(t)], where

$0 \le H(t) < \infty$. R(t) = $e^{-H(t)} = e^{-\int_0^t h(x)\,dx}$, E(T) = $\int_0^\infty R(t)\,dt$, f(t) = R(t)×h(t), and the average failure rate over the interval $[t_1, t_2]$ is given by $\bar{\lambda} = \int_{t_1}^{t_2} h(x)\,dx \,/ (t_2 - t_1) = \dfrac{\ln R(t_1) - \ln R(t_2)}{t_2 - t_1}$, read as average number of failures per unit (of time), or the average failure rate (AFR) during $[t_1, t_2]$. Thus the AFR during the mission interval (0, t] is given by AFR(0, t) = $\dfrac{\ln 1 - \ln R(t)}{t}$ = H(t)/t.

Properties of the Hazard function h(t)

(a) h(t) ≥ 0 for all t. (b) h(0) = $\dfrac{f(0)}{R(0)}$ = f(0) → h(0) must be finite at t = 0 unless f(0) = ∞ such as the W(δ, θ, β = 1/2). (c) $\lim_{t \to \infty} h(t) = \infty$ must be infinite, simply implying that any man-made system must have a finite life (or a finite TTF), i.e., no man-made system can last forever! (d) $\int_0^{\infty} f(t)dt$ = $\int_0^{\infty} h(t)e^{-H(t)}dt$ = 1.

Example 5. A device has the TTF pdf, f(t) = $te^{-t^2/2}$, which is the Weibull pdf W(δ = 0, θ = $\sqrt{2}$, β = 2). (a) Find the RE and hazard functions. (b) If N_0 = 50 such items are placed on life test at time zero and only 27 are still operating after 1 hour, find the approximate expected number of failures in the interval (1, 1.10 hour) using the hazard function.

(a) R(t) = $\int_t^{\infty} xe^{-x^2/2}dx$ = $e^{-t^2/2}$, $0 \le t < \infty$. h(t) = f(t) /R(t) = t, $t \ge 0$.

(b) The exact procedure : Pr($1 \le T \le 1.1 \mid T > 1$) = $\dfrac{R(1) - R(1.1)}{R(1)}$ =

$1 - e^{-1.1^2/2} / e^{-1/2}$ = 0.09968 → E($N_f \mid 1 \le T \le 1.1$) = 27×0.09968 = 2.691 units.

The Approximate Procedure

h(t) = t, assuming Δt = 0.02 → we are dividing the interval = (1, 1.1) into 5 subintervals each of length Δt = 0.02, and the number of survivors at the end of 1 hour is N_s = 27. The 5 subintervals are (1, 1.02), (1.02, 1.04), (1.04, 1.06), (1.06, 1.08), and (1.08, 1.10).

For subinterval (1, 1.02), h(1.01) × Δt = 1.01 × 0.02 = 0.0202 → E(N_f) = 0.0202 × 27 = 0.5454. For subinterval (1.02, 1.04), h(1.03) × Δt = 1.03 × 0.02 = 0.0206 →

E(N_f) = 0.0206 × 26.4546= 0.5450

h(1.05) × Δt = 1.05 ×0.02 = 0.021 → E(N_f) = 0.021 × 25.910 = 0.5441

h(1.07) × Δt = 1.07 ×0.02 = 0.0214 → E(N_f) = 0.0214 × 25.366 = 0.5428

h(1.09) × Δt = 1.09 ×0.02 = 0.0218 → E(N_f) = 0.0218 × 24.823 = 0.5410 →

E($N_f \mid 1 \le T \le 1.1$) = 2.7184. Note that the amount of error 2.7184 – 2.6910 = 0.0274 is perhaps due to the fact that the value of Δt = 0.02 is probably too large. To better understand the above procedure, set Δt at 0.01 and obtain E($N_f \mid 1 \le T \le 1.1$) = 2.704739.

Exercise 6. An automobile manufacturer would like to determine the length of time, measured in miles, for its warranty period such that it will allow only up to 1% engine failure. From past history, the lifetime distribution of its engines is W(5000, 40000, 2.75). Determine the Warranty period for the manufacturer's engines. ANS: More than 11000 miles.

The RE Measures for a Gamma pdf

By definition $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. Dividing both sides of this last definition by $\Gamma(\alpha)$ we obtain: $1 = \int_0^\infty \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx$, and per force the integrand $f(x;\alpha) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx$ must be a density function called the Standard Gamma density. Ebeling uses the uncommon notation γ for α. When α is a positive integer, the common notation for the shape parameter is n.

One integration by parts of Γ(n) will show that Γ(n) = (n – 1)Γ(n – 1). After (n – 1) integration by parts, we obtain Γ(n) = (n – 1) (n – 2) .... Γ(1) = (n–1)!. Inserting n = 1 into the definition of Γ(n) yields $\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx = \int_0^\infty e^{-x}dx = 1$; therefore, Γ(n) = (n – 1) (n – 2) ...1 = (n – 1)!. $\rightarrow$ 0! = Γ(1) = 1. Further, this last result also implies that Γ(n + 1) = n Γ(n); it can be proven that $\Gamma(1/2) = \sqrt{\pi}$ and hence Γ(3/2) = (1/2)Γ(1/2), Γ(5/2) = Γ(3/2 + 1) = $\frac{3}{2}\Gamma(\frac{3}{2})$, etc. For example, Γ(10) = 9! = 362880. To obtain the Gamma pdf, we make the transformation x = λt in the definition $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. $\rightarrow$

$$\Gamma(\alpha) = \int_0^\infty (\lambda t)^{\alpha-1}e^{-\lambda t}\,\lambda\, dt \rightarrow 1 = \frac{1}{\Gamma(\alpha)}\int_0^\infty (\lambda t)^{\alpha-1}e^{-\lambda t}\lambda dt \rightarrow \int_0^\infty \frac{1}{\Gamma(\alpha)}(\lambda t)^{\alpha-1}e^{-\lambda t}\lambda dt = 1 \qquad (14)$$

Equation (14) clearly shows that the integrand must be a pdf over the range [0, ∞) because its integral over [0, ∞) yields 100%. The function under the integral in (14) is called the Gamma pdf in statistical literature with the rate-parameter λ and the shape parameter α.

$$f(t) = \frac{\lambda(\lambda t)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda t} \qquad (15)$$

The meaning of these 2 parameters will be made clear in the application examples provided below, where λ is also called the scale parameter. Note that Ebeling on p. 85 uses the uncommon notations $1/\alpha$ for λ. The major application of the gamma pdf occurs from the fact that the sum of n independent and identical exponential rvs, $T_n = t_1 + t_2 + ... + t_n$, has a gamma time to failure density function with parameters n and λ, where each t_i is distributed according to $\lambda e^{-\lambda t_i}$. Further, the gamma density has application in maintenance scheduling where the amount of deterioration during an interval $[t_1, t_2]$ has a gamma pdf with scale parameter λ and shape $n = \gamma(t_2 - t_1)$, where $\gamma > 0$ is a constant of proportionality.

The first 4 moments of the Gamma pdf are given by $E(T_n) = E(t_1 + t_2 + ... + t_n) = n/\lambda$, $V(T_n) = V(t_1 + t_2 + ... + t_n) = n/\lambda^2$, and you may verify that $MO = (n - 1)/\lambda$, $\alpha_3 = 2/\sqrt{n}$, $\alpha_4 = 3 + (6/n)$, and hence he kurtosis is equal to $\beta_4 = 6/n$. Note that the values of $\alpha_3 = 2/\sqrt{n}$ and $\alpha_4 = 3 + 6/n$ clearly show that the limiting distribution (i.e., as $n \to \infty$) of the Gamma density is the Gaussian $N(n/\lambda, n/\lambda^2)$. Unfortunately, n must exceed 150 (because the exponential is highly skewed) before the normal approximation to Gamma becomes fairly adequate. If α is not a positive integer, then replace n by α in the above formulas.

To obtain the RE function for the Gamma pdf, we make use of its relationship with the Poisson pmf. As an example, suppose that in a data communication system messages arrive at a node at a rate $\lambda = 45$ messages per minute, or the rate $\lambda = 0.75$ messages per second. However, messages arriving at the node are bundled into packets of 5 messages each before they are transmitted over the network. Since the arrival rate $\lambda = 0.75$/sec is a constant, then the interarrival (or intervening) times of individual messages must follow an exponential pdf and as a result the number of arrivals during an interval of length t seconds, denoted by X(t), must have a Poisson pmf with mean $\lambda t = 0.75t$. Suppose we now ask the question "what is the Pr that a packet is formed in less than t = 10 seconds?". I will answer this question one message at a time. The Pr of the 1st arrival (from time 0) in less than t = 10 seconds is given by $Pr(T_1 < 10 \text{ sec}) = Pr[X(10) \geq 1] = 1 - Pr[X(10) = 0] = 1 - e^{-\lambda t} = 0.999447$. This last Pr is also the Pr that the 2nd message arrives in less than 10 seconds measured from the arrival time of the 1st message, and so on for the 3rd, 4th and 5th messages. Next what is the Pr that the 2nd message arrives in less than 10 seconds measured from time zero? $Pr(T_2 < 10) = Pr[X(10) \geq 2] = 1 - Pr[X(10) \leq 1] = 1 - e^{-7.5} - 7.5\, e^{-7.5} = 0.995299$. This is where the connection between the Gamma density and the Poisson pmf occurs. Because we can use the fact that $T_2 = t_1 + t_2$ has a Gamma pdf with rate $\lambda = 0.75$/sec but the parameter n = 2 arrivals, and therefore, $\Pr(T_2 < 10) = \int_0^{10} \frac{0.75}{\Gamma(2)}(0.75t)^1 e^{-0.75t}dt = 1 - e^{-7.5} - 7.5\,e^{-7.5} = 0.995299$, where one integration by parts yields this last result. Similarly, $\Pr(T_5 < 10 \text{ seconds}) = \Pr[X(10) \ge 5 \text{ arrivals}] = 1 - \Pr[X(10) \le 4 \text{ arrivals}] = 1 - \sum_{k=0}^{4} e^{-\lambda t}(\lambda t)^k / k! = 1 - \sum_{k=0}^{4} e^{-7.5}(7.5)^k / k! = 1 - 0.132062 = 0.867938$. If we wish to compute the Pr that the time to form a new packet exceeds 5 seconds, then we can make use of the fact that the two events $[T_5 > 5 \text{ seconds}]$ and $[X(5) \le 4 \text{ arrivals}]$ are equivalent, i.e., $\Pr(T_5 > 5 \text{ seconds}) = \Pr[X(5) \le 4 \text{ arrivals}] = \sum_{k=0}^{4} \frac{(3.75)^k}{k!} e^{-3.75} = 0.677548$. Again the direct integration of the Gamma pdf, $\int_5^{\infty} \frac{0.75}{\Gamma(5)}(0.75t)^4 e^{-0.75t}dt$, will also give the same Pr of 0.677548, i.e., $\int_5^{\infty} \frac{0.75}{\Gamma(5)}(0.75t)^4 e^{-0.75t}dt = 0.677548$. The above discussions now will enable us to obtain a RE function for a Gamma TTF distribution only when n is a positive integer, as illustrated below where T_n denotes the time (from zero) to the n^{th} failure.

$$R(t) = \Pr(T_n > t) = \Pr[X(t) \le (n-1) \text{ failures}] = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad (16)$$

As a direct application of the Gamma RE function in equation (16), consider a 4-unit standby system where the n = 4 units have identical failure rates of $\lambda = 0.00025$/hour. At time t = 0, unit 1 is brought on line using a switch, which is assumed to have a RE = 1; ASA unit 1 fails, unit 2 is put on line with the aid of the sensing switch, and so on until the 4^{th} unit is put on line. Once the 4^{th} unit fails, then the system fails; note that there are 3 cold (or inactive) spares in this example. Our objective is to compute the system RE for a mission of t = 500 hours. Using Eq. (16), we obtain $R_{sys}(500 \text{ hours}) = \Pr(T_4 > 500) = P[X(500) \le 3 \text{ failures}] = \sum_{k=0}^{3} \frac{(0.125)^k}{k!} e^{-0.125} = 0.9999908$. So far, we know the expressions for the Gamma pdf, the RE function, and the MTTF = n/λ. The MTTF for the above standby system is given by E(T) = 4/0.00025 = 16,000 hours. To obtain the value of the hazard function at t = 500 hours, we use the fact that h(t) = f(t)/ R(t) so that h(500) =

$\frac{0.00025(0.125)^{4-1}}{\Gamma(4)}e^{-0.125}$ /0.9999908 = 7.1818443×10^{-8} per hour.

Example 6. The time to failure of an electronic component follows a gamma pdf with mean 600 hours and σ^2 = 12,000 hrs^2. Determine the RE and its hazard rate at 400 hours. Solution: E(T) = n/λ = 600 hours and $\sigma^2 = 12000 = n/\lambda^2$ $\rightarrow$ λ = 0.05, and n = 30 $\rightarrow$ From Eq. (16), R(400) = $\sum_{k=0}^{29}\frac{(20)^k}{k!}e^{-20}$ = 0.978182. From Eq. (15), f(400) = $\frac{0.05(20)^{30-1}}{\Gamma(30)}e^{-20}$ = 0.00062576522 $\rightarrow$ h(400) = f(400)/ R(400) = 0.000639723 per hour.

When the shape α is not a positive integer, then the RE function of the gamma density is given by $R(t) = \int_t^{\infty}\frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x}dx = \int_{\lambda t}^{\infty}\frac{1}{\Gamma(\alpha)}y^{\alpha-1}e^{-y}dy = 1-\frac{1}{\Gamma(\alpha)}\int_0^{\lambda t}y^{\alpha-1}e^{-y}dy$. In Matlab use $R(t) = 1-\frac{1}{\Gamma(\alpha)}$gamcdf($\lambda$t, α). In MS Excel, R(t) = 1–GAMMA.DIST(λt, α, 1, True).

Estimation of RE Measures From Randomly Observed Data

(a) Case of Small to Moderate n (say n < 30), i.e., Raw Data

Consider the example where n = 8 springs are cycled to failure resulting in the failure data 190, 245, 265, 300, 320, 325, 370, and 400 kilocycles. Let $t_{(i)}$ be the ith order statistic, and $p_i = F[t_{(i)}]$. Then p_i is the proportion of the population failing just prior to the ith order statistic $t_{(i)}$. It can be shown that $E(p_i) = \frac{i}{n+1}$ and the median of p_i is $\tilde{p}_i \cong \frac{i-0.30}{n+0.40}$. In RE literature, these are called the mean and median ranks, respectively. For example, for $t_{(3)}$ = 265 KCs, the mean rank is 3/(8 + 1) = 1/3, implying that an average of 33.33333% of the population is expected to fail by the 3rd order statistic $t_{(3)}$ = 265 KC. The value of the median rank for 265 KCs is $\tilde{p}_3 \cong \frac{3-0.30}{8+0.40}$ = 0.32143, implying that we are 50% confident that at most 32.143% of the population will fail by $t_{(3)}$ = 265 KCs. In RE estimation, it is a common practice to use the median ranks

$$\tilde{p}_i = \hat{F}[t_{(i)}] = \frac{i-0.30}{n+0.40} \qquad (17a)$$

as point estimate of the proportion of the population failing just prior to the $t_{(i)}$ and hence a point estimate of RE at $t_{(i)}$ is $\hat{R}[t_{(i)}] = 1 - \frac{i - 0.30}{n + 0.40} = \frac{n - i + 0.70}{n + 0.40}$. Thus, at $t_{(3)}$ = 265 KC a point estimate, a point estimate of RE is $\hat{R}[t_{(3)}] = 1 - \hat{F}[t_{(3)}] = 0.67857$. In order to estimate the hazard rate, we 1st need to obtain a point estimate of f(t). Since by definition f(t) = dF(t)/dt, then a point estimate of f(t) is given by

$$\hat{f}(t) = \frac{\hat{F}[t_{(i+1)}] - \hat{F}[t_{(i)}]}{t_{(i+1)} - t_{(i)}}, \qquad t_{(i)} \le t \le t_{(i+1} \tag{17b}$$

For the interval $t_{(3)} \le t \le t_{(4)}$ = [265, 300], the point estimate of f(t) from Eq. (17b) is $\hat{f}(t) = \frac{0.44048 - 0.32143}{300 - 265}$ = 0.0034, and thus $\hat{h}(t) = \frac{\hat{f}(t)}{\hat{R}[t_{(3)}]} = \frac{0.0034}{0.67867}$ = 0.00501 failures per KC.

(b) Case of Large n or Grouped Data (n > 40)

As an example, consider the failure data given in Table 1.1 on page 7 of Elsayed ("Reliability Engineering by E. A. Elsayed, Addison Wesley Longman, Inc, 1996, ISBN:0-201-63481-3), reproduced atop the next page, where N_0 = 200 items were put on test at time t = 0. Note that the data are not in the raw form but are in grouped format, and N_0 = 200 > 40. The point estimates of the 3 RE measures R(t), f(t) and h(t) can be computed as follows. To understand how to obtain these point estimates, I will select one of the subintervals, say [2001, 3000], and will compute the point estimates $\hat{R}(t)$, $\hat{f}(t)$, and $\hat{h}(t)$ independent of the author's computations. For the 3rd interval of the Table, N_s(t = 2,000 hrs) = 60, N_f(t = 2,000) = 140, N_s(t = 3,000) = 40, and N_f(t =3,000) = 160. Therefore, $\hat{R}(2000)$ = 60/200 = 0.30, $\hat{F}(2000) = 0.70$, $\hat{R}(3000)$ = 40/200 = 0.20, and $\hat{F}(3000)$ = 0.80. For t ≅ 2500, $\hat{f}(t) = \frac{0.80 - 0.70}{1000}$ = 0.0001. To estimate $\hat{h}(t)$ at t ≅2500, we use the fact that $\hat{h}(t) = \hat{f}(t)/\hat{R}(2000)$ = 0.0001/0.30 = $0.0003333\overline{3}$. The complete estimates, computed by me, are summarized in the following Table.

Table 1.1 of E. A. Elsayed (2012, p. 6)

Interval in Hours	0-1000	1001-2000	2001-3000	3001-4000	4001-5000	5001-6000	6001-7000
N_f (= no. of failures)	100	40	20	15	10	8	7

Time in hours	0-1000	1001-2000	2001-3000	3001-4000	4001-5000	5001-6000	6001-7000
N_f	100	40	20	15	10	8	7
$\hat{R}(t)$	(1, 0.5)	(0.50,0.30)	(0.30, 0.20)	(0.20, 0.125)	(0.125, 0.075)	(.075, 0.035)	(0.035, 0)
$\hat{F}(t)$	(0, 0.5)	(0.50, 0.70)	(0.70, 0.80)	(0.80, 0.875)	(0.875, 0.925)	(.925, .965)	(0.965, 1)
$\hat{f}(t)$	0.0005	0.0002	0.0001	0.000075	0.00005	0.00004	0.000035
$\hat{h}(t)$	0.0005	0.0004	0.000333	0.000375	0.0004	0.000533	0.001

The Lognormal (LGN) Failure Model

A continuous rv, T, is said to have a lognormal pdf iff its natural logarithm, X = ln(T), is normally distributed. That is, $X = \ln(T) \sim N(\mu, \sigma^2)$, where $\mu = E(X)$, and $\sigma^2 = V(X)$, $0 \le T < \infty$, while $-\infty < X < \infty$. To obtain the pdf of T, we 1st graph X = ln(T) versus T, as shown in Figure 2. Note that the function x = ln(t) is concave downward because $d^2x/dt^2 = -t^{-2} < 0$ for t > 0. Figure 2 clearly shows that $\Pr(x_1 < X < x_2) = \Pr(t_1 < T < t_2)$. Then by definition of these Prs, it follows that

$$\int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx = \int_{t_1}^{t_2} f(t)\,dt\,, \qquad (18)$$

where f(t) represents the mortality density of T. We now make the transformation x = ln(t) in the integral on the LHS of Eq. (18), where dx = dt/t. Eq. (19) shows that the integrands of the two sides must equal because the limits are identical and both integrations are carried out wrt t,

$$\int_{t_1}^{t_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\ln t-\mu}{\sigma})^2} dt/t = \int_{t_1}^{t_2} f(t)\,dt \qquad (19)$$

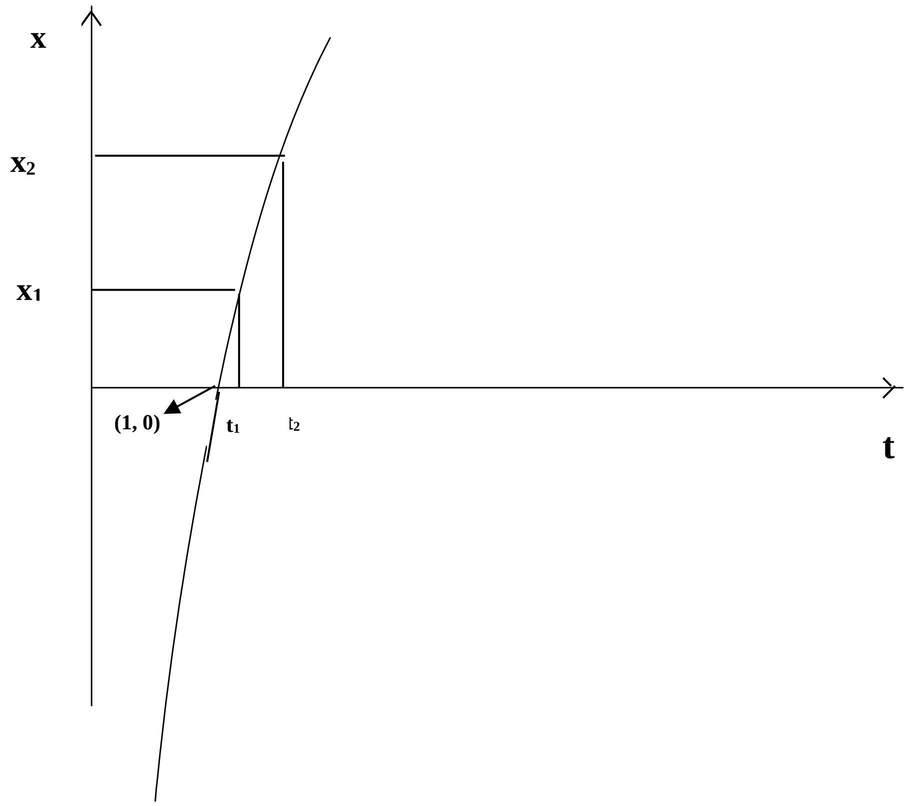

Figure 2. The graph of x = ln(t) versus t

and hence the lognormal (LGN) pdf, LGN, is given by

$$f(t) = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\ln t-\mu}{\sigma})^2} = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln t - \ln(t_{0.5})}{\sigma}\right]^2} = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(t/t_{0.5})}{\sigma}\right]^2} \qquad (20)$$

Note that Ebeling uses the awkward uncommon notation s for σ. The lognormal generally describes time to restore or repair (TTR) of an electrical or mechanical component, and also has applications in accelerated life testing. The lognormal distribution is positively skewed ($\alpha_3 > 0$) as depicted in Figure 4.4(a) on page 74 of Ebeling. The MTTF is computed as follows. The parameter μ is location, while σ is the scale parameter; this is unlike what Ebeling claims atop his p. 81.

$E(T) = E(e^X) = \int_{-\infty}^{\infty} \frac{e^x}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$; letting $Z = \frac{x-\mu}{\sigma}$ in this last integral results in $dZ = dx/\sigma$,

and $E(T) = E(e^X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mu+\sigma z} e^{-z^2/2} dz = e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[(z-\sigma)^2 - \sigma^2]} dz$

$$\rightarrow E(T) = MTTF = e^{\mu+\sigma^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma)^2} dz = e^{\mu + \sigma^2/2} \qquad (21)$$

Similarly, $\mu'_2 = E(T^2) = E(e^{2X}) = e^{2(\mu + \sigma^2)}$, $\mu'_3 = E(T^3) = E(e^{3X}) = e^{3\mu + 4.5\sigma^2}$,

and $\mu'_4 = E(T^4) = E(e^{4X}) = e^{4\mu + 8\sigma^2}$. These origin moments lead to the following (central) moments

$$\mu_2 = V(T) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \qquad (22a)$$

$$\alpha_3 = \mu_3/\sigma^3 = \frac{e^{3\sigma^2} - 3e^{\sigma^2} + 2}{(e^{\sigma^2} - 1)^{1.5}} > 0 \qquad (22b)$$

$$\alpha_4 = \mu_4/\sigma^4 = \frac{e^{6\sigma^2} - 4e^{3\sigma^2} + 6e^{\sigma^2} - 3}{(e^{\sigma^2} - 1)^2}. \qquad (22c)$$

Both the skewness $\alpha_3 > 0$ and kurtosis ($\beta_4 = \alpha_4 - 3$) are increasing functions of σ^2. The value of lognormal skewness becomes equal to that of exponential at $\sigma^2 = 0.30402386326$, i.e., α_3(lognormal) = 2.00 when $\sigma = 0.5513835899444$ and exceeds that of the exponential when $\sigma^2 > 0.30402386326$. Its kurtosis at this value of σ^2 is equal to 7.8634625 as compared to $\alpha_4 - 3 = 6$ for the exponential density. The RE function of the lognormal pdf is obtained from R(t) = Pr(T > t) = Pr[lnT > ln(t)] = Pr[X > ln(t)] = Pr[$\frac{X-\mu}{\sigma} > \frac{\ln(t)-\mu}{\sigma}$] = Pr[Z > $\frac{\ln(t)-\mu}{\sigma}$]. Hence, the survivor function is given by

$$R(t) = 1 - \Phi\left[\frac{\ln(t)-\mu}{\sigma}\right] = \Phi\left[\frac{\mu-\ln(t)}{\sigma}\right] = \Phi\left[\frac{\ln(t_{0.5})-\ln(t)}{\sigma}\right] = \Phi\left[\frac{\ln(t_{0.5}/t)}{\sigma}\right] \qquad (23)$$

where $\mu = \ln(t_{0.5})$ because $R(t_{0.5}) = 0.50$ shows that $\mu - \ln(t_{0.50}) = 0$. Further, the hazard function is given by h(t) = f(t)/R(t), where f(t) is given in equation (20), R(t) is obtained from (23), and Φ is the cdf of a standardized normal variate. As an application, study Example 4.10 on page 84 of Ebeling. In general, the lognormal h(t) is both DFR and IFR depending on μ, σ and the value of t.

The percentiles, t_p, of the lognormal pdf are obtained by letting $R(t_p) = 1 - p$ and using the RE function in Eq. (23) as shown below.

$F(t_p) = \Phi[\frac{\ln(t_p) - \mu}{\sigma}] = p \rightarrow \frac{\ln(t_p) - \mu}{\sigma} = Z_{1-p}$, where $Z_{1-p} > 0$ only if $p > 0.50$. For example, $Z_{0.05}$ = 1.644854 (Ebeling does not present a notation for normal inverse) while $Z_{0.95} = -1.644854$. Therefore, the p^{th} quantile is given by

$$t_p = e^{\mu + Z_{1-p} \times \sigma} \tag{24}$$

From Eq. (24), the 25th percentile (i.e., the 25% failure point) for the lognormal distribution is given by $t_{0.25} = e^{\mu - 0.67449 \times \sigma}$, the 50th percentile (or the median) is $t_{0.50} = e^{\mu}$, and the 75th percentile is given by $t_{0.75} = e^{\mu + 0.67449 \times \sigma}$. Since the lognormal pdf is positively skewed and $\sigma > 0$, then for certain E(T) $= e^{\mu + \sigma^2/2} > t_{0.50} = e^{\mu} >$ MO, where the modal point MO $= e^{\mu - \sigma^2}$. Further, Eq. (24) shows that the lognormal inverse function is given by

$$F^{-1}(x) = e^{\mu + (Z_{1-x})\sigma}, \qquad 0 \le x \le 1.$$

It must be emphasized that if a rv U ~ $N(\mu, \sigma^2)$, then the ln(U) does not have a $LGN(\mu, \sigma^2)$ distribution, but the variate $e^U = T$ has the $LGN(\mu, \sigma^2)$ pdf.

The Beta Model as the Underlying Life Distribution

The Beta distribution has widespread applications in the fields of RE engineering, QC and Project Management. Almost invariably, the pdf of a proportion (such as highway sections needing repair) follows the standard Beta distribution given by

$$g(x) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, & 0 \le x \le 1, \\ 0, & \text{elsewhere} \end{cases} \tag{25a}$$

where the rv X = some proportion or fraction, the parameters a , b must > 0, E(X) = a/(a+b) , V(X) = $ab/[(a+b)^2(a+b+1)]$ and MO = (a – 1)/(a+b – 2) for $a \ge 1$ and a + b > 2. Further, the Beta distribution has also widespread applications in the field of Bayesian Statistics. The skewness and the 4th standardized central moment (α_4) are given by

$$\alpha_3(\text{Beta}) = 2(b-a)(a+b+1)^{1/2}/[(a+b+2)\times(ab)^{1/2}]$$

$$\alpha_4(\text{Beta}) = \frac{3(a+b+1)(a^2b + ab^2 + 2a^2 + 2b^2 - 2ab\ )}{ab(a+b+2)(a+b+3)}.$$

Further, the Beta skewness > 0 iff b > a, and when a = b, $\alpha_3 = 0$ and its $\alpha_4 = 3(2a + 1)/(2a + 3) < 3$. It seems that at a = 1, the CV $\to$ 100% as b $\to \infty$. For any statistical distribution, the ranges of α_3 and α_4 are $-\infty < \alpha_3 < +\infty$, $1 < \alpha_4 < +\infty$, and my conjecture is that $\alpha_4 > 1 + 1.0375\alpha_3^2$ for almost all statistical distributions! Both parameters a & b are shape parameters. Like the gamma pdf, the standard Beta pdf has its roots in the definition of Beta function defined by the integral

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ a, b > 0. \qquad (25b)$$

The proof of $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is extremely difficult and is far from the intension of these notes. Given the validity of Eq. (25b), then one should easily see that the g(x) in (25a) is indeed a pdf, called the Standard Beta density.

In RE engineering the Beta pdf is used to model components with limited lifetimes (such as compressors and light bulbs). If the lifetime of a device, such as a light bulb, has a beta distribution over the range (δ, U) = (100 hours, 1000 hours) and parameters a = 3 and b = 2, then the Beta rv in (25a) has to be linearly transformed as $T = \delta + (U - \delta)X$, where $0 \le x \le 1$, and δ is the minimum life. Note that the standard Beta density ranges from x = 0 to x = 1. For our light bulb example, T = 100 + 900x, and the pdf of T (the transformed Beta) is given by

$$f(t) = \begin{cases} \frac{\Gamma(a+b)}{900\Gamma(a)\Gamma(b)}\left(\frac{t-100}{900}\right)^2\left(\frac{1000-t}{900}\right), & 100 \le t \le 1000\,\text{hours} \\ 0\ , & \text{elsewhere} \end{cases}$$

I obtained the above lifetime pdf for light bulbs by letting F(t) be the cdf of T, G(x) be the cdf of X, a = 3 and b = 2. Then by definition, $F(t) = \Pr(T \le t) = \Pr(100 + 900\,X \le t) = \Pr(X \le \frac{t-100}{900}) = G(x)$, where $x = \frac{t-100}{900}$. Therefore, $f(t) = dF(t)/dt = dG(x)/dt = \frac{dG(x)}{dx} \times \frac{dx}{dt} = \frac{g(x)}{dt/dx} = g(x)/(900)$.

When a = b = 1, the Beta pdf reduces to a rectangular (or uniform) U(0,1) distribution. The RE function of the Beta rv is simple to obtain only when the parameters a and b are positive integers; otherwise, R(t) will have the form of an incomplete Beta function. The modal point of the Beta density is given by MO = (a – 1)/(a + b – 2). The hazard rate is increasing (IFR) only when (a – 1) > (a + b – 2)t because the 1st derivative is positive when (a – 1) > (a + b – 2)t. The percentiles of the Beta distribution are not easy to obtain when the parameters a and b are non-integers. However, when both a and b are positive integers, such as a = 3 and b = 2, then the 25th percentile of the B(3, 2, x) distribution is obtained by solving the integral equation

$$\int_0^{x_{0.25}} \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} x^2(1-x)dx = 0.25 \rightarrow 4x_{0.25}^3 - 3x_{0.25}^4 = 0.25 \rightarrow$$

$$3x_{0.25}^4 - 4x_{0.25}^3 + 0.25 = 0$$

This last polynomial is not easily solved, showing that the Beta cdf is not directly invertible (unlike the exponential, uniform, triangular, or Weibull which are directly invertible). The solution of the above polynomial of order 4 is $x_{0.25} = 0.45632171$ (this is the 25% failure point). Similarly, $x_{0.50}$ = Median = 0.61427242, $x_{0.75} = 0.75697791$, and the IQR = 0.30065620. I used the Matlab functions betainv(0.25, 3, 2), betainv(0.5, 3, 2), and betainv(0.75, 3, 2) to verify the values of these percentiles that I computed.

The RE function of the transformed Beta pdf with maximum life U has a complicated form and is obtained from integrating the density function given below:

$$f(t) = \begin{cases} \frac{\Gamma(a+b)}{(U-\delta)\Gamma(a)\Gamma(b)}\left(\frac{t-\delta}{U-\delta}\right)^{a-1}\left(\frac{U-t}{U-\delta}\right)^{b-1}, & \delta \le t \le U \text{ hours,} \\ 0\ , & \text{elsewhere} \end{cases}$$

and thus

$$R(t) = Pr(T> t) = \int_t^U \frac{\Gamma(a+b)}{(U-\delta)\Gamma(a)\Gamma(b)}\left(\frac{y-\delta}{U-\delta}\right)^{a-1}\left(\frac{U-y}{U-\delta}\right)^{b-1}dy\,. \qquad (26a)$$

The proportion of the population failing by time t is given by

$$F(t) = 1 - R(t) = \int_\delta^t \frac{\Gamma(a+b)}{(U-\delta)\Gamma(a)\Gamma(b)}\left(\frac{y-\delta}{U-\delta}\right)^{a-1}\left(\frac{U-y}{U-\delta}\right)^{b-1}dy \qquad (26b)$$

Making the transformation $x = \frac{y-\delta}{U-\delta}$ in Eq. (26b) results in

$$F(t) = \int_0^{\frac{t-\delta}{U-\delta}} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}dx = \text{Incomplete-Beta}(\frac{t-\delta}{U-\delta}, a, b) \quad (26c)$$

Eq. (26c) shows that the RE function of the transformed Beta density, using MS Excel, is given by R(t)

$$= \int_{\frac{t-\delta}{U-\delta}}^{1} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}dx = 1 - \text{Incomplete-Beta}(\frac{t-\delta}{U-\delta}, a, b) = 1 - \text{betadist}(\frac{t-\delta}{U-\delta}, a, b)$$

(26d)

For the light-bulb example, assuming that the parameters a = 3 and b = 2, the value of the RE function at t = 600 hours is given by R(600) = Pr(T > 600) = 1 – $F_T(600)$ =1 – betadist($\frac{600-100}{1000-100}$, 3, 2) = 1 – betadist(5/9, 3, 2) = 1 – 0.40009145 = 0.59990855. In Matlab, the syntax for the incomplete Beta function is given by betainc(x, a, b). Since a = 3 and b = 2 are integers, the RE function can easily be obtained from

$$R(t) = \int_{\frac{t-100}{900}}^{1} \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} x^2(1-x)dx = \int_{\frac{t-100}{900}}^{1} \frac{4!}{2!1!} x^2(1-x)dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{\frac{t-100}{900}}^{1}$$

$$= \left[4x^3 - 3x^4\right]_{\frac{t-100}{900}}^{1} = 1 - 4(\frac{t-100}{900})^3 + 3(\frac{t-100}{900})^4$$

And as a result, the R(600) = $1 - 4(\frac{600-100}{900})^3 + 3(\frac{600-100}{900})^4$ = $1 - 4(5/9)^3 + 3(5/9)^4$ = 0.59990855052583, as before. It seems that the RE function of the translated Beta distribution is an increasing function of the parameter a and decreasing function of b. This is due to the fact that MTTF = $\delta + (U - \delta)a/(a+b)$. In Project Management the transformed Beta is used to model the time required to complete an activity, where U is the very pessimistic time (i.e., everything that could go wrong will go wrong) and δ is the very optimistic time (i.e., if everything goes very well as anticipated). The cdf of the transformed beta in that case is given by Eq. (26b).

The Extreme-Value (EXTV) Distributions

Ebeling covers this failure-time distribution on pp. 74-76. As stated by Ebeling in his Example

4.7, this distribution is used to model the failure of the weakest-stress point of a component or a manufactured part. By definition, if a rv, X, has a W(0, θ, β) TTF distribution, then its $\log_e$ = ln has an extreme–value (EXTV) distribution, i.e., T = ln(X) has the EXTV distribution. Unlike the relationship between the N(μ, σ^2) and LGN(μ, σ^2), the rv T =TTF has the EXTV distribution iff X = e^T has the W(0, θ, β > 0) distribution. In order to arrive at the RE function, R(t), as given in Eq. (4.21) on p. 74 of Ebeling, we let F(x) represent the cdf of W(0, θ, β) and G(t) represent the cdf of t = ln(x), and R(t) is the RE function of lifetime T. Then by definition, G(t) = Pr(T$\leq$ t) = Pr(ln(X) $\leq$ t) = Pr(X $\leq e^t$) = F(e^t) $\rightarrow R_T(t)$ = $R_X(e^t)$ = Exp[$-(e^t/\theta)^\beta$] = Exp[$-(e^t)^\beta(1/\theta)^\beta$] = Exp[$-e^{t\beta}(\theta)^{-\beta}$] = Exp[$-e^{\beta t}$ $(e^{\ln(\theta)})^{-\beta}$] = Exp[$-e^{\beta t}e^{-\beta\ln(\theta)}$] = Exp[$-e^{\beta(t-\ln\theta)}$] = Exp[$-e^{(t-\mu)/\alpha}$] , where $\alpha = 1/\beta$ is called the scale parameter and $\mu = \ln\theta$ is called the location parameter. This last expression for R(t) is identical to Eq. (4.21) of Ebeling. Note that the EXTV distribution has no shape parameter just like the N(μ, σ^2), and hence all EXTV pdfs have the same-shape graphs. The density is obtained from f(t) = −dR(t)/dt as shown in Eq. (4.22) on p. 74 of Ebeling. The MTTF is given by E(T) = $\mu - \gamma\alpha$, where γ = 0.57721566490153 is the irrational Euler's constant defined by $\gamma = \lim_{n\to\infty}\left[\sum_{i=1}^{n}(1/i) - \log_e(n)\right]$. At n = 100,000, the value of the expression $[\sum_{i=1}^{n}(1/i)] - \log_e(n)$ = 0.577221. The Var of T is given by $\sigma_T^2 = \pi^2/(6\beta^2)$, β > 0, so that $\sigma_T = \pi\alpha$/sqrt(6) $\approx 1.2825498302\alpha$. For the application of the Min-EV distribution, see the Example 4.7 on p. 76 of Ebeling. Further, it can be proven that the modal point of T occurs at MO = μ = lnθ, and hence, R(t) = Exp[$-e^{(t-MO)/\alpha}$] = Exp[$-e^{\beta(t-MO)}$]. It seems that the EXTV distribution is negatively skewed because MTTF < Median-life < MO (see figure 4.3(b)) on p. 75 of Ebeling. Further, because the EXTV distribution is directly invertible, its quantile function is given by $t_p = \mu + \ln[\ln(1-p)^{-1}]/\beta = \mu + \alpha \ln[\ln(1-p)^{-1}] = \mu + \alpha \ln[-\ln(R)]$. For example, the 25th percentile is given by $t_{0.25} = \mu + \alpha \ln[\ln(1-0.25)^{-1}]$, and for the Example 4.7 on p. 76 of Ebeling, the value of B-25 life is $t_{0.25}$ = 5 + 0.40$\times$ (–1.245899324) = 4.5016403 years. The values of μ =MO = 0 & α =1 lead to the Standard EXTV distribution, i.e., EXTV(0, 1). The reader should be careful not to confuse MO = 0 & α =1 with E(T) = $-\gamma$ & $\sigma_T = \pi/\sqrt{6}$. It can easily be proven that if X has the pdf e^{-x}, then Y = ln(X) has the Standard EXTV distribution.

The standard Largest EXTV distribution can be obtained from the fact that f(x) = e^x is a density over the range $(-\infty, 0)$ with cdf $F(x) = e^x$. Letting $Y = -\ln(-X)$ leads to the cdf of Y as $G_Y(y) = \exp(-e^{-y})$, whose density is given by $g(y) = dG_Y(y)/dy = e^{-y}\exp(-e^{-y}) = \exp(-y - e^{-y})$. Now, letting $Y = (T-\mu)/\alpha$ gives the density of the Largest EXTV distribution as $f_T(t) = \frac{1}{\alpha}e^{-(t-\mu)/\alpha}\exp[-e^{-(t-\mu)/\alpha}]$. The moments of this density are $E(T) = \mu + \gamma\alpha$, $V(T) = (\alpha\pi)^2/6$, $\alpha 3 = 1.13955$, and its kurtosis is 2.40. The modal point is $MO = \mu$, the points of inflection occur at $t = \mu \pm \alpha\times\ln[0.5(3+\sqrt{5})]$, and its quantile function is $t_p = \mu-\alpha\ln[-\ln(p)]$. For an application example, see the Exercise 7.42 on page 170 of Ebeling.

The Mean Residual (or Excess, or Left-over) Life at Age T_0

Consider a device whose lifetime distribution anew is given by f(t). Therefore, its unconditional mean life (starting from time zero) is given by $MTTF = E(T) = \int_0^\infty R(t)dt$. Suppose now the device has survived a length of time T_0, i.e., its age is T_0, but still functioning properly. The question that now arises is what is the expected conditional remaining (or residual, or leftover) lifetime of the component? The rv $(T - T_0 \mid T > T_0)$ is called the residual life at age T_0 and its expectation is called the mean residual life (MRL); thus, for a fixed T_0 (or a given age T_0)

$$\text{MRL(at age } T_0) = E(T - T_0 \mid T > T_0) = \int_{T_0}^\infty (x - T_0) f(x \mid T > T_0)\, dx \,. \tag{27}$$

Recall that $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ for any two events in the same universe. Then by the same token, $f(x \mid T > T_0) = \frac{f(x)}{\Pr(T > T_0)} = \frac{f(x)}{R(T_0)}$. Substituting from this last equality into (27) yields $MRL(T_0)$ =

$$\int_{T_0}^\infty (x - T_0)[f(x)/R(T_0)]dx = \frac{1}{R(T_0)}\int_{T_0}^\infty x f(x) dx - \frac{T_0}{R(T_0)}\int_{T_0}^\infty f(x) dx$$

$$\rightarrow \quad \text{MRL(at age } T_0) = \frac{1}{R(T_0)}\int_{T_0}^\infty x f(x) dx - T_0 \tag{28a}$$

Ebeling gives a slightly but equivalent expression for $MRL(T_0)$, atop p. 34 in his Eq. (2.18), which is also derived as follows: MRL(T_0) = $\int_{T_0}^{\infty}(x - T_0)[f(x)/R(T_0)]dx$ = $\int_{T_0}^{\infty}(x - T_0)[-dR(x)/R(T_0)]$ =

$$\frac{1}{R(T_0)}\left\{[-(x - T_0)R(x)]_{T_0}^{\infty} + \int_{T_0}^{\infty}R(x)dx\right\} = \frac{1}{R(T_0)}\int_{T_0}^{\infty}R(x)dx \qquad (28b)$$

Elsayed (*Reliability Engineering, E. A. Elsayed* , ISBN: 978-1-118-13719-2) provides a good application of Eq. (28a) in Example 1.20 on his pp. 70-71. The application involves rotary compressors that provide cooling liquid for a power generator. It is assumed that the lifetime of a compressor has a Beta (4, 2, t) TTF pdf, $0 \le t \le 1$ year. Thus, f(t) = $\frac{\Gamma(6)}{\Gamma(4)\Gamma(2)}t^3(1 - t) = 20t^3(1 - t)$, $0 \le t \le 1$, and R(t) = $1 - 5t^4 + 4t^5$. The reliability at 9 months is equal to R(0.75) = $1 - 5(0.75)^4 + 4(0.75)^5$ = 0.3671875; thus the mean residual life is given by MRL(0.75) = $\frac{1}{0.3671875}\int_{0.75}^{1}(1 - 5t^4 + 4t^5)dt$ = 0.0961879433 yrs.

Example 7. Suppose the failure rate of a heater switch is h(t) = 0.0005 failures/hour (CFR). Since the hazard function is a constant λ = 0.0005, then its lifetime pdf must be the exponential given by f(t) = 0.0005 $e^{-0.0005t}$. Further, we have the information that the switch has already lasted 300 hours (i.e., its age is 300 hours) and still functioning perfectly. Before we proceed to compute its MRL(at 300 hours), we 1st compute its unconditional MTTF (when the unit is brand new, or from t = 0) given by E(T) = 1/0.0005 = 2000 hours. The MRL at the age of 300 hours is obtained from Eq. (28), by using integration by parts, as MRL(300) = $\frac{1}{e^{-0.15}}\int_{300}^{\infty}x\lambda e^{-\lambda x}dx - 300$ =

$$\frac{1}{e^{-0.15}}\left[-x e^{-\lambda x}\Big|_{300}^{\infty} - \int_{300}^{\infty} -e^{-\lambda x}dx\right] - 300 = \frac{1}{e^{-0.15}}\left[300e^{-0.15} + \frac{1}{0.0005}e^{-0.15}\right] - 300 = 2000$$

hours. We should obtain the same result using Eq. (2.18E) of Ebeling because MRL(300)

$$= \frac{1}{R(300)}\int_{300}^{\infty}R(x)dx = \frac{1}{e^{-0.15}}\int_{300}^{\infty}e^{-\lambda t}dt = \frac{1}{e^{-0.15}}\left[-\frac{1}{\lambda}e^{-\lambda t}\right]_{300}^{\infty} = \frac{1}{e^{-0.15}}[\frac{1}{\lambda}e^{-300\lambda}] = \frac{1}{\lambda} = E(T) =$$

Unconditional MTTF of 2000 hours as before. This last development indeed shows that the exponential density is memoryless because its MRL is equal to its unconditional MTTF. By memoryless, we mean that the Pr(T > 1000 $|$T > 300) = $\dfrac{\Pr(T>1000)}{\Pr(T>300)}$ = $\dfrac{e^{-1000\lambda}}{e^{-300\lambda}}$ = $e^{-700\lambda}$ = Pr(T > 700 hours) = R(700) = R(1000$|$ T > 300). Thus for the exponential pdf, the conditional reliability Pr(T > 1000 $|$T > 300) is equal to the unconditional reliability at 700 hours starting from time zero, i.e., R(700) = Pr(T > 700) = Pr(T > 1000 $|$T > 300) = R(300+7000$|$component's age is 300 hours). Note that in general, the MRL(T_0) is not equal to MTTF +T_0.

<u>**Exercise 7.**</u> The underlying lifetime distribution of an electrical device is Rayleigh (see p. 8 of my notes) with characteristic life θ = 50,000 hours. (a) Obtain its unconditional RE function. (b) Compute the hazard rate at t = 20000 hours. (c) Given that such a device has an age of $T_0 \equiv 20000$ hours (i.e., it is known that T > 20000), compute its expected excess life. ANS: (b) h(20000) = 0.0000160/hour, (c) MRL(20000) = 29723.50983 hours.

Appendix

Proof of a strictly increasing F(t) having a U(0, 1) Distribution

Let Y = F(t); then the cdf of Y is given by Pr(Y≤ y) = Pr(F(t) ≤ y) = Pr[F^{-1}F(t)≤ F^{-1}(y)] = Pr(T ≤ F^{-1}(y)] = F_T[F^{-1}(y)] = y → Because the Pr(Y≤ y) = F_Y(y) = y and the cdf of U(0,1) is also given by y, then Y = F(t) is distributed according to U(0,1).

The variances of the TTF distributions in my Table 2, respectively, from top to bottom, are $1/\lambda^2$, $(\theta-\delta)^2\,[\Gamma(1+\frac{2}{\beta})-\Gamma^2(1+\frac{1}{\beta})]$, n/λ^2, $e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$, $(U-\delta)^2ab/[(a+b)^2(a+b+1)]$, and $\pi^2/(6\beta^2)$. Four out of the listed six statistical distributions are positively skewed; the EXTV distribution is negatively skewed and the Weibull only with 0 < β < 1, which has a DFR, is also negatively skewed. The median values, in order of listings, are $t_{0.50} = \ln(2)/\lambda$, $\delta + (\theta-\delta)(\ln 2)^{1/\beta}$, not directly invertible, e^{μ}, betainv(0.5, a, b), and for the (Smallest) EXTV-dist $t_{0.50} = \mu - 0.366512921\alpha$. The modal points are MO = 0, $\delta + (\theta-\delta)\times[(\beta-1)/\beta]^{1/\beta}$, β ≥ 1, (n – 1)/λ, $e^{\mu-\sigma^2}$, [(a – 1)U + (b – 1)δ]/(a+b – 2) for a ≥ 1 and a + b > 2, and MO = μ – 0.366512921/β (where 1/β =α). Finally, the shape parameter β of the Weibull is called the slope because $R(t) = e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$, $t>\delta$ shows that $\ln[R(t)] = -(\frac{t-\delta}{\theta-\delta})^{\beta}$ → ln[1/R(t)]

$= (\frac{t-\delta}{\theta-\delta})^{\beta} \rightarrow \ln\{\ln[1/R(t)]\} = \beta \ln[(t-\delta)/(\theta-\delta) \rightarrow \ln\{\ln[1/R(t)]\} = \beta[\ln(t-\delta) - \ln(\theta-\delta)] =$

$\beta \ln(t-\delta) - \beta \ln(\theta-\delta) = \beta \ln(t-\delta) + C$, where $C = -\beta \ln(\theta-\delta) = \beta \ln[1/(\theta-\delta)]$ is a constant .

Letting $Y = \ln[\ln(R^{-1})] = \ln[-\ln(R)]$, $X = \ln(t-\delta)$ yields $Y = \beta X + C$, which represents a line with slope β and Y-intercept C. Because all Weibull distributions have a RE value of e^{-1} at their characteristic life $t_c = \theta$, the Weibull graph paper has two ordinates, where the right scale gives R(t) and the opposing left scale gives F(t). Thus, $Y = \ln[\ln(R^{-1})]$ is the right coordinate scale and its value at $t = \theta$ is equal to $Y(\theta) = \ln[\ln(e^{-1})^{-1}] = \ln[\ln(e)] = \ln(1) = 0$ while the corresponding left ordinate is obtained

Table 2. Summary of Reliability Measures

Lifetime Distribution	Failure Density f(t)	Survival Function R(t)	Hazard Rate Function	MTTF
Exponential	$\lambda e^{-\lambda t}$	$e^{-\lambda t}$	λ	$1/\lambda$
Weibull	$\frac{\beta}{\theta-\delta}(\frac{t-\delta}{\theta-\delta})^{\beta-1} e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$	$e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$	$\frac{\beta}{\theta-\delta}(\frac{t-\delta}{\theta-\delta})^{\beta-1}$	$\delta + (\theta - \delta)\times \Gamma[(1/\beta) + 1]$
Gamma	$\frac{\lambda}{\Gamma(n)}(\lambda t)^{n-1} e^{-\lambda t}$	$\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, when n is a pos. integer	f(t)/R(t)	n/λ
Lognormal	$\frac{1}{\sigma t\sqrt{2\pi}} e^{-(\frac{\ln t-\mu}{\sigma})^2}$	$\Phi[\frac{\mu - \ln(t)}{\sigma}]$	f(t)/R(t)	$e^{\mu+\sigma^2/2}$
Beta, $t = \delta + (U - \delta) x$, $0 \le x \le 1$	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$ Standard Beta pdf	$1 - \text{betadist}(\frac{t-\delta}{U-\delta}$, a, b)	f(t)/R(t)	$\delta + (U - \delta)a/(a+b)$
EXTV (Smallest)	See Eq. (4.22) p. 74 of Ebeling	$\text{Exp}[-e^{\beta(t-\mu)}]$	See Eq. (4.20) on p. 74 of Ebeling	$\mu - \gamma/\beta$
EXTV (Largest)	$h(t)\times R(t)$	$\exp[-e^{-(t-\mu)/\alpha}]$	$\frac{1}{\alpha} e^{-(t-\mu)/\alpha}$	$\mu + \gamma\alpha$

from 0 = ln[ln(1/R)] = $\ln[\ln(\frac{1}{1-F})] \rightarrow \ln(\frac{1}{1-F})$ = e^0 = 1 → $\frac{1}{1-F}$ = e → 1 – F = $e^{-1} \cong 0.3679$ →

F = 0.6321 → The 63.21% failure point on the left scale corresponds to the characteristic life θ on the abscissa because ln[ln(1/R(θ))] = 0. Finally, the RE function of the Weibull for the same values of δ and θ is an increasing function of the slope β up to the characteristic life θ and then becomes a decreasing function of β for t values beyond θ.

The RE function of the Weibull for the water pump Example is graphed below. The total area under the RE function (see page 7-10 of these notes) and the abscissa, T, is equal to its MTTF, where time is measured in km.

For the Lognormal distribution, I will show below as to why the location parameter $\mu = \ln(t_{0.50})$. $\Pr(T \le t_{0.50}) \equiv 0.50$, where $T \sim LGN(\mu, \sigma^2) \rightarrow \Pr[\ln T \le \ln(t_{0.50})] \equiv 0.50 \rightarrow \Pr[X \le \ln(t_{0.50})] \equiv 0.50$, where $X \sim N(\mu, \sigma^2) \rightarrow \Pr(X \le \mu) \equiv 0.50$ because for any $N(\mu, \sigma^2)$, $\mu = x_{0.50}$; thus, $\Pr(X \le \mu) = \Pr(\ln T \le \ln t_{0.50}) \equiv \Pr[X \le \ln(t_{0.50})] \equiv 0.50 \rightarrow \mu = \ln(t_{0.50})$.

Finally, the skewness of the EXTV distribution is approximately equal to $\alpha_3 = -1.13955$, and its kurtosis is $\beta_4 = \alpha_4 - 3 = 2.40$. Just like the normal, exponential, & the LGN, the EXTV distribution has no shape parameter and hence the graphs of all EXTV distributions look alike.