Interference Theory for Stress and Strength **Reference : Chapter 7 of Ebeling** **Maghsoodloo**

Let the rv Y represent the strength of a component and the rv X represent the stress applied on the component. Then the component reliability is given by

$$\mathbf{R = P(Y > X) = \begin{cases} P(Y - X > 0), \\ \text{or } P(Y/X > 1) \end{cases} = \begin{cases} P(Y - X > 0), \\ \text{or } P(\eta > 1) \end{cases}} \qquad \textbf{(89)}$$

where the random variable $\eta = Y/X$ is called the Factor of Safety (or Safety Factor). We assume that both X and Y are independent continuous rvs with pdfs f(x) and g(y), respectively. To obtain a general expression for RE, we can use any one of the following two approaches depending on the expressions for f(x) and g(y).

1. Suppose that the stress acting on a device is in the neighborhood of x_0. Then the device incremental RE is given by $\mathbf{P[(x_0 - dx \le x \le x_0 + dx) \cap (Y > x_0)] =}$

$$\mathbf{P(x_0 - dx \le x \le x_0 + dx) \times P(Y > x_0 | \, x_0 - dx \le x \le x_0 + dx) = f(x_0)dx \int_{x_0}^{\infty} g(y)dy} \, .$$

where $P(Y > x_0 | \, x_0 - dx \le x \le x_0 + dx) = P(Y > x_0)$ because Y is independent of X. Since the value of stress can range from $-\infty$ (in reality from 0) to ∞, we can remove the condition on x_0 by integrating over all possible values of x_0, i.e., the unconditional RE is given by

$$\mathbf{R = \int_{-\infty}^{\infty} [f(x_0) \int_{x_0}^{\infty} g(y)dy]dx = \int_{-\infty}^{\infty} [f(x) \int_{x}^{\infty} g(y)dy]dx = \int_{-\infty}^{\infty} G_y(x)f(x)dx} \qquad \textbf{(90a)}$$

where $G_y(x) = 1 - F_y(x)$ and $F_y(x)$ is the cdf of Y at the stress value x so that $G_y(x)$ is the exceeding pr of strength at the stress value x.

2. Suppose the strength of a device is around the value y_0. Then the device will be RE iff the stress acting on it is less than y_0. Letting $F_x(y)$ represent the cdf of stress at the strength y and applying similar logic as above, we deduce that

$$\mathbf{R = \int_{-\infty}^{\infty} F_x(y)g(y)dy} \qquad \textbf{(90b)}$$

Example 13. Suppose a component's strength, Y, ~ N(100 MPa, 100) and the stress acting on the component is exponentially distributed with mean μ_x = 50 MPa. Since the exponential cdf is directly invertible while the Gaussian is not, then it is best to use Eq. (90b) because the cdf of the exponential is given by $F_X(y) = 1 - e^{-y/50}$, while the normal cdf is an integral. Hence, from (90b)

$$R = \int_{-\infty}^{\infty}(1-e^{-y/50})g(y)dy = \int_{-\infty}^{\infty}(1-e^{-y/50})\frac{e^{-\frac{1}{2}(\frac{y-100}{10})^2}}{10\sqrt{2\pi}}dy = 1-\int_{-\infty}^{\infty}\frac{e^{-\frac{4y}{200}-\frac{y^2-200y+10^4}{200}}}{10\sqrt{2\pi}}dy =$$

$$1-\int_{-\infty}^{\infty}\frac{e^{-\frac{y^2-196y+10^4}{200}}}{10\sqrt{2\pi}}dy = 1-\int_{-\infty}^{\infty}\frac{e^{-\frac{y^2-196y+10^4}{200}}}{10\sqrt{2\pi}}dy = 1-\frac{1}{10\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{y^2-196y+10^4}{200}}dy =$$

$$1-\frac{1}{10\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{[(y-98)^2+396]}{200}}dy = 1-\frac{e^{-1.98}}{10\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(\frac{y-98}{10})^2}dy = 1-e^{-1.98} = 0.861931.$$

The central factor of safety is given by $n_c = \mu_y/\mu_x$ = E(Y)/E(X) = 2, while the mean factor of safety $\bar{n}$ = E(Y/X) = E(η) cannot be directly computed; however it can be shown that $\bar{n} \cong n_c(1 + CV_x^2)$, where $n_c = \mu_y/\mu_x$ is called the central factor of safety and CV_x is the coefficient of variation of stress. For our example, $CV_x = \sigma_x/\mu_x$ = 1 (or 100%) so that the expected factor of safety is approximately $\bar{n} \cong 2(1+1^2) = 4$.

Suppose now in the above problem the strength Y was deterministic at a constant value of 100 MPa but stress was still exponential. Then the component RE reduces to R = P(X < 100) = $F_X(100)$ = $1- e^{-100/50}$ = 0.864665. Conversely, if stress were deterministic at the constant value of x = 50 MPa, then R = P(Y > 50) = P(Z > $\frac{50-100}{10}$) = P(Z > –5) = 0.99999971334843.

Normal Stress and Strength

Suppose the strength of a device Y ~ N(μ_y, σ_y^2) and is subject to the stress X which is also N(μ_x, σ_x^2). Then from Eq. (89), we deduce that R = P(Y > X) = P(Y – X > 0) = P(W > 0), where W =

Y – X is a LC of normally and independently distributed rvs and hence itself is Gaussian with μ = E(W) = $\mu_y - \mu_x$ and variance V(W) = V(Y) + V(X) = $\sigma_y^2 + \sigma_x^2$, depicted in Figure 18. Figure 18 shows that

$$\mathbf{R = P(W > 0) = P(Y - X > 0) = P\left(Z > \frac{0-\mu}{\sigma_w}\right) = P\left(Z > \frac{-\mu}{\sqrt{\sigma_y^2+\sigma_x^2}}\right) = = P\left(Z > \frac{-(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}}\right) =}$$

$$\mathbf{P(Z > -Z_0) = \Phi(Z_0), \text{ where } Z_0 = \frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}}}.$$

<u>Example14.</u> The strength of a component is N(μ_y, 400 MPa^2) and the stress acting on it is also Gaussian N(μ_x, 625 MPa^2). Determine the value of central factor

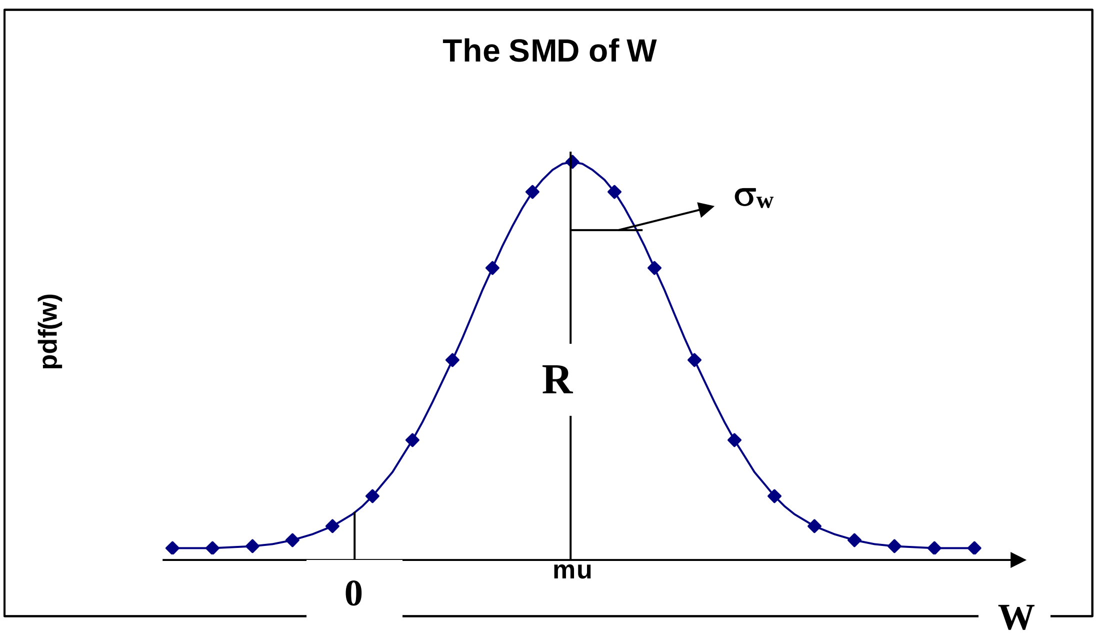


Figure 18. The SMD of W = Y – X

of safety, $n_c = \mu_y/\mu_x$, such that the component RE is at least 0.999 if CV(Y) = 0.10.

Solution: CV(Y) = 0.10 → 0.10 = σ_y/μ_y = 20/μ_y → μ_y = 200 MPa. $\sigma_w = \sqrt{\sigma_y^2+\sigma_x^2} = \sqrt{1025} =$

32.10562; $\Phi(Z_0) = 0.999 \rightarrow Z_{0.001} = \dfrac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}} \rightarrow 3.090232 = \dfrac{200 - \mu_x}{32.01562} \rightarrow \mu_x = 101.0643 \rightarrow n_c =$ 200/101.0643 =1.9789383 and CV_x = 25/101.0643 = 0.2473673. Thus, on the average the mean strength has to be nearly at least twice the mean stress to attain a RE of at least 0.999.

In the normal case, the device RE can also be expressed in terms of CV_y and CV_x as follows:

$$Z_0 = \frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}} = \frac{\mu_y / \mu_x - 1}{\sqrt{\sigma_y^2/\mu_x^2 + \sigma_x^2/\mu_x^2}} = \frac{n_c - 1}{\sqrt{(\sigma_y^2/\mu_y^2)(\mu_y^2/\mu_x^2) + \sigma_x^2/\mu_x^2}} =$$

$$= \frac{n_c - 1}{\sqrt{CV_y^2(n_c^2) + CV_x^2}} \rightarrow \mathbf{R = P(Z > -Z_0) = \Phi(Z_0).}$$

Further, it can be shown that given the values of $\bar{n} = E(Y/X) = E(\eta)$ and σ_η, the maximum UNRE value, regardless of the underlying distributions, is given by

$$Q = \bar{R} \leq \frac{(\bar{n})^2 CV_\eta^2}{(\bar{n})^2 CV_\eta^2 + (\bar{n} - 1)^2}$$

where $CV_\eta = \sigma_\eta / \bar{n}$. Conversely, if the desired value of R and the CV_η are given, then the minimum required mean factor of safety, $\bar{n}$, is given by

$$\bar{n} \geq \frac{1}{1 - CV_\eta \sqrt{R/Q}}$$

where $Q = \bar{R}$ and $CV_\eta = \dfrac{\sqrt{CV_y^2 + CV_x^2}}{1 + CV_x^2}$. For example, if r = 0.999 is required and it is known that CV(Y/X) = 0.03, then the minimum required factor of safety is given by

$\bar{n} \geq \dfrac{1}{1 - 0.03\sqrt{0.999/0.001}} =$ 19.3081. Note that this required $\bar{n} \geq$ 19.3081 makes no assumptions about f(x) and g(y); for example, if both underlying distributions are normal as in Example 14, then $\bar{n} \cong n_c(1 + CV_x^2)$ =1.9789383×(1 +0.2473673²) = 2.100031.

On the other hand, if no assumptions are made about f(x) and g(y), and the design value of R is given along with CV_y and CV_x, then the minimum required value of central safety factor is given by

$$n_c = \frac{\mu_y}{\mu_x} \geq \frac{1}{1 + CV_x^2 - \sqrt{R(CV_y^2 + CV_x^2)/\bar{R}}}$$

For example, suppose the design value of R = 0.999, and it is known that CV_y = 2.4% while CV_x = 1.8%, then the minimum central safety factor is given by $n_c \geq$

$$\frac{1}{1+(0.018)^2 - \sqrt{0.999(0.024^2 + 0.018^2)/0.001}} = 19.1883.$$

Exponential Strength and Uniform Stress

As an example suppose Y ~ Exp(μ_y = 1000 MPa) and stress acting on the device is U(200, 500 MPa). Our objective is to compute component RE.

$$R = \int_{-\infty}^{\infty} g(y)dy \times F_x(y) = \int_{-\infty}^{\infty} 0.001e^{-0.001y}\,\frac{y-200}{300}dy = \int_{-\infty}^{0} 0 \times 0dy +$$

$$\int_{0}^{200} 0.001e^{-0.001y} \times 0dy + \int_{200}^{500} 0.001e^{-0.001y}\,\frac{y-200}{300}dy + \int_{500}^{\infty} 0.001e^{-0.001y}(1)dy =$$

$$\frac{1}{300}\left[-(y-200)e^{-0.001y}\Big]_{200}^{500} + \int_{200}^{500} e^{-0.001y}dy\right] + \left[-e^{-0.001y}\right]_{500}^{\infty} = \frac{1}{300}[-300e^{-0.5}$$

$$+(e^{-0.20} - e^{-0.50})/0.001] + 0.60653066 = 0.100803 + 0.60653066 = 0.707333645.$$

We could also arrive at the same RE as follows: $R = \int_{-\infty}^{\infty} G_y(x)f(x)dx = \int_{-\infty}^{\infty} e^{-0.001x}dx/300$

$$= \int_{200}^{500} e^{-0.001x}dx/300 = \frac{-1}{0.30}e^{0.001x}\Big]_{200}^{500} = \frac{1}{0.30}(e^{-0.20} - e^{-0.50})$$ = 0.707333645, as before.

Exercise 25. Suppose stress acting on a component has an exponential distribution with mean μ_x = 100 MPa and its strength is Weibull with minimum strength δ = 600 MPa, characteristic strength θ = 900 MPa and slope β = 2.00. Compute the device RE. ANS: R = $R_1 + R_2$ = $P(X< 600)\times P(Y \geq 600)$ + $P(X \geq 600)\times P(Y \geq x)$= 0.9996405635.

Lognormal Stress and Strength

Suppose X and Y are independent and lognormally distributed with parameters $x_{0.50} = e^{\mu_x}$, σ_x, $y_{0.50} = e^{\mu_y}$ and σ_y. Then, R = P(Y/X > 1) = P(lnY – lnX > 0) = P(W > 0), where W = lnY – lnX is normally distributed with mean E(W) = E(lnY) – E(lnX) = $\mu_y - \mu_x = \ln(y_{0.50}) - \ln(x_{0.50}) = \ln(y_{0.50}/x_{0.50})$ and V(W) = V(lnY – lnX) = $\sigma_y^2 + \sigma_x^2$. Hence, R = P(W > 0) = P[Z > $\dfrac{0-\ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2+\sigma_x^2}}$]

$$= 1 - \Phi\left[\frac{-\ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2+\sigma_x^2}}\right] = \Phi\left[\frac{\ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2+\sigma_x^2}}\right].$$

As an example consider the Example 7.11 on page 134 of Ebeling where $y_{0.50}$ = 8.1, σ_y = 0.07, $x_{0.50}$ = 5.5, and σ_x = 0.15. Then, R = $\Phi[\ln(8.1/5.5)/\sqrt{0.0049+0.0225}]$ = 0.990323303, which match's that of Ebeling's to 2 decimal accuracy.

RE Estimation from Strength/Stress Data

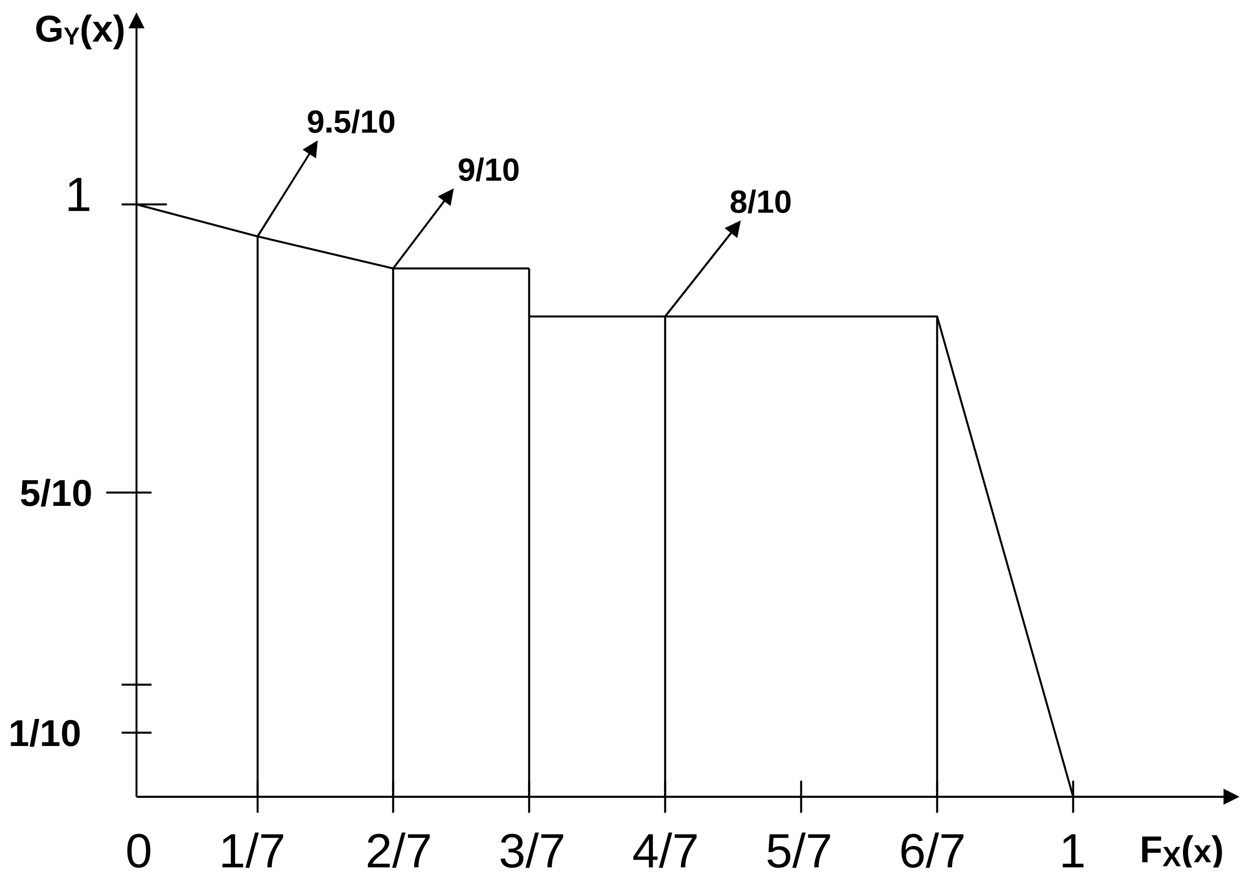
Recall from Eq. (90a) that R = $\int_{-\infty}^{\infty} f(x)dxG_y(x)$; making the transformation f(x) = $dF_x(x)/dx$ in this integral results in f(x)dx = $dF_x(x)$ and R = $\int_0^1 G_y(x)dF_x$; thus, if we graph $G_y(x)$ versus $F_x(x)$, the area under $G_y(x)$ and the abscissa $F_x(x)$-axis from zero to 1 will give the approximate reliability.
As an example suppose that stress and strength analysis performed on a randomly selected component resulted in the following simulated data.

Stress (MPa): 8 15 12 13 14 17 15

Strength (MPa): 14 10 17 18 20 19 23 22 25 19

We first tabulate the estimates of $F_x(x)$ and $G_y(x)$ as shown below.

KPa	**0**	**8**	**10**	**12**	**13**	**14**	**15**
$F_x(x)$	**0**	**1/7**	**1/7**	**2/7**	**3/7**	**4/7**	**6/7**
$G_y(x)$	**1**	**1**	**9/10**	**9/10**	**9/10**	**8/10**	**8/10**
KPa	**17**	**18**	**19**	**20**	**22**	**23**	**25**
$F_x(x)$	**7/7**	**1**	**1**	**1**	**1**	**1**	**1**
$G_y(x)$	**7/10**	**6/10**	**4/10**	**3/10**	**2/10**	**1/10**	**0**



The above Figure shows that the approximate value of RE is given by

$$\hat{\mathbf{R}} = \frac{1}{7}\left[\frac{1+0.95}{2}+\frac{0.95+0.90}{2}+0.90+3(0.8)+0.8/2\right] = 0.80.$$

Random Loads (Section 7.3.2 on pp. 157-158 of Ebeling)

Because random loads represent realistic life situations, we will illustrate the use of the Poisson distribution through the Exercise 7.8 on p. 163 of Ebeling (2nd Ed.). The stated parameters are maximum strength capacity of the dam at y_U = 20feet, and the distribution of flood levels, when they do occur, is exp(λ_x = 0.25/feet) so that the mean flood levels is 1/0.25 = 4 feet. The Poisson occurrence rate, r, of floods is 0.50floods/year, i.e., r = 0.50/year. In order to compute the RE over 10 years, we first compute the $\mu = E(N_{floods}) = 10\times0.50 = 5$floods. Thus, the Pr that exactly i floods occur in the next 10 years is given by $P_i = \frac{5^i}{i!}e^{-5}$, i = 0, 1, 2, 3, 4, 5, 6, …. Secondly, each time a flood occurs the Pr that its level does not exceed 20 feet is given by $R = 1-e^{(-0.25\times20)} = 0.993262053$, and hence the collapse Pr is given by Q = 1–R = 0.006737947. Therefore, the system RE over t-years is given by

$$R_{Sys}(t) = P_0 + P_1\times R + P_2\times R^2 + P_3\times R^3 + \ldots = e^{-\mu} + \mu e^{-\mu}\times R + \frac{\mu^2}{2!}e^{-\mu}\times R^2 + \frac{\mu^3}{3!}e^{-\mu}\times R^3 + \ldots.$$

$$= e^{-\mu}\left[1 + \mu R + \frac{(\mu R)^2}{2!} + \frac{(\mu R)^3}{3!} + \ldots\right] = e^{-\mu}\sum_{i=0}^{\infty}\frac{(\mu R)^i}{i!} = e^{-\mu}e^{\mu R} = e^{-\mu(1-R)} = e^{-\mu Q},$$ where $Q = e^{-\lambda_x y_U}$. Hence, $R_{Sys}(10\text{ years}) = e^{-rt\times Q} = e^{-5\times0.006737947} = 0.9668714445$. For a 20year-span, $R_{Sys}(20) = e^{-10\times0.006737947} = 0.9348404$.

Suppose now we wish to have a Sys RE of least 99% at 20 years instead of 0.9348404. How much larger the dam's maximum capacity must be in order to attain this Sys RE, i.e., $R_{Sys}(20) = e^{-20r\times Q} \geq 0.99 \rightarrow -20rQ \geq \ln(0.99) \rightarrow -20\times0.50Q \geq -0.01005034 \rightarrow Q \leq 0.10005034 \rightarrow e^{-0.25y_U} \leq 0.10005034 \rightarrow y_U = 27.611$ feet.