INSY 7380 System Reliability (Static Models) Maghsoodloo

Reference: Chapter 5 of C. E. Ebeling

Throughout this chapter, it is assumed that the reliability of all system components (or subsystems) practically stays constant (or static) for the duration of mission time t (i.e., the length of mission-time t is short enough so that such an assumption is tenable). In part (2) of this chapter we will consider dynamic models where, in general, there will be reliability degradation of subcomponents with respect to time.

The Series Static (or Serial) System

Such a system is reliable only if all subsystems are reliable. ASA one subsystem fails, the system fails. For the sake of illustration, consider a series system with n = 4 components (or subsystems) each with a RE = 0.995 (such as the system of 4 tires on a passenger car with trip duration of t = 500 miles). The RE block diagram is shown below.

$$\boxed{0.995} \longrightarrow \boxed{\text{Unit 2}} \longrightarrow \boxed{\text{Unit 3}} \longrightarrow \boxed{0.995}$$

$$\mathbf{R_{Sys} = (0.995)^4 = 0.98015}$$

In the above RE computation, we are assuming that the 4 tire failures are independent. In general, if the series system has n units with independent failures, then

$$R_{Sys} = \prod_{i=1}^{n} R_i , \qquad (29)$$

where R_i is the RE of the its subsystem. It is clear that $R_{sys} \le \text{Min}\{R_i\}$, i = 1, 2,..., n because each R_i in Eq. (29) is less than 1.

The Pure Parallel Systems (Static Models)

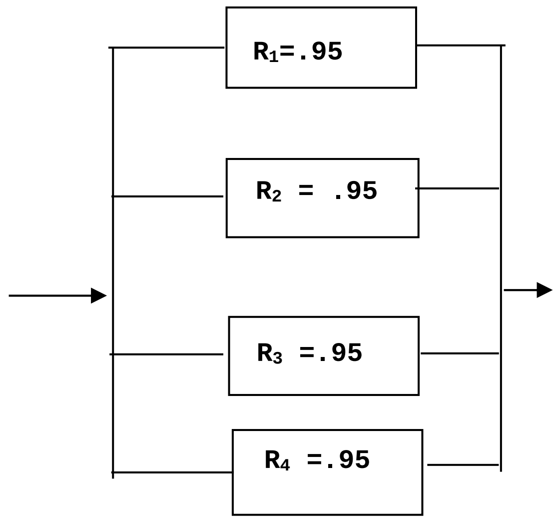
Such a system is reliable only if at least one component is reliable (i.e., n hot spares in parallel redundancy). The RE (reliability) block diagram for a pure parallel system with n = 4 units is given below, where R_i = 0.95 for i = 1, 2, 3, 4. The system reliability $R_{Sys} = 1 - Q_{Sys}$, where Q_{Sys} represents system unreliability $\overline{R}_{Sys}$. A pure-parallel system is reliable only if at least one unit is reliable. Or, the system is unreliable only if all n = 4 components fail. Hence, system unreliability is given by: Q_{Sys} = System Failure Pr = $(0.05)^4$ = 0.00000625 = $\bar{R}_{Sys}$

$R_{Sys} = 1 - Q_{Sys} = 1 - \bar{R}_{Sys} = 1 - 0.0^5625 = 0.9^5375 = 0.99999375$, in contrast to 0.81451 = $(0.95)^4$ for a 4-unit static series system. In general, for a pure parallel system with n hot spares, first the system unreliability is obtained followed by system RE as shown below:

$$\mathbf{Q_{Sys} = \prod_{i=1}^{n} Q_i = \prod_{i=1}^{n}(1-R_i) = \bar{R}_{Sys}}$$

$$\mathbf{R_{Sys} = 1 - Q_{Sys} = 1 - \prod_{i=1}^{n}(1-R_i).}$$

In a pure parallel system, $R_{Sys} \geq \text{Max}(R_1, R_2, ..., R_n)$.



The RE Block Diagram for a 4-unit Pure-Parallel System

In Exercise 5.2 on page 117 of Ebeling, by common-mode failure it is meant that the n components of the pure-parallel system do not operate independently of each other but share one common source of failure so that $R_{Sys} = [1 - \prod_{i=1}^{n}(1-R_i)] \times R_C$, where C denotes common.

<u>Exercise 8</u>. (a) Consider a system with n components in series with REs: $R_1, R_2, ..., R_n$. Assuming that it is possible to increase the RE of only one of the n components, determine which design component you would select to improve in order to optimally increase R_{Sys}. Repeat for a pure parallel system. In both cases provide a proof. [Hint: Examine the partial derivative of R_{Sys} wrt R_i]. (c) Estimate the failure rate of a component with approximately a constant RE during mission interval (0, t).

Next, consider the above 4-unit parallel system where $R_i = 0.95$, i = 1, 2, 3, 4, but suppose that at least 2 components must survive for the system to be reliable. Let P(i) =

the Pr that exactly i (i = 1, 2, 3, 4) units will be reliable. Then R_{Sys} = R(2; 4, 0.95) = 1 – P(0) – P(1) = 1 – $(0.05)^4$ – $_4C_1(0.95)(0.05)^3$ = 0.99951875, as compared to 0.99999375 for a pure-parallel system. Note that this problem can also be solved directly by summing P(2), P(3), and P(4) because these are 3 mutually exclusive possibilities. As a review of the binomial pmf, the reader should verify R(2; 4, 0.95) = $\sum_{i=2}^{4} P(i)$. This example represents a 2-out-4 parallel system (no longer a pure-parallel system) redundancy.

<u>Exercise 9.</u> Compute the reliability for a 3-out-of-5 redundant system, where R_i = R = 0.95 for all 5 units (see p. 104 of Ebeling). ANS: 0.998842

Generalizing recall that the binomial Pr mass function (pmf) as b(x; n, p) = $_nC_x\, p^x q^{n-x}$ gives the Pr of exactly x successes in n Bernoulli (independent) trials. The cdf of the binomial is given by B(x; n, p) = $\sum_{i=0}^{x} {}_nC_i\, p^i (q^{n-i})$. For the sake of illustration, consider a 3-out-of-4 parallel system each unit with success Pr R_i = p, and failure Pr q_i = 1 – p. Assuming that the units failure modes are independent, the system RE is obtained from b(3; 4, p) + b(4; 4, p) = $_4C_3p^3q^{4-3} + p^4 = 4p^3(1-p) + p^4 = 4p^3 - 3p^4$ = 1 – B(2; 4, p). In general, the RE of a k-out-of-n redundant System is given by

$$\mathbf{R(k;\, n,\, p) = \sum_{r=k}^{n} {}_nC_r p^r (q^{n-r}) = 1 - B(k-1;\, n,\, p)} \qquad \textbf{(30)}$$

As yet another example, consider a 3-out-of-7 parallel redundant system, where each hot spare has a RE of R = p = 0.975. Then from equation (30), we have R(3; 7, 0.975) = 1 – B(2; 7, 0.975) = 1 – binocdf(2,7,0.975) = 1 – 0.0000001966247558593755 =
0.99999980337524, where I have invoked the Matlab function binocdf(x, n, p). The Excel function for B(k–1; n, p) is BINOMDIST(k – 1, n, p, True). The False option will give the point mass probability.

Reliability Computation For Mixed Parallel Systems

Ebeling gives a good discussion of Series –Parallel (low-level redundancy), Parallel-

Series (high-level), and Mixed-Parallel systems on pp. 102-105 (Section 5.3). To illustrate RE computations, consider Figure 5.3 at the bottom of his page 102 of Ebeling, where the six units in the combined Series-Parallel system have $R_1 = R_2 = 0.90$, $R_3 = R_6 = 0.98$ and the 2-unit series subsystem has $R_4 = R_5 = 0.99$. We wish to compute the system reliability denoted by $R_{Sys,}$ where Ebeling uses the notation R_s for system reliability.

Step 1: Reduce the subsystem-B to a 2-unit series subsystem, where $R_1 = R_2 = 0.90$. Then $Q_{SubsysA} = (0.10)^2 = 0.01 \rightarrow R_{SubsysB} = (1 - 0.01)\times 0.98 = 0.97020$.

Step 2: Reduce the RE of the 2-unit series subsystem-C to a single unit RE in pure parallel redundancy with subsystem B.

$$R_{SubsysC} = (0.99)^2 = 0.98010$$

Step 3: We now have a 2-unit pure-parallel subsystem consisting of B and C. The UNRE (unreliability) of this subsystem in parallel redundancy is $Q_{SubSys1} = (1 - 0.97020)(1 - 0.98010) = 0.00059302$. $\rightarrow R_{SubSys1} = 1 - Q_{SubSys1} = 0.9994069800 \rightarrow R_{Sys} = 0.9994069800 \times R_6 = 0.979419 = R_s$, where $R_6 = 0.98$.

Ebeling covers low-level (see Figure 5.4 p. 103) and high-level redundancy (see Figure 5.5, at the bottom of p. 103), where the low-level redundancy always yields larger RE than high-level redundancy, which is counterintuitive. To illustrate this fact and as an exercise, draw the diagram for a 3-unit (A, B, and C) low-level and high-level redundancy and show that indeed this is the case, where all units have RE = p.

RE Analysis For Complex Systems

For the sake of illustration, consider the RE block diagram of Figure 5.6(a) on page 105 of Ebeling. The block diagram shows that the system is reliable thru a total of six paths $A\cap C$, $A\cap E\cap C$, $A\cap E\cap D$, $B\cap D$, $B \cap E\cap D$ or $B \cap E\cap C$, where $R_A = R_B = 0.90$, $R_C = R_D = 0.95$ and $R_E = 0.80$. We present two methods of computing R_{Sys}: (1) The Decomposition, (2) The Path-Tracing Method where duplications are eliminated using the least number of components so that the six paths are reduced to the four distinct paths $A\cap C$, $A\cap E\cap D$, $B\cap D$, or $B \cap E\cap C$.

The Decomposition Method

This method begins with selecting a keystone component, and for Figure 5.6 on page 115 of Ebeling the most logical keystone component is E which links the RE structure of the system. From the law of Conditional Total Probability, we have

$$\mathbf{R_{Sys} = R_E \times R(Sys \mid E \text{ is reliable}) + Q_E \times R(Sys \mid \overline{E})} \qquad \textbf{(31)}$$

where the event $\overline{E}$ implies the failure of component E so that F(E) = Q(E) = 0.20 = Pr($\overline{E}$). From now on, I am invoking the notation AC = $R_{A\cap C}$, BD = $R_{B\cap D}$, $R_{A\cap B\cap C\cap D}$ = ABCD, etc only for convenience and less writing.

R(Sys$\mid$E) = AC + AD + BC + BD – ABCD – ACD – ABD – ABC – BCD – ABCD + 4ABCD – ABCD = 0.98752500. R(Sys$\mid\overline{E}$) = $R_{A\cap C}$ + $R_{B\cap D}$ – $R_{A\cap B\cap C\cap D}$ = (0.9)(0.95) +(0.9)(0.95) – $(0.9)^2(0.95)^2$ = 0.97897500 $\rightarrow$ R_{Sys} = 0.80×0.98752500 + 0.20×0.97897500 = 0.98581500. Ebeling's answer at the bottom of Table 5.2 on his page 106 is accurate only to 4 decimals, while mine is accurate to 6 decimals.

The Path-Tracing Method

As stated earlier for Figure 5.6a (p. 105) there are 4 minimal paths A∩C, A∩E∩D, B∩D, or B ∩E∩C (with least number of components) thru which the complex system in Figure 5.6 can be reliable. Of the 2^5 = 32 total paths listed in Ebeling's Table 5.2, p.106, only 16 are RE paths but they lead to duplications. Recall from STAT3600 that if E_i (i = 1, 2, 3, 4 are four events, then

$$\Pr\left(\bigcup_{i=1}^{4} E_i\right) = \sum_{i=1}^{4} \Pr(E_i) - \sum_{i\neq j}^{4} \Pr(E_i \cap E_j) + \sum_{i\neq j\neq k}^{4} \Pr(E_i \cap E_j \cap E_k) - \Pr(E_1 \cap E_2 \cap E_3 \cap E_4) \qquad \textbf{(32)}$$

<u>Exercise 10.</u> Use the above formula (32) to recompute R_{Sys} for the Figure 5.6 of Ebeling. Again to minimize the amount of writing, you should use the notation AD= R(A∩D), CE = R(C∩E), etc. ANS: 0.985815.

Tie-Set and Cut-Set Methods for Complex System RE Computations

A tie-set (T) is a complete path that leads to system RE thru the RE block diagram. The minimum tie-set contains no duplications within the set, i.e., all the RE paths in the minimum tie-set contain the minimum number of components for system RE. As an example, again consider Figure 5.6(a) on page 105 of Ebeling. As stated above, this system has 4 tie-sets $T_1 = A \cap C$, $T_2 = A \cap E \cap D$, $T_3 = B \cap D$, and $T_4 = B \cap E \cap C$ (using the least number of components), where $R_{Sys} = \Pr(T_1 \cup T_2 \cup T_3 \cup T4)$ and $T_i = E_i$ as in Eq. (32).

A (minimal) cut-set (C), on the other hand, consists of the minimum number of components whose unreliabilities, $\bar{R}$, lead to system failure. For the Figure 5.6 of Ebeling the minimum cut-sets are $C_1 = \bar{A} \cap \bar{B}$ (i.e., both A and B do not work), $C_2 = \bar{C} \cap \bar{D}$, $C_3 = \bar{A} \cap \bar{E} \cap \bar{D}$, and $C_4 = \bar{B} \cap \bar{E} \cap \bar{C}$. Hence, the system UNRE is given by

$$\bar{\mathbf{R}}_{\mathbf{Sys}} = \mathbf{Q_{Sys}} = \mathbf{Pr}(\bigcup_{i=1}^{4} \mathbf{C_i}) = \sum_{i=1}^{4} \mathbf{P(C_i)} - \sum_{i \neq j}^{4} \mathbf{Pr(C_i \cap C_j)} + \sum_{i \neq j \neq k}^{4} \mathbf{Pr(C_i \cap C_j \cap C_k)} -$$

$$\mathbf{Pr(C_1 \cap C_2 \cap C_3 \cap C_4)} = (0.10)^2 + (0.05)^2 + (0.1)(0.20)(0.05) + (0.1)(0.20)(0.05) - (0.10)^2(0.05)^2 - (0.10)^2(0.20)(0.05) - (0.10)^2(0.20)(0.05) - (0.10)(0.20)(0.05^2) - (0.10)(0.20)(0.05^2) - (0.10^2)(0.20)(0.05^2) + 4(0.10^2)(0.20)(0.05^2) - (0.10^2)(0.20)(0.05^2) =$$

$0.01418500 \rightarrow R_{Sys} = 1 - \bar{R}_{Sys} = 1 - 0.01418500 = 0.98581500$ as before!

Ebeling provides a good example of the minimum tie-sets and cut-sets on page 109 in his Example 5.8. You should study Example 5.8 and verify $R_{Sys} = 0.79760$ using both T_i's and C_i's.

Bounds on System Reliability

Ebeling covers this topic in section 5.4.3 (pp. 110-112). To obtain the greatest lower bound (glb) on R_{Sys}, it follows that the system unreliability, Q_{Sys}, can never be larger than putting all its cut-sets in serial configuration [see Figure 5.9 (a) atop page 111 of Ebeling]. Thus,

$Q_{Sys} \leq$ Pr(at least one cut-set occurs) = 1 – Pr(zero cut-set occurs) $\rightarrow$

$1 - R_{Sys} \leq 1 -$ Pr(zero cut-set occurs) $\rightarrow$ $-R_{Sys} \leq -$ Pr(zero cut-set occurs) $\rightarrow$

$R_{Sys} \geq$ Pr(zero cut-set occurs) = glb(on R_{Sys}) = R_L

The above inequality gives the glb on R_{Sys} and is the wording version of Eq. (5.18) on page 111 of Ebeling. As an example, consider the mixed series-parallel system in low-level redundancy of figure 5.8 atop page 109 of Ebeling. Clearly the cut-sets are $C_1 = AC$, $C_2 = AD$, $C_3 = BC$, and $C_4 = BD$ with Prs $\Pr(C_1) = 0.10 \times 0.20 = 0.02$, $\Pr(C_2) = 0.10 \times 0.30 = 0.03$, $\Pr(C_3) = 0.20 \times 0.40 = 0.08$, and $\Pr(C_4) = 0.40 \times 0.30 = 0.12 \rightarrow$ Pr(zero cut-set occurs) =

$0.98 \times 0.97 \times 0.92 \times 0.88 = 0.7696057600 \rightarrow 0.7696057600 = \text{glb}(R_{Sys}) = R_L \leq R_{Sys} = 0.7976$
To obtain the least upper bound (lub) on R_{Sys}, we first put all the minimal tie- sets in pure-parallel redundancy (see Figure 5.9 (b), p. 111). Thus, $R_{Sys} \leq$ R(at least one minimal tie-set, or minimal path) = 1 – Q(all tie-sets) = R_u. The inequality $R_{Sys} \leq 1 -$ Q(all tie-sets) is basically the Eq. (5.19) on page 111 of Ebeling. Figure 5.8 on page 109 of Ebeling has two tie-sets $T_1 = A \cap B$ and $T_2 = C \cap D$ so that Q(all tie-sets) = (1 – 0.54)(1 – 0.56) = 0.202400 so that $R_{Sys} \leq$ 1 – Q(all tie-sets) = 1 – 0.202400 = 0.797600, the equality occurring because there are only two minimal paths.

Network RE Computations Using the Factoring Algorithm

To Illustrate this concept, I am making use of the very good Example 2.22 borrowed from the text by E. A. Elsayed (2012) entitled *" Reliability Engineering"* published by Wiley ISBN: 978-1-118-13719-2, pp. 129-130. In order to comprehend RE computations for a simple network, I will go through the Example 2.22 of Elsayed in a stepwise fashion.

A graph, G, is a pictorial representation of a network which consists of nodes and edges. Elsayed's Figure 2.21 on his page 129 is reproduced below. Node A is where gas is vaporized, and nodes B, C, D, E are distribution centers where gas is received from node A and must be delivered to destination node F where critical services are provided for a city. Graph G in Figure 2.21 of Elsayed has 8 edges e_i , i = 1, 2, ..., 8, (in actuality an edge in this example is a transmission gas pipeline) and 6 nodes A, B, C, D, E, F. Since I have not done research in network RE, my understanding of the subject is limited and it seems from studying Elsayed's text that nodes are always reliable and that only edges (or gas pipe lines in this case) in a graph can fail. For the example 2.22 of Elsayed, node A is where gas is vaporized, and nodes B, C, D, E, F are distribution centers. The objective is to get from node A through the eight edges to node F, which provides gas to critical services of a city.

Step 1: Select an arc (or edge, or a pipe), say edge 1 (which connects node A to B) and construct 2 subgraphs G_1 and $\overline{G}_1$, where G_1 implies that e_1 is reliable and $\overline{G}_1$ implies that edge 1 has failed. The two subgraphs are reproduced below. Letting p_i represent the RE of e_i, then from the law of total Pr we have

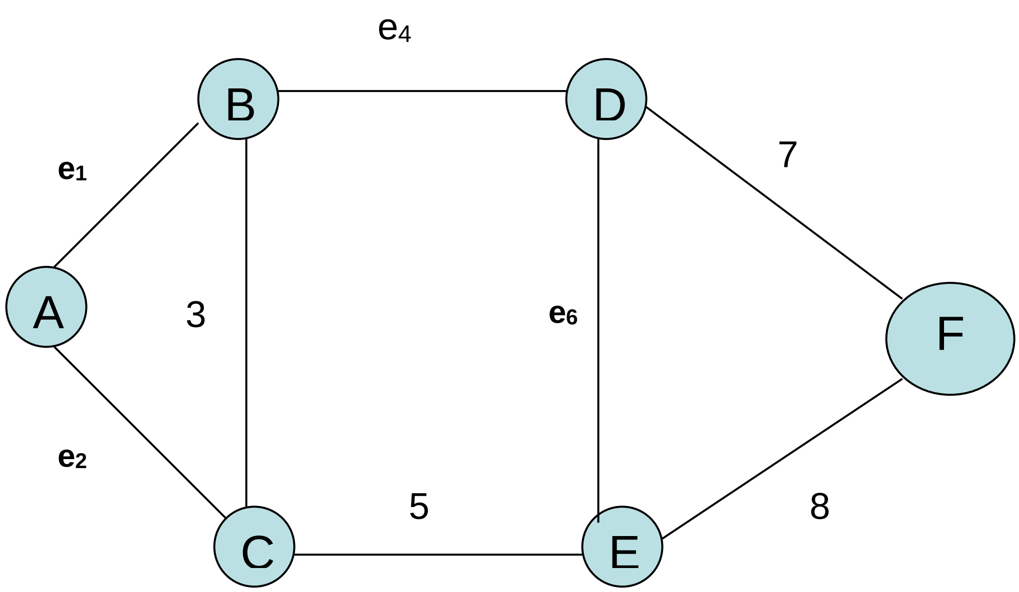
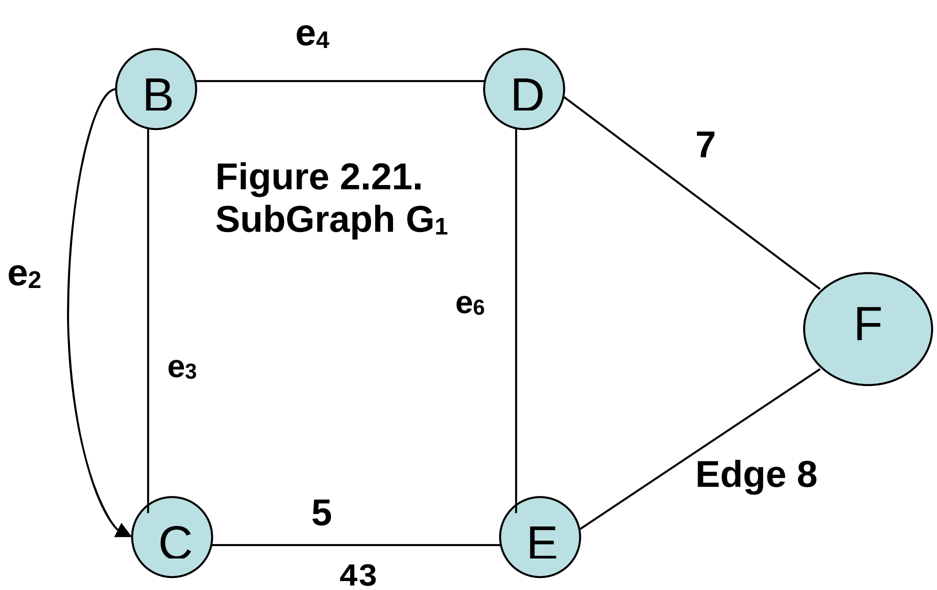


Figure 2.21 of E. A. Elsayed (2012)

$$R_{Sys} = R(G) = p_1\, R(G \mid e_1) + q_1\, R(G \mid \bar{e}_1) = p_1 R(G \mid e_1) + (1 - p_1) R(G \mid \bar{e}_1) \qquad (33)$$

Note that equation (33) is based on 2 mutually exclusive events: either e_1 is reliable or e_1 is unreliable, i.e., these 2 events have no intersection.

Step 2: Divide G_1 further into two subgraphs $G_{1,4}$ and $G_{1,\bar{4}}$, where subgraph $G_{1,4}$ consists of only nodes and arcs where both e_1 and e_4 are working reliably, while $G_{1,\bar{4}}$ implies that e_1 is working reliably while e_4 has failed. These two subgraphs are shown on pp. 44-45..



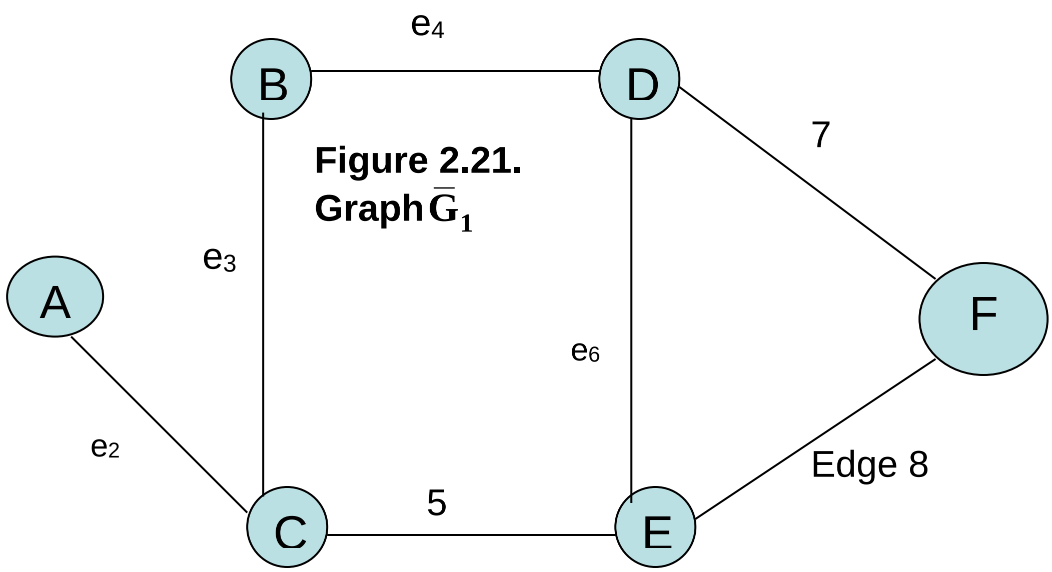


Figure 2.21.
Graph $\overline{G}_1$

Therefore, the 1st term on the RHS of (33) can be further dissected as

$$\mathbf{R(G|e_1) = p_4R(G|e_1, e_4) + (1 - p_4)R(G|e_1, \overline{e}_4) = p_4R(G|e_1, e_4) + q_4R(G|e_1, \overline{e}_4)} \qquad \textbf{(34a)}$$

I will 1st compute $R(G|e_1, e_4)$, followed by $R(G|e_1, \overline{e}_4)$, and then substitute these into (34a) to obtain $R(G|e_1)$. To compute $R(G|e_1, e_4)$, consider the subgraph $G_{1,4}$ below.

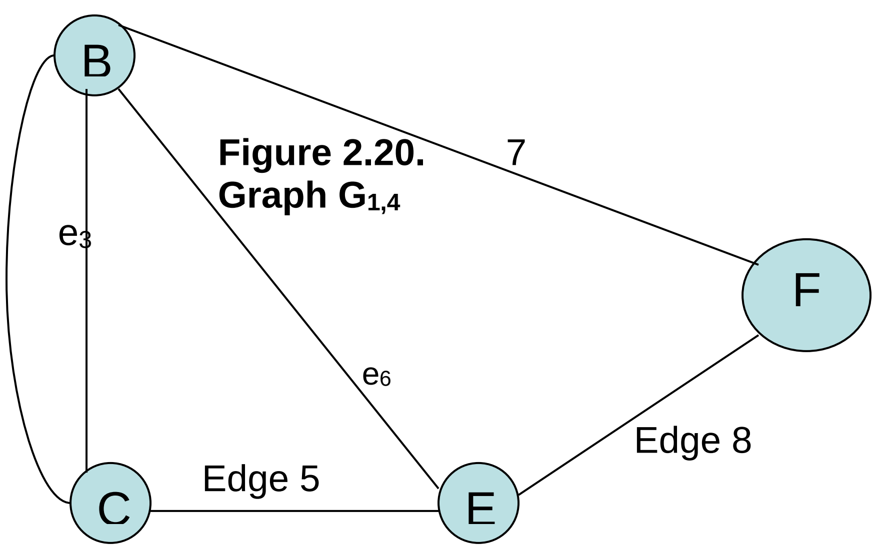


Figure 2.20.
Graph $G_{1,4}$

$$\mathbf{R(G_{1,4}) = R(G \mid e_1, e_4) = (p_7 + p_6p_8 - p_7p_6p_8) + (p_2 + p_3 - p_2p_3)p_5p_8 - p_7(p_2 + p_3 - p_2p_3)p_5p_8 - (p_6p_8 - p_7p_6p_8)(p_2 + p_3 - p_2p_3)p_5} \quad \textbf{(35a)}$$

When and if all e_i's have the same reliability p, then equation (35a) reduces to

$$\mathbf{R(G \mid e_1, e_4) = p(1 + p + p^2 - 5p^3 + 4p^4 - p^5)} \quad \textbf{(35b)}$$

Next we compute $R(G \mid e_1, \overline{e}_4)$, which is a part of the 2nd term on the RHS of (34a). Now concentrate on subgraph $G_{1,\overline{4}}$ below.

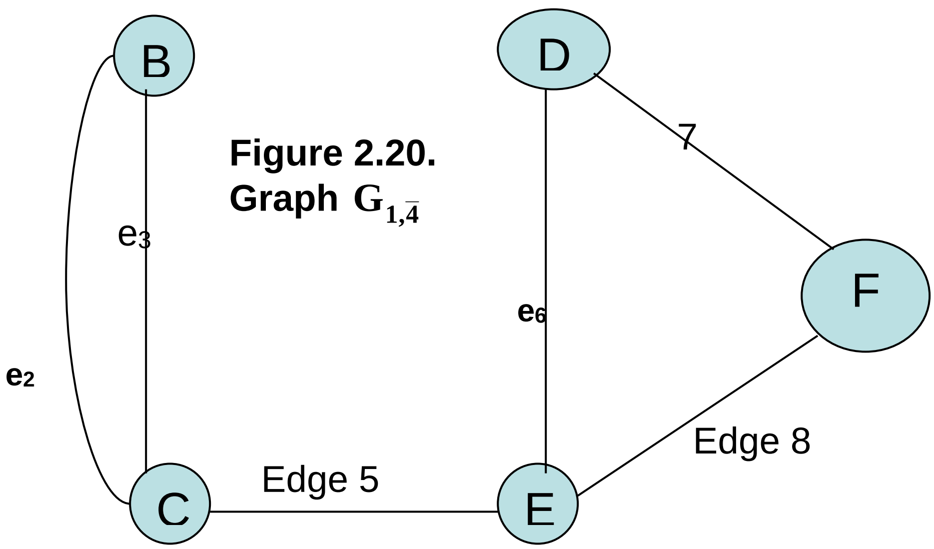


Figure 2.20.
Graph $G_{1,\overline{4}}$

$$\mathbf{R(G_{1,\overline{4}}) = R(G \mid e_1, \overline{e}_4) = (p_2 + p_3 - p_2p_3)p_5(p_8 + p_6p_7 - p_6p_7p_8)} \quad \textbf{(36a)}$$

When all e_i's have the same reliability p, then equation (36a) reduces to

$$\mathbf{R(G \mid e_1, \overline{e}_4) = p^3(2 + p - 3\,p^2 + p^3)} \quad \textbf{(36b)}$$

Substitution of equations (36b) and (35b) into (34a) yields

$$\mathbf{R(G \mid e_1) = p^2(1 + p + p^2 - 5p^3 + 4p^4 - p^5) + (1 - p)p^3(2 + p - 3p^2 + p^3)}$$
$$\mathbf{= p^2(1 + 3p - 9p^3 + 8\,p^4 - 2p^5)} \quad \textbf{(34b)}$$

Step 3: Dissect subgraph $\overline{G}_1$ further into 2 subgraphs $G_{\overline{1},7}$ and $G_{\overline{1},\overline{7}}$; the subgraph $G_{\overline{1},7}$ implies that edge 1 has failed but arc 7 is reliable, while $G_{\overline{1},\overline{7}}$ represents the subgraph where both arcs 1 and 7 have failed. Again using the law of total Pr, the last term on the

RHS of Eq. (33) becomes

$$\mathbf{R(G \mid \bar{e}_1) = p_7\, R(G \mid \bar{e}_1, e_7) + q_7\, R(G \mid \bar{e}_1, \bar{e}_7) = p_7\, R(G \mid \bar{e}_1, e_7) + (1 - p_7)\times}$$

$$\mathbf{R(G \mid \bar{e}_1, \bar{e}_7).} \qquad \mathbf{(37a)}$$

In order to compute $R(G \mid \bar{e}_1, e_7)$, concentrate on $G_{\bar{1},7}$ drawn below.

$$\mathbf{R(G \mid \bar{e}_1, e_7) = p_2[p_5(p_8 + p_6 - p_6p_8) + p_3p_4 - p_3p_4p_5(p_8 + p_6 - p_6p_8)]} \qquad \mathbf{(38a)}$$

Note that when e_8 fails the above subgraph can still be reliable through the path $e_2e_5e_6$ because e_7 is 100% reliable. In the case of $p_1 = p_2 = ... = p_8$, equation (38a) reduces to

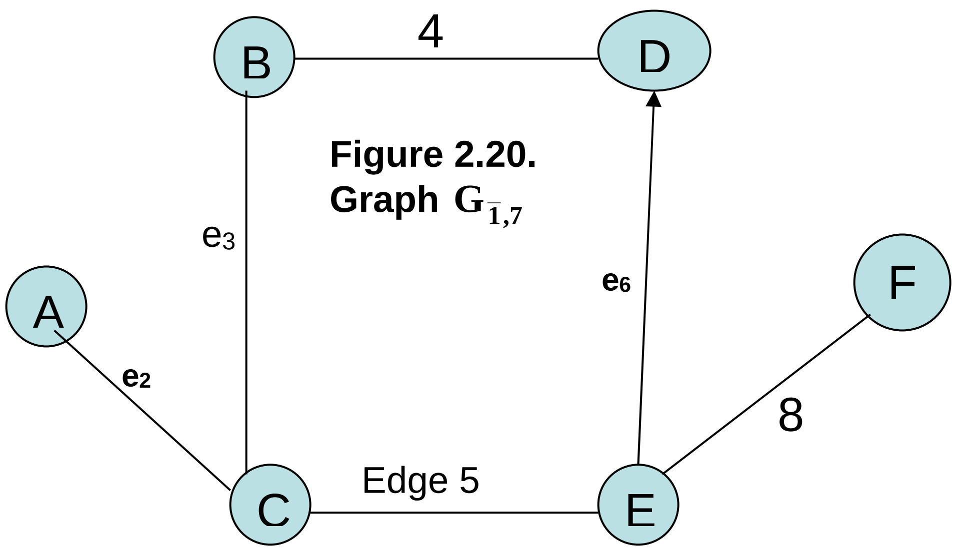


Figure 2.20.
Graph $G_{\bar{1},7}$

$$\mathbf{R(G \mid \bar{e}_1, e_7) = p^3(3 - p - 2\,p^2 + p^3)} \qquad \mathbf{(38b)}$$

In order to compute the last term on the RHS of (37a), concentrate on the graph $G_{\bar{1},\bar{7}}$.

$$\mathbf{R(G \mid \bar{e}_1, \bar{e}_7) = p_2(\, p_5p_8 + p_3p_4p_6p_8 - p_3p_4p_5p_6p_8)} \qquad \mathbf{(39a)}$$

For the case of all equal reliabilities, equation (39a) reduces to

$$\mathbf{R(G \mid \bar{e}_1, \bar{e}_7) = p^3(1 + p^2 - p^3)} \qquad \mathbf{(39b)}$$

Substituting equations (39b) and (38b) into (37a), and assuming that all p_i's are equal, results in

$$\mathbf{R(G \mid \bar{e}_1) = p^4(3 - p - 2\,p^2 + p^3) + (1 - p)p^3(1 + p^2 - p^3) =}$$

$$\mathbf{= p^3(1 + 2p - 4p^3 + 2p^4)} \qquad \mathbf{(37b)}$$

Finally, substituting (37b) and (34b) into equation (33) we obtain the following result, which is valid only for the case when all p_i's are equal to p.

$$\mathbf{R_{Sys} = R(G) = p[p^2(1 + 3p - 9p^3 + 8p^4 - 2p^5)] + (1 - p)p^3 \times}$$
$$\mathbf{(1 + 2p - 4p^3 + 2p^4) = p^3(2 + 4p - 2p^2 - 13p^3 + 14p^4 - 4p^5)}$$

This last system RE is identical to the one given in equation (2.52) on page 130 of E. A. Elsayed (2012). When all p_i's are all equal to p = 0.995, then R_{Sys} = R(G) = 0.99992450566.

Three-State Models for a Series System

As Ebeling describes in his section 5.6, p. 113, almost all electrical devices have 3 states: (1) operating reliably, (2) failing open, or (3) failing short. For example, when a diode fails open, then current flow in all direction becomes almost zero (i.e., impedance becomes almost infinite in all directions), while if the diode fails short, then resistance in all directions becomes almost zero. Other devices such as valves and switches can also fail in more then one way; a switch can fail open (so that current will not flow), or vice a versa can fail short so that it will be impossible to stop current flow. Similarly, a resistor or transistor can fail open (i.e., will not let current through), or may fail short (i.e., resistance becomes almost zero in all directions). Following Ebeling's notation, to this end, let R_i = the i^{th} unit is reliable, q_{oi} = the Pr that the i^{th} unit fails open , and q_{si} = the Pr that the i^{th} unit fails short; therefore, $R_i = 1 - q_{oi} - q_{si}$. Mechanical devices, such as valves and flaps, can also fail while they are in the open and shut states. If a valve fails (to) open, then it is impossible to open in order to allow liquid to flow. If it fails shut (or fails short), regardless of its initial state, then it is impossible to shut in order to stop liquid flow.

Now consider n = 3 units (such as switches) in series for closing a circuit (for current to flow), where each unit can be in 3 different states : (1) functioning reliably, (2) failing open (o = state 2), and (3) failing short (s = state 3). The system will be reliable iff all 3 units are reliable. However, because the 3 units are in series, the system will fail open, if at least one of the 3 switches opens (unexpectedly) and totally stops current flow. Therefore, the Pr that the system will fail open, denoted by F_o(Sys), is given by F_o(Sys) = P[at least one unit fails open] = 1 – P[no units fail open] →

$$\mathbf{F_o(Sys) = 1 - \prod_{i=1}^{3}(1 - q_{oi})} \qquad \textbf{(40a)}$$

In general, a serial system fails iff at least one unit fails open, or all units fail short. Further the system becomes short-circuited, iff all 3 units fail short; thus, the Pr that the system fails short is given by

$$\mathbf{F_s(Sys) = P[\text{all 3 units fail short}] = \prod_{i=1}^{3} q_{si}} \qquad \textbf{(40b)}$$

Eqs. (40a&b) clearly show that system unreliability is given by $\bar{R}_{Sys} = Q_{Sys} =$ $F_o(Sys) + F_s(Sys)$ because the two failure modes are mutually exclusive. Therefore for a 3-unit system, we obtain

$$\mathbf{R_{Sys} = 1 - F_o(Sys) - F_s(Sys) = 1 - [1 - \prod_{i=1}^{3}(1 - q_{oi}) + \prod_{i=1}^{3} q_{si}\,] \rightarrow}$$

$$\mathbf{R_{Sys} = \prod_{i=1}^{3}(1 - q_{oi}) - \prod_{i=1}^{3} q_{si}} \qquad \textbf{(41a)}$$

Equation (41a) is identical to Ebeling's (5.20) near the bottom of page 114 for the general case of n units in series. The argument for the general case is identical to the above. If all 3 units are functionally identical (i.e., $q_{oi} = q_o$ for all i, and $q_{si} = q_s$), then equation (41a) reduces to

$$\mathbf{R_{Sys} = (1 - q_o)^3 - q_s^3} \qquad \textbf{(41b)}$$

As Ebeling mentions, unlike the case of a 2-state serial system where the addition of more units in series will diminish system RE, the addition of more units to a 3-state series will not necessarily decrease system RE. Therefore, the objective now is to obtain the optimal number of units, n_o, that will maximize $R_{Sys} = (1 - q_o)^n - q_s^n$ for the case of n identical units. To accomplish this task, we need to set $\partial R_{Sys}/\partial n$ equal to zero and solve the resulting equation for n_o.

$$\frac{\partial R_{Sys}}{\partial n} = \frac{\partial}{\partial n}[(1-q_o)^n - q_s^n] = \frac{\partial}{\partial n}(1-q_o)^n - \frac{\partial}{\partial n}q_s^n = \frac{\partial}{\partial n}a^n - \frac{\partial}{\partial n}b^n, \qquad (42)$$

where for convenience we have let $1 - q_o = a$, and $q_s = b$. We need to be reminded of the fact that any real number $x \geq 0$ can be written as $x = e^{\ln(x)}$ because of the fact that the exponential and natural logarithm are inverse functions. Invoking this information into equation (42) results in

$$\frac{\partial R_{Sys}}{\partial n} = \frac{\partial}{\partial n}(e^{\ln(a)})^n - \frac{\partial}{\partial n}(e^{\ln(b)})^n = \frac{\partial}{\partial n}(e^{n\ln(a)}) - \frac{\partial}{\partial n}(e^{n\ln(b)}) = e^{n\ln(a)}(\ln a) -$$

$$e^{n\ln(b)}\times(\ln b) = a^n(\ln a) - b^n(\ln b) = (1-q_o)^n[\ln(1-q_o)] - q_s^n(\ln q_s)$$

$$\xrightarrow{\text{Set equal to}} 0 \qquad (43)$$

Equation (43) shows that at the optimal n, denoted by n_o, we must have

$$(1-q_o)^{n_o}[\ln(1-q_o)] = q_s^{n_o}(\ln q_s) \rightarrow \left(\frac{1-q_o}{q_s}\right)^{n_o} = \frac{\ln q_s}{\ln(1-q_o)} \rightarrow$$

$$n_o = \frac{\ln[\ln q_s/\ln(1-q_o)]}{\ln[(1-q_o)/q_s]} \qquad (44)$$

The optimal solution n_o in equation (44) is a decreasing function of q_o for a fixed value of q_s but is an increasing function of q_s at a fixed value of q_o.

Three-State Parallel Systems

For illustrative purposes, consider a 4-switch pure parallel system where each switch has 3 states (1 = functioning reliably, 2 = fail open, and 3 = fail short). Since the n = 4 units are in parallel, then the system will fail in the open mode only if all 4 units fail open. Thus, using the notation of the previous section, $F_o(Sys) = \prod_{i=1}^{4} q_{oi}$. However, the system will fail short only if at least one unit fails short, i.e., $F_s(Sys) = 1 - \prod_{i=1}^{4}(1-q_{si})$. Hence,

$$R_{Sys} = 1 - F_o(Sys) - F_s(Sys) = 1 - [\prod_{i=1}^{4} q_{oi} + 1 - \prod_{i=1}^{4}(1-q_{si})] \rightarrow$$

$$R_{Sys} = \prod_{i=1}^{4}(1-q_{si}) - \prod_{i=1}^{4} q_{oi} \qquad (45)$$

Equation (45) is identical to Ebeling's (5.21 on page 115) for the general case n. When all n units are identical (i.e., $q_{si} = q_s$, and $q_{oi} = q_o$), equation (45) generalizes to $R_{Sys} = (1 - q_s)^n - q_o^n$. The optimum number of units, n_o, in parallel redundancy is obtained by partially differentiating $R_{Sys} = (1 - q_s)^n - q_o^n$ with respect to n and setting the resulting derivative equal to zero. You may verify that the solution is given by

$$n_o = \frac{\ln[\ln q_o / \ln(1-q_s)]}{\ln[(1-q_s)/q_o]}.$$

Unlike the series system, n_o is an increasing function of q_o for a fixed q_s but is a decreasing function of q_s at a fixed value of q_o.

Three-State Parallel-Series Systems

Consider an (n = 2, m = 3) Parallel-Series (high-level redundancy) System as depicted in Figure 3 on the next page. We assume that units of the same letter are identical, i.e., $q_{oa1} = q_{oa2} = q_{oa}$, $q_{sa1} = q_{sa2} = q_{sa}$; similarly for units B and C. Thus all 6 units can fail open or fail short. The A-series fails open if at least one A-unit fails open; similarly, for the B- and C-series. However, for any of the 3 series to fail short, both units in the series must fail short. Further, in order to arrive at R_{Sys}, we simplify the notation by letting $q_{oa} = q_{o1}$, $q_{sa} = q_{s1}$, $q_{ob} = q_{o2}$, $q_{sb} = q_{s2}$, $q_{oc} = q_{o3}$, and $q_{sc} = q_{s3}$. In the following development, I will use the notation consistent with the previous 2 sections. Then it follows that $F_o(A) = 1 - (1-q_{o1})^2$, $F_o(B) = 1 - (1-q_{o2})^2$, $F_o(C) = 1 - (1-q_{o3})^2 \rightarrow F_o(Sys) = F_o(A)\times F_o(B)\times F_o(C)$. Similarly, $F_s(A) = (q_{s1})^2$, $F_s(B) = (q_{s2})^2$, and $F_s(C) = (q_{s3})^2 \rightarrow F_s(Sys) = 1 - [1-(q_{s1})^2]\times[1-(q_{s2})^2]\times[1-(q_{s3})^2]$. Hence,

$$R_{Sys} = 1 - F_o(Sys) - F_s(Sys) = \prod_{i=1}^{3}(1-q_{si}^2) - \prod_{i=1}^{3}[1-(1-q_{oi})^2] \qquad (46)$$

When $q_{si} = q_s$ and $q_{oi} = q_o$ for all i, equation (46) reduces to

$$R_{Sys} = (1-q_s^2)^3 - [1-(1-q_o)^2]^3$$

When there are n units in series in m-parallel redundancy, then Eq. (46) generalizes to

$$R_{Sys} = \prod_{i=1}^{m}(1-q_{si}^{n}) - \prod_{i=1}^{m}[1-(1-q_{oi})^{n}] \qquad \textbf{(47a)}$$

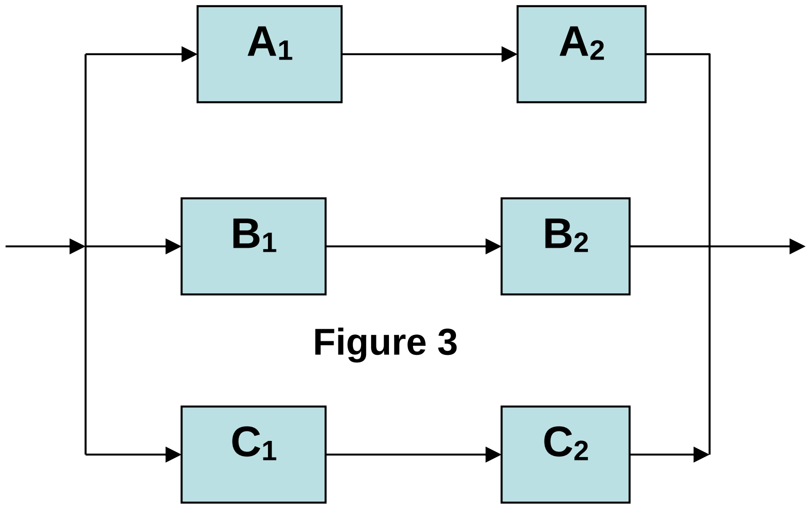


Figure 3

When all units are identical in their failure modes, Eq. (47a) for system RE reduces to

$$R_{Sys} = (1-q_{s}^{n})^{m} - [1-(1-q_{o})^{n}]^{m} \qquad \textbf{(47b)}$$

Which is identical to that of Elsayed's (2.59) near the bottom of page 116.

<u>**Exercise 11.**</u> Derive the RE expression for a Three-State Series-Parallel system (low-level redundancy) for the case (m = 2 parallel units in n = 3 series) and then generalize to any n and m.

We will not discuss System structure function because Ebeling does not use it, as far as I know, much anywhere in his text except for pp. 108-110. However, it is an interesting mathematical tool for system Reliability computations.