Dynamic Models

Static models (chapter 5 Part 1) are applicable only if components' RE practically stay constant (i.e., do not change appreciably during mission time). If components' RE do decrease as a function of time, then we must apply dynamic models (i.e., the hazard function is increasing during mission time and hence R(t) is a decreasing function of time as expected). There are two types of such models: (1) Non-repairable, and (2) Repairable.

Exercise 12. Consider a system whose RE stays almost constant during the fixed mission time t. Determine the approximate value of its hazard function h(t). Hint: Use the fact that f(t) = –dR/dt.

The Series (or Serial) Nonrepairable Dynamic Models

This is a system where all n components must function reliably during the mission interval (0, t) in order to complete the mission; further, the RE of the i[th] subsystem (i = 1, 2, ..., n), $R_i(t)$, is a decreasing function of time. The system RE is given by

$$R_{Sys}(t) = \prod_{i=1}^{n} R_i(t) \qquad \textbf{(48a)}$$

Clearly, $R_{Sys}(t) \le \text{Min}[R_i(t)]$, i = 1, 2, ..., n.

Next, let $h_i(t)$ be the hazard function (HZF) of the i[th] component and $h_{Sys}(t)$ be the HZF of the system. Then (from Chapter 2)

$$R_i(t) = e^{-\int_0^t h_i(x)dx}, \text{ and } \quad R_{Sys}(t) = e^{-\int_0^t h_{Sys}(x)dx}. \qquad \textbf{(48b)}$$

Substituting these last 2 equations into (48a) results in

$$R_{Sys}(t) = e^{-\int_0^t h_{Sys}(x)dx} = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} e^{-\int_0^t h_i(x)dx} = e^{-\int_0^t \sum_{i=1}^{n} h_i(x)dx} \qquad \textbf{(48c)}$$

Hence,

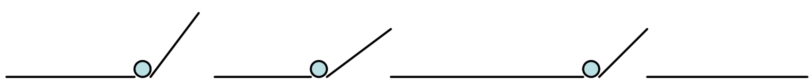
$$\mathbf{h_{Sys}(t)} = \sum_{i=1}^{n} h_i(t) \quad \textbf{and} \quad \mathbf{R_{Sys}(t)} = e^{-\sum_{i=1}^{n}\int_0^t h_i(x)dx} = e^{-\sum_{i=1}^{n} H_i(t)} \qquad \textbf{(49)}$$

From equation (48b), we can compute the MTTF = E(T) = $\int_0^{\infty} R_{Sys}(t)\,dt = \int_0^{\infty} e^{-\int_0^t h_{Sys}(x)dx}\,dt =$

$$\int_0^{\infty} e^{-\int_0^t \sum_{i=1}^{n} h_i(x)dx}\,dt = \int_0^{\infty}\left[e^{-\sum_{i=1}^{n}\int_0^t h_i(x)dx}\right]dt$$

$$\textbf{Or:} \qquad \mathbf{MTTF_{Sys}} = \int_0^{\infty}\left[e^{-\sum_{i=1}^{n} H(t)}\right]dt \qquad \textbf{(50a)}$$

Example 8. An electrical system has n = 3 circuit breakers in series for the purpose of closing the circuit (in order for the current to flow), as depicted below.



Each breaker has the same constant failure rate λ = 0.00005/hour. (a) Obtain the HZF and compute system RE for a mission of length t = 400 hours. (b) Compute the MTTF. (c) Obtain the system failure density function. Note that if the objective were to interrupt current, then we would need at least one unit to open reliably.

$h_{Sys}(t) = \sum_{i=1}^{3} h_i(t) = \sum_{i=1}^{3} 0.00005$ = 3×0.00005 = 0.00015/hour $\rightarrow$

H(t) = 0.00015t $\rightarrow$ $R_{Sys}(t) = e^{-H(t)} = e^{-0.00015t}$

Note that this last RE function for the n = 3 series system can also be obtained from $R_{Sys}(t)$ = $R_1(t)\times R_2(t)\times R_3(t) = e^{-\lambda t}\,e^{-\lambda t}\,e^{-\lambda t} = e^{-3\lambda t} = e^{-0.00015t}$ $\rightarrow$ $R_{Sys}(400) = e^{-0.06}$ = 0.9417645. (b)

MTTF = $\int_0^{\infty} R_{Sys}(t)\,dt = \int_0^{\infty} e^{-0.00015t}dt = \frac{1}{0.00015}$ = 6666.6667 hours.

(c) $f_{Sys}(t) = -dR_{Sys}/dt = 0.00015e^{-0.00015t} \rightarrow \int_0^{\infty} 0.00015e^{-0.00015t}dt = 1.00000$

Exercise 13. Suppose each component of a n-unit series system has a constant failure rate $h(t) = \lambda_i$ (i =1, 2, ..., n), i.e., each lifetime T_i is independently and exponentially distributed. (a) Prove that in general for such a series system

$$\mathbf{R_{Sys}(t) = e^{-(\sum_{i=1}^{n}\lambda_i)t}, \quad and \quad MTTF_{Sys} = 1/(\sum_{i=1}^{n}\lambda_i).} \qquad \textbf{(50b)}$$

(b) Use the results of part (a) to show that the TTF of a series system, whose individual component lifetimes, T_i (i = 1, 2, ..., n), are independently and exponentially distributed with failure rates λ_i, is also exponentially distributed with a failure rate $\lambda = \sum_{i=1}^{n}\lambda_i$. This proof also shows, a well-known result in stochastic processes, that the sum of n independent Poisson streams at rates $\lambda_1, \lambda_2, ..., \lambda_n$ is also Poisson at the arrival rate $\lambda = \sum_{i=1}^{n}\lambda_i$. Moreover, if all failure rates λ_i (i = 1, 2, ..., n) are equal to λ, then $MTTF_{Sys} = 1/(n\lambda)$.

Example 3.1 on Page 172 of A. E. Elsayed (2012). A series system consists of n = 5 components, three of which have constant failure rates $\lambda_1 = 5\times10^{-6}$, $\lambda_2 = 3\times10^{-6}$ and $\lambda_3 = 9\times10^{-6}$. The remaining components c_4 and c_5 have Weibull lifetime distributions with minimum lives $\delta_4 = \delta_5 = 0$, characteristic lives $\theta_4 = 1.5\times10^4$, $\theta_5 = 2.5\times10^4$, and slopes (or shapes) $\beta_4 = 2.2$, $\beta_5 = 2.1$. Compute system RE at t = 1000 hours. In order to use Eq. (49), we must first obtain the HZFs of c_4 and c_5 using the result of Exercise 5(c) in Chapter 2, which showed that for a zero minimum life Weibull the HZF is $h(t) = \frac{\beta}{\theta}(t/\theta)^{\beta-1}$, $t \geq 0$.

Hence, $h_4(t) = \frac{2.2}{1.5\times10^4}(t/1.5\times10^4)^{1.2} = 1.4289661\times10^{-9}\,(t^{1.2})$, $h_5(t) = \frac{2.1}{2.5\times10^4}(t/2.5\times10^4)^{1.1} = 1.2205211\times10^{-9}\,(t^{1.1}) \rightarrow h_{Sys}(t) = \sum_{i=1}^{n}h_i(t) = 17\times10^{-6} + +$

$1.4289661\times10^{-9}\,(t^{1.2}) + 1.2205211\times10^{-9}\,(t^{1.1}) \rightarrow H(t) =$

$$\int_0^t [17\times10^{-6} + 1.4289661\times10^{-9}\,(x^{1.2}) + 1.2205221\times10^{-9}(x^{1.1})]dx \rightarrow$$

$$\mathbf{R_{Sys}(t) = e^{-[17\times10^{-6}\,t + 6.49530023\times10^{-10}\,t^{2.2} + 5.812005\times10^{-10}\,t^{2.1}]}}. \qquad \textbf{(51)}$$

To evaluate the RE at 1000 hours, simply put t = 1000 hours in equation (51). This yields R_{Sys}(1000 hours) = $e^{-0.020745473}$ = 0.9794682339, which is consistent with Elsayed's answer of 0.9795 on his p. 172.

The RE Function for a Nonrepairable Series System with a Weibull TTF

Consider an n-unit series system each unit of which has the HZF $h_i(t) = \frac{\beta_i}{\theta_i}(t/\theta_i)^{\beta_i-1}$, i = 1, 2, ..., n. Eq. (49) yields $h_{Sys}(t) = \sum_{i=1}^{n} \lambda_i\beta_i(\lambda_i t)^{\beta_i-1}$ where $\lambda_i = 1/\theta_i$. From equation (48c), we obtain $R_{Sys}(t) = e^{-\sum_{i=1}^{n} H(t)} = e^{-\sum_{i=1}^{n}\int_0^t (\beta_i/\theta_i)(x/\theta_i)^{\beta_i-1}dx} =$

$$= e^{-\sum_{i=1}^{n}(t/\theta_i)^{\beta_i}} = e^{-\sum_{i=1}^{n}(\lambda_i t)^{\beta_i}} \qquad \textbf{(50c)}$$

If the n units have identical Weibull distributions, then (50c) reduces to $R_{Sys}(t) = e^{-n(t/\theta)^{\beta}} = e^{-n(\lambda t)^{\beta}}$. The MTTF of a serial dynamic system with n different Weibull components is not easy to obtain because this author could not find a closed-form antiderivative for the $R_{Sys}(t)$ in (50c) when β_i's are different. I surmise that such an antiderivative may not exist in closed-form and numerical integration has to be applied once the system parameters n, θ_i, and β_i (i = 1, 2, ..., n) are specified. However, when the n units have the same Weibull slope $\beta = \beta_i$ for all i, then from Eq. (50c) $MTTF_{Sys} = E(T) = \int_0^\infty R_{Sys}(t)\,dt = \int_0^\infty e^{-t^{\beta}\sum_{i=1}^{n}(1/\theta_i)^{\beta}}\,dt =$

$\int_0^\infty e^{-t^\beta \lambda}\, dt = \int_0^\infty e^{-\lambda t^\beta} dt$, where $\lambda = \sum_{i=1}^{n} (1/\theta_i)^\beta = \sum_{i=1}^{n} (\theta_i)^{-\beta}$. If we make the transformation $x = \lambda t^\beta$ in the integral $\int_0^\infty e^{-\lambda t^\beta} dt$, then after 8 long and tedious steps we will obtain

$$\text{MTTF}_{\text{Sys}} = \lambda^{-1/\beta}\ \Gamma(1+\frac{1}{\beta}) = \left[\sum_{i=1}^{n} \theta_i^{-\beta}\right]^{-1/\beta} \Gamma(1+\frac{1}{\beta}) \qquad \textbf{(50d)}$$

The MTTF of a series system has to be a decreasing function of n, which is exhibited by Eq. (50d). Moreover, if all characteristic lives, θ_i, are identical to θ, i.e., iid Weibull lifetimes, then (50d) further reduces to

$$\text{MTTF}_{\text{Sys}} = n^{-1/\beta} \times \theta \times \Gamma(1+\frac{1}{\beta}) = \frac{\theta\Gamma(1+1/\beta)}{n^{1/\beta}} \qquad \textbf{(50e)}$$

Eq. (50e) shows that the MTTF is a decreasing function of n while it is an increasing function of both θ and β.

The Example 3.6 on Page 180 of A. E. Elsayed(2012). For this example, n = 6 components, β = 1.75, and the vector $\theta = 10^5\times[7 \quad 8.2 \quad 4.6 \quad 6.5 \quad 6.8 \quad 5.0]^T$. Matlab computations using equation (50d) yields MTTF = 192262.7302989619 hours.

Pure Parallel Nonrepairable Dynamic Models (Hot Spares)

In a pure parallel n-unit redundant system, at least one subsystem must function reliably in order for the system to complete the specified mission of length t. Furthermore, it is tacitly assumed that component failures are completely independent. Therefore, the system fails during the interval (0, t) only if all n subsystems (or components) fail, i.e.,

$$R_{Sys}(t) = 1 - Q_{Sys}(t) = 1 - F_{Sys}(t) = 1 - \prod_{i=1}^{n} Q_i(t) = 1 - \prod_{i=1}^{n} [1 - R_i(t)] \qquad \textbf{(52a)}$$

where $Q_i(t)$ is the failure (cumulative) Pr of the ith component c_i (i = 1, 2, ..., n). If all n parallel units have independent and exponential lifetimes with failure rates λ_i, then $R_i(t)$

$= e^{-\lambda_i t}$ and as a result equation (52a) reduces to

$$\mathbf{R_{Sys}(t) = 1 - \prod_{i=1}^{n}[1 - R_i(t)] = 1 - \prod_{i=1}^{n}[1 - e^{-\lambda_i t}]} \qquad \textbf{(52b)}$$

Further, if all n components also have identical failure rates $\lambda = \lambda_i$ for all i, then (52b) reduces to

$$\mathbf{R_{Sys}(t) = 1 - \prod_{i=1}^{n}[1 - e^{-\lambda t}] = 1 - (1 - e^{-\lambda t})^n = 1 - \sum_{k=0}^{n} {}_nC_k\,(-e^{-\lambda t})^k\,(1)^{n-k}}$$

$$\mathbf{= -\sum_{k=1}^{n} {}_nC_k\,(-e^{-\lambda t})^k = \sum_{k=1}^{n} {}_nC_k\,(-1)^{k+1}(e^{-k\lambda t})} \qquad \textbf{(52c)}$$

<u>Example 9.</u> Consider the 3 circuit breakers of Example 8 that were in series for the purpose of closing the circuit, but now the objective is to stop the current from flowing. Then the system has n = 3 hot spares where at least one breaker must open circuit W/O failure to stop the current flow at time t = 400 hours, where each unit has the same failure rate λ_i = 0.00005/hour = λ. The use of equation (52c) with n = 3 yields $R_{Sys}(t) = 3\,(+1)e^{-\lambda t} + 3\,(-1)\,e^{-2\lambda t} + (1)(1)e^{-3\lambda t} = 3e^{-\lambda t} - 3e^{-2\lambda t} + e^{-3\lambda t}$. Note that this last expression is the same as $e^{-3\lambda t} + {}_3C_2\,e^{-2\lambda t}(1 - e^{-\lambda t}) + {}_3C_1 e^{-\lambda t}(1 - e^{-\lambda t})^2$. Therefore, system RE at 400 hours is given by $R_{Sys}(400) = 3e^{-0.02} - 3e^{-0.04} + e^{-0.06} = 0.99999223604754$.

The HZF for a pure parallel system is obtained by using the fact that $h_{Sys}(t) = f_{Sys}(t)/R_{Sys}(t)$, where $R_{Sys}(t)$ is given in equation (52a). First, we need to obtain the system failure density function by $f_{Sys}(t) = -dR_{Sys}(t)/dt$. Therefore,

$$\mathbf{h_{Sys}(t) = \frac{\frac{d}{dt}\prod_{i=1}^{n}[1 - R_i(t)]}{1 - \prod_{i=1}^{n}[1 - R_i(t)]}} \qquad \textbf{(53a)}$$

If the pure parallel system consists of n identical, independent, and exponential lifetimes each at the failure rate λ, then equation (53a) reduces to

$$h_{Sys}(t) = \frac{\lambda \sum_{k=1}^{n} {}_nC_k (-1)^{k+1} k(e^{-k\lambda t})}{\sum_{k=1}^{n} {}_nC_k (-1)^{k+1} (e^{-k\lambda t})} \qquad \textbf{(53b)}$$

Equation (53b) clearly shows that the HZF of a pure n-unit parallel system is a decreasing function of time (i.e., the failure rate is not a constant), and hence the system TTF is not exponentially distributed (i.e., this is not a Poisson process).

For the Example 9 above, the hazard rate for the parallel system at t = 400 is equal to

$h_{Sys}(400) = \dfrac{0.00005 \sum_{k=1}^{3} {}_nC_k (-1)^{k+1} k(e^{-0.02k})}{0.99999223604754} = 5.764973554342638\times10^{-8}$, while h(1000) = $3.394241026928207\times10^{-7}$ > h(400).

The MTTF of a pure parallel system is obtained by integrating the $R_{Sys}(t)$ from 0 to ∞.

$$\textbf{Thus,} \quad E(T) = MTTF_{Sys} = \int_0^\infty \left[1 - \prod_{i=1}^{n} [1 - R_i(t)]\right] dt \qquad \textbf{(54a)}$$

In the special case of identical exponential lifetimes at the rate λ equation (54a) reduces to

$$E(T) = MTTF_{Sys} = \int_0^\infty \sum_{k=1}^{n} {}_nC_k (-1)^{k+1} (e^{-k\lambda t})\, dt = \sum_{k=1}^{n} [{}_nC_k (-1)^{k+1} \int_0^\infty e^{-k\lambda t}\, dt]$$

$$= \sum_{k=1}^{n} [{}_nC_k (-1)^{k+1} / k\lambda] = \frac{1}{\lambda} \sum_{k=1}^{n} [{}_nC_k (-1)^{k+1} / k] \qquad \textbf{(54b)}$$

Eq. (54b) shows that the MTTF of the pure parallel system of Example 9 is given by E(T) = 3/λ – 3/(2λ) + 1/(3λ) = 36666.66666666667 hours. Moreover, (54b) clearly shows that E(T) is an increasing function of n but a decreasing function of lambda. If we increase λ from 0.00005 to 0.00007 in Example 9, then E(T) reduces to 26190.47619047619, while if we increase n from 3 to 5 at the same λ = 0.00005, then E(T) increases to 45666.66666666667 hours. Finally, equation (54b) can be used to determine the number of units in pure parallel redundancy needed to achieve a desired MTTF = E(T). If we wish to increase the MTTF of the Example 9 from 36666.66666666667 to 70000.0000 hours, then an n of at least 19 units are needed in pure parallel redundancy.

Exercise 14. Consider a pure parallel system of n = 4 units where each unit has the same constant hazard function λ = 0.00005. Obtain the expression for the RE function and compute the value of R(500 hours). (b) Use R(t) from your part (a) to compute the $MTTF_{Sys}$. (c) Compute the value of the HZF at t = 500 hours.

Pure Parallel Nonrepairable Dynamic Models with Weibull TTF

Since the RE function for a Weibull TTF is given by R(t) = $e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$ and in case δ = 0, then the RE function for the i^{th} unit with zero minimum life is $R_i(t)$ = $e^{-(t/\theta_i)^{\beta_i}}$ = $e^{-(\lambda_i t)^{\beta_i}}$, where θ_i is the characteristic life t_c of component c_i and β_i is the Weibull slope of c_i (i = 1, 2, ..., n). Substituting for $R_i(t)$ in equation (52b) results in

$$R_{Sys}(t) = 1 - \prod_{i=1}^{n}\left[1-e^{-(\lambda_i t)^{\beta_i}}\right] \quad \textbf{(55a)}$$

When all n units have identical t_c's and slopes, then equation (55a) reduces to

$$R_{Sys}(t) = 1 - \left[1-e^{-(t/\theta)^{\beta}}\right]^{n} = 1 - \sum_{k=0}^{k=n} {}_{n}C_{k}\,(1)^{n-k}\,(-e^{-(t/\theta)^{\beta}})^{k} =$$

$$= \sum_{k=1}^{k=n} {}_{n}C_{k}\,(-1)^{k+1}\,e^{-k(\lambda t)^{\beta}} \quad \textbf{(55b)}$$

I could not find a closed-form antiderivative for the system RE function in equation (55a) in order to obtain an expression for $MTTF_{Sys}$ = E(T) for the general n, different θ_i's and β_i's. So, it seems that the following integral in (56a) has to be evaluated for specified values of n, θ_i's and β_i's.

$$MTTF_{Sys} = E(T) = \int_{0}^{\infty}\left\{1 - \prod_{i=1}^{n}\left[1-e^{-(t/\theta_i)^{\beta_i}}\right]\right\}dt \quad \textbf{(56a)}$$

However, if we take the special case of $\theta_i = \theta$ and $\beta_i = \beta$ for all i (i.e., identical and ind. Weibull TTFs), then after 20 tedious steps equation (56a) reduces to

$$MTTF_{Sys} = \theta\,\Gamma(1+\frac{1}{\beta})\sum_{k=1}^{k=n}\left[\frac{(-1)^{k+1}\,{}_{n}C_{k}}{k^{1/\beta}}\right]$$

$$= \theta\Gamma(1+\frac{1}{\beta})\sum_{k=1}^{k=n}\left[(-1)^{k+1}\,{}_nC_k\ (1/k)^{1/\beta}\right] \qquad \textbf{(56b)}$$

Example 3.7 on pages 181 of Elsayed(2012). For this example, n = 4, and h(t) = $3.5\times10^{-6}t$ for all four components. Then the RE function of each c_i (i =1, 2, 3, 4) is given by

$R_i(t) = e^{-\int_0^t \lambda x dx} = e^{-\lambda t^2/2} = e^{-(\frac{t\sqrt{\lambda}}{\sqrt{2}})^2}$, where for convenience I have let $3.5\times10^{-6} = \lambda$.

Clearly this is the RE function for a Weibull with $\delta = 0$, $(1/\theta) = \frac{\sqrt{\lambda}}{\sqrt{2}}$, and the slope $\beta = 2$.

Hence, the value of the $t_c = \theta = \frac{\sqrt{2}}{\sqrt{\lambda}} = \frac{\sqrt{2}}{\sqrt{3.5\times10^{-6}}}$ = 755.9289460184544 hours.

Substituting n = 4, θ = 755.9289460184544 hours, and $\beta = 2$ into equation (56b) yields MTTF = 1049.611304905619 hours. This answer does not match that of Elsayed's (970.184 hours) on his page 181?

Example 3.8 on Pages 182 of Elsayed (2012). For this example n = 3, $h_1(t) = \lambda_1 t^{1.5}$, $h_2(t) = \lambda_2 t^{1.5}$ and $h_3(t) = \lambda_3 t^{1.5}$, where $\lambda_1 = 0.25\times10^{-6}$, $\lambda_2 = 0.20\times10^{-6}$, and $\lambda_3 = 0.24390244\times10^{-6}$. Therefore, the three RE functions are $R_1(t) = e^{-(\lambda_1/2.5)t^{2.5}}$, $R_2(t) = e^{-(\lambda_2/2.5)t^{2.5}}$, and $R_3(t) = e^{-(\lambda_3/2.5)t^{2.5}}$, and θ_1 = 630.9573444802, θ_2 = 689.864830731, and θ_3 = 637.22021701 hours. From (55a) $R_{Sys}(t) = 1 - Q_1(t)\times Q_2(t)\times Q_3(t)$; thus,

$$\mathbf{R_{Sys}(t) = 1 - (1 - e^{-(t/\theta_1)^{2.5}})(1 - e^{-(t/\theta_2)^{2.5}})(1 - e^{-(t/\theta_3)^{2.5}})} \qquad \textbf{(57a)}$$

Note that the all 3 units in (57a) have Weibull TTFs with $\beta_i = \beta = 2.5$, and t_c values given above $\rightarrow$ $MTTF_1$ = 559.8256221451 hours, $MTTF_2$ = 612.0921032752 and $MTTF_3$ = 5.6538244233402hours. In order to evaluate the system MTTF by integrating the RE function in (57a), we must first expand the last term on the RHS so that the integration can be carried out. This leads to

$$\mathbf{R_{Sys}(t) = R_1(t) + R_2(t) + R_3(t) - R_1(t)\times R_2(t) - R_1(t)\times R_3(t) - R_2(t)\times R_3(t) + R_1(t)\times R_2(t)\times R_3(t)} \qquad \textbf{(57b)}$$

Equation (57b) has 7 terms, each of which is a Weibull RE function, that have to be integrated from 0 to ∞ one-by-one in order to compute the MTTF. I will illustrate the integration for $R_1(t)$ = $e^{-(\lambda_1/2.5)t^{2.5}}$ and you may verify the other 6 in (57b).

In the integral $\int_0^\infty e^{-(\lambda_1/2.5)t^{2.5}}dt$, make the change of variable, x = $(\lambda_1/2.5)t^{2.5}$ $\rightarrow$ dx =

$(\lambda_1)t^{1.5}$dt . Making the substitutions dt = $\frac{dx}{\lambda_1 t^{1.5}}$, t = $(\frac{2.5x}{\lambda_1})^{1/2.5}$ = $(\frac{2.5x}{\lambda_1})^{0.40}$ and

$t^{1.5}$ = $(\frac{2.5x}{\lambda_1})^{0.60}$ into the last integral, we obtain $\int_0^\infty e^{-x}\frac{dx}{\lambda_1(2.5x/\lambda_1)^{0.60}}$ =

$$\frac{1}{\lambda_1}\int_0^\infty e^{-x}(2.5x/\lambda_1)^{-0.60}dx = \frac{1}{\lambda_1}\int_0^\infty e^{-x}(\lambda_1/2.5)^{0.60}(x^{-0.60})dx = \frac{1}{\lambda_1^{0.40}(2.5)^{0.60}}$$

$$\int_0^\infty e^{-x}(x^{0.40-1})dx = \frac{\Gamma(0.40)}{(\lambda_1)^{0.40}(2.5)^{0.60}} = \frac{0.40\,\Gamma(0.40)}{(\lambda_1)^{0.40}(2.5)^{-0.40}} = \frac{\Gamma(1+0.40)}{(0.40\lambda_1)^{0.40}} = 559.8256221451$$

hours. Similar calculations yield $MTTF_{Sys}$ = 795.60317214 hours, which matches Elsayed's answer of 795.62 hours.

The k-Out-Of-n Non-reparable Parallel Systems (k < n)

As in the previous two cases, it is assumed that all n units fail independently of each other, but the system is reliable iff at least 1 < k < n units operate successfully during the mission interval (0, t). The RE expression when the n units are not identical is not simple, and each case has to be obtained for the specific process parameters n, k, $R_i(t)$ where at least two $R_i(t)$'s are different. As A. E. Elsayed points out on pages 157-158, for most such systems the n independent units have identical HZF $h_i(t)$

= h(t) for all i = 1, 2, ..., n, and thus $R_i(t)$ = $e^{-\int_0^t h(x)dx}$ = $e^{-H(t)}$ for all i. In case all n units also possess exponential lifetimes, then $R_i(t)$ = $e^{-\lambda t}$, where each unit has the same constant failure rate λ. In this latter exponential case, the RE expression is given by

$$\mathbf{R_{Sys}(t) = R(k;\, n,\, e^{-\lambda t}) =} \sum_{r=k}^{n} {}_nC_r\,(e^{-\lambda t})^r\,(1-e^{-\lambda t})^{n-r} = \sum_{r=k}^{n} {}_nC_r\,e^{-r\lambda t}\,(1-e^{-\lambda t})^{n-r}$$

(58a)

The failure density function for a k-out-of-n Parallel System, with identical exponential lifetimes, is obtained from $f_{Sys}(t) = -dR_{Sys}/dt$, where $R_{Sys}(t)$ is given in (58a).

$$\mathbf{f_{Sys}(t)} = \lambda \sum_{r=k}^{n} \left[{}_{n}C_{r}\, e^{-r\lambda t} (1-e^{-\lambda t})^{n-r-1} (r - n e^{-\lambda t}) \right] \qquad \textbf{(58b)}$$

The corresponding HZF is given by $h_{Sys}(t) = f_{Sys}(t)/R_{Sys}(t)$, where $f_{Sys}(t)$ is given in (58b) and $R_{Sys}(t)$ is computed from (58a).

<u>Examples 3.4&3.5 on Page 177 of Elsayed (2012).</u> For this example n = 3, k = 2, $\lambda_i = \lambda = 0.00003$ and t = 1000 hours. Substitution into Equation (58a) yields $R_{Sys}(1000) = 0.99743123021029$, which is identical to Elsayed's answer to 4 decimals. The value of hazard rate by Matlab computations is $f_{Sys}(1000)/R_{Sys}(1000) = 5.022905383950688\times10^{-6}$. For Elsayed's Example 3.5, let us change system parameter requirements to a more real-life situation. Keeping lambda at 0.00004/hour, we wish to determine the number of parallel units, n, for a 2-out-of-n system such that $R_{Sys}(2000) \geq 0.99950$. Using equation (58a), Matlab computations yields $R(2;4, e^{-0.08}) = 0.99828695648591$, while $R(2;5, e^{-0.08}) = 0.99983604029385$; thus, n = 5 identical units are needed to guarantee a RE of at least 0.9995 for a mission of duration 2000 hours.

To obtain the MTTF for a k-out-of-n system with n identical exponential lifetimes, we must integrate the $R_{Sys}(t)$ in equation (58a) from 0 to ∞.

$$\mathbf{MTTF_{Sys}} = \int_0^\infty \sum_{r=k}^{n} {}_{n}C_{r} (e^{-r\lambda t})(1-e^{-\lambda t})^{n-r}\, dt = \sum_{r=k}^{n} \left[{}_{n}C_{r} \int_0^\infty e^{-r\lambda t}(1-e^{-\lambda t})^{n-r}\, dt \right] \qquad \textbf{(59a)}$$

The integral in equation (59a) inside the brackets is tedious to compute for general n, but I made two transformations in (59a) in order to obtain the integration result; the change of variables are $x = \lambda t$ followed by $n - r = j$. After 32 careful steps, equation (59a) integrates to

$$\mathbf{MTTF_{Sys}} = \frac{1}{\lambda} \sum_{j=0}^{n-k} \left[{}_{n}C_{n-j} \sum_{r=0}^{j} [{}_{j}C_{r} (-1)^{r} / (n + r - j)] \right] \qquad \textbf{(59b)}$$

<u>Example 3.9 on Page 183 of Elsayed(2012).</u>

For this example, n = 4, k = 2, and $\lambda = 8.5\times10^{-8}$ failures/hour. Inserting these values into (59b), Matlab computations yield $MTTF_{Sys}(2;4, 8.5\times10^{-8}) = 127450.9804$ hours, which is consistent to Elsayed's answer to 5 significant figures.

The RE function for a k-Out-Of-n Nonrepairable Weibull System

As before, we assume that the failures of the n units are independent and all units possess a minimum life $\delta = 0$, $t_c = \theta = 1/\lambda$, and the same slope β. Then each RE function $R_i(t) = e^{-(t/\theta)^\beta} = e^{-(\lambda t)^\beta}$ for all i. Then equation (58a) becomes

$R_{Sys}(t) = R(k; n, e^{-(\lambda t)^\beta}) = \sum_{r=k}^{n}\left\{ {}_nC_r [e^{-(\lambda t)^\beta}]^r [1 - e^{-(\lambda t)^\beta}]^{n-r}\right\} =$

$$= \sum_{r=k}^{n} {}_nC_r \, e^{-r(\lambda t)^\beta} [1 - e^{-(\lambda t)^\beta}]^{n-r}, \qquad \textbf{(60)}$$

where only for simplicity we have let $1/\theta = \lambda$ in equation (60). The MTTF is obtained from

$$\mathbf{MTTF_{Sys}} = \int_0^\infty \sum_{r=k}^{n} {}_nC_r \, e^{-r(\lambda t)^\beta} [1 - e^{-(\lambda t)^\beta}]^{n-r} dt = \sum_{r=k}^{r=n}\left[{}_nC_r \int_0^\infty e^{-r(\lambda t)^\beta} [1 - e^{-(\lambda t)^\beta}]^{n-r} dt \right]$$

(61a)

This last integral inside brackets is not easy to carry out for general n and k. It took me 5 pages of tedious calculations to carry out the integral in (61a). I first made the transformation $(\lambda t)^\beta = x$. Then I used the fact that $(1 - e^{-x})^{n-r} = \sum_{j=0}^{n-r} {}_{n-r}C_j (-e^{-x})^j$ and made a 2nd transformation $u = (r + j)x$ to obtain the following result:

$$\mathbf{MTTF_{Sys}} = \frac{\Gamma(1 + 1/\beta)}{\lambda} \sum_{r=k}^{r=n}\left[{}_nC_r \sum_{j=0}^{n-r} \frac{(-1)^j \; {}_{n-r}C_j}{(r + j)^{1/\beta}} \right] \qquad \textbf{(61b)}$$

<u>Example 3.10 on page 184 of Elsayed(2012).</u> From the problem statement n = 4, k = 2, and $h(t) = 2.7\times10^{-4}t$. The RE function for each unit is $R_i(t) = e^{-\int_0^t 2.7\times10^{-4} x\,dx} = e^{-1.35\times10^{-4} t^2}$ for all i = 1, 2, 3, 4, or $R(t) = e^{-(\sqrt{1.35}\times10^{-2} t)^2}$. Therefore, the TTF of each unit has a W(0, 100/$\sqrt{1.35}$, 2) distribution so that $\lambda = \sqrt{1.35}/100$. Inserting n = 4, k = 2, $\lambda = \sqrt{1.35}/100$, and $\beta = 2$ into my equation (61b), Matlab computations result in MTTF =

85.71996308005328 hours. This answer is smaller that of Elsayed's by a factor of 1000. The system RE given by A. E. Elsayed is $R_{Sys}(t) = 6\mathbf{e}^{-k t^2} - 8\mathbf{e}^{-1.5k t^2} + 3\mathbf{e}^{-2k t^2}$ and $MTTF_{Sys} = \int_0^\infty (6e^{-k t^2} - 8e^{-1.5k t^2} + 3e^{-2k t^2})dt = \sqrt{\pi}\ (6/\sqrt{4k} - 8/\sqrt{6k} + 3/\sqrt{8k})$ are indeed correct but there seems some glitch in the final answer. I also used Matlab to compute the expression in the middle of page 184 of Elsayed (2012), and it also yielded the value of $MTTF_{Sys} = \sqrt{\pi}\ (6/\sqrt{4k} - 8/\sqrt{6k} + 3/\sqrt{8k})$ = 85.71996308005338, which is identical to the value from equation (61b).

I surmise that by now you have gotten the message about how to compute the MTTF of a time-dependent non-repairable system. The 1st key step is to obtain the RE function, $R_{Sys}(t)$, and then integrate the system RE function always from zero to ∞, no matter what the value of minimum life $t_0 = \delta \ge 0$ is.

Exercise 15. Consider a nonrepairable parallel redundant system of n = 4 components each with a constant failure rate of λ = 0.00005/hour. The system is reliable only if at least 2 units function W/O failure during the mission time of t = 400 hours. Obtain the general form of the reliability function $R_{Sys}(t)$ and then compute $R_{Sys}(t)$ at t = 400. (b) Compute the hazard function rate at t = 400 hours, and the MTTF. (a) 0.9999694054. (b) $2.260379378172426\times10^{-7}$, MTTF = 21666.666666667.

Repairable Systems

So far we have dealt with system RE W/O repair and maintenance, and therefore, system availability at time t, A(t), was simply the same as system RE at time t, R(t). Examples of nonrepairable components are light bulbs, resistors, computer chips and batteries, while complex systems such as airplanes, cars and air conditioning systems have many repairable components. If a system is repairable, then there are two important performance criteria: MTBF (Mean Time Between Failures), and steady-state (or long-term) availability. Unfortunately, a detailed discussion of repairable systems will have to wait until Chapter 9 of Ebeling because such systems (where Time to Repair, TTR, is generally much smaller than TTF and as a result MTTR << MTBF) require a thorough knowledge of Markov Chains and Renewal theory, which I will cover once we arrive at Chapter 6. By renewal we mean that a system fails but upon failure the failed component is either replaced with a brand new unit or is repaired to its original condition. This is called a renewal process. A renewal process is the generalization of a Poisson process where the interarrival (or intervening) times between two successive events (failures) can have any pdf instead of just the exponential. Thus, a Poisson process is the simplest renewal process because its renewal density function is a constant λ, and as a result the renewal function $E[N_f(t)]$ = Expected-value of number of Poisson events = λt.

If a failed component is immediately replaced with a new one (i.e., if its Mean-Time-to-Repair, MTTR, is negligible compared to MTBF), then the long-term availability of the system is almost 100%, and its point availability at time t, A(t), is equal to $R_{Sys}(t)$. Otherwise, if the TTR has a specified distribution, such as exponential with repair rate λ_r (or r) then the steady-state (i.e., as $t \to \infty$) inherent availability will be shown to equal the expression given below.

$$A_I = \frac{\text{MTBF}}{\text{MTBF}+\text{MTTR}} \qquad \textbf{(62a)}$$

If the system failure rate is also a constant λ, then (62a) reduces to

$$A_I = \frac{1/\lambda}{1/\lambda + 1/\lambda_r} = \frac{\lambda_r}{\lambda_r + \lambda}. \qquad \textbf{(62b)}$$

Example 3.13 of Elsayed(2012) on his pages 189-190. The TTF is Weibull with $\delta = 0$, $t_c = 5\times10^6$ hours and slope $\beta = 2.15$, and the repair time is exponential at the rate $\lambda_r = 10^{-4}$ per hour. Therefore, the failure density function is given by $f(t) = \frac{2.15}{5\times10^6}(\frac{t}{\theta})^{1.15} \times e^{-(t/\theta)^{2.15}}$, where $\theta = 5\times10^6$ hours, and the TTR has the pdf $g(t) = \lambda_r e^{-\lambda_r t}$, where $\lambda_r = 10^{-4}$. Using equation (7) of chapter 2, the MTBF = $\theta\,\Gamma(1 + \frac{1}{\beta}) = 5\times10^6\Gamma(1 + \frac{1}{2.15}) = 4.428041951947659\times10^6$, and MTTR = $1/10^{-4}$ = 10,000 hours. Substituting these mean times into equation (62a) gives $A_I = 0.99774675406220$.

Before we discuss standby redundant systems, we must alert you to the fact that in RE engineering the repair time may consist of several phases: (1) Detection and Diagnosis, (2) Delay in obtaining parts, (3) The actual active repair itself, (4) Testing time at the repair facility before returning the item to service. Therefore, the repair rate λ_r may be an ensemble of the all the aforementioned phases which are also referred to as administrative and logistic times.

Standby Redundant Systems (Cold or Inactive Spares)

For example, a dc power supply generator, a sensing switch and 3 batteries (in standby) would form a 4-unit standby system. The three batteries are called cold (or inactive) spares, and it is generally assumed that their idle (or quiescent, de-energized, or off-line) failure rates are almost zero. Figure 4 depicts the 4-unit standby system. Only unit 1 is on-line (or energized) at time 0, while units 2, 3 and 4 are in idle standby (i.e., cold spares). First, we consider the case of perfect switching where $R_{sw} = 1$. We first consider the simpler case of four identical units, where each unit for our standby system has $\lambda_i = \lambda = 0.001$/hour and we wish to compute $R_{Sys}(t)$, specifically $R_{Sys}(t)$ at t = 500 hours, as depicted

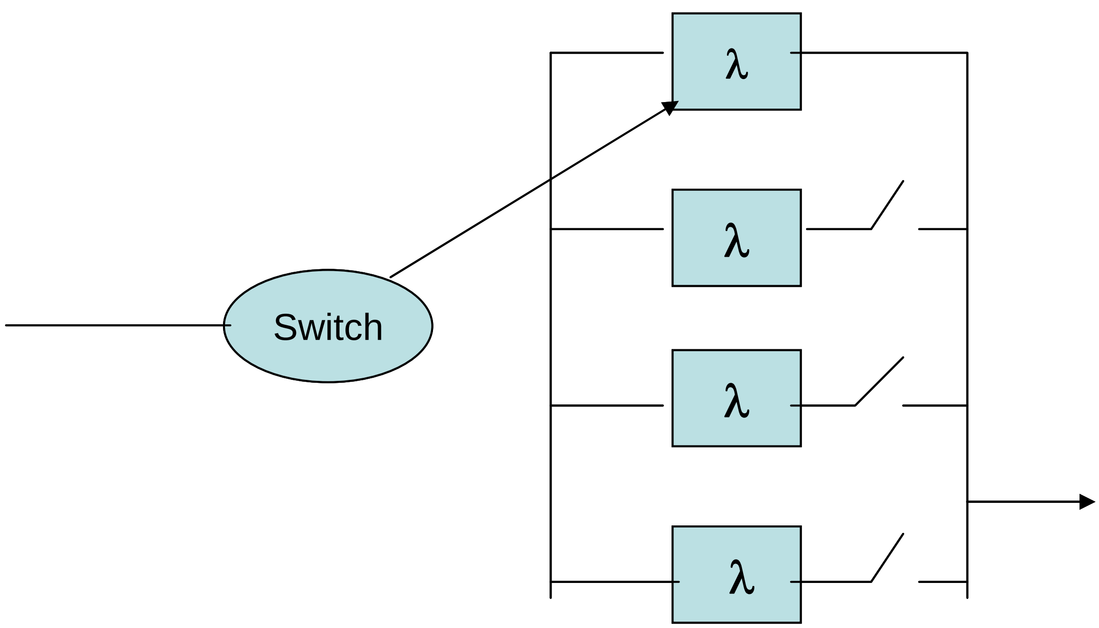


Figure 4. A 4-unit Standby System with identical failure rates

in Figure 4. Clearly the system lifetime, $T_{Sys} = T_4$, is given by

$$\mathbf{T_{Sys} = T_4 = TTF_1 + TTF_2 + TTF_3 + TTF_4} \quad \mathbf{(63)}$$

where TTF_i = the time to failure of unit i (i =1, 2, 3, 4). In equation (63) the notation T_4 means the Sum of 4 independent rvs. Note that for a 4-unit pure parallel system equation (63) does not hold because in a parallel system all 4 units are simultaneously energized at time zero; further, for a standby system equation (63) is a good approximation only because we are assuming that the quiescent failure rate λ_d (d for de-energized) of a standby unit is far much smaller than its energized failure rate. For the three batteries in standby, for example, the value of off-line failure rate, λ_d, could be as small as 10^{-7} per hour. Note that Ebeling on pp. 130-132 performs a Markov-analysis of a 2-unit Standby system, where λ_d is not small enough to be ignored. Since each TTF_i has an exponential density given by $f_i(t)$ = $0.001\mathbf{e}^{-0.001t}$, then the Pr density of T_4 is the 4-fold convolution of $0.001\mathbf{e}^{-0.001t}$ with itself, i.e., the pdf of T_4 is given by $g_{T_4}(t) = f_1(t)^* f_2(t)^* f_3(t) * f_4(t)$, where * denotes convolution. I showed in chapter 2 that this 4-fold convolution is simply a gamma pdf with parameters n = 4 and λ = 0.001, i.e., the pdf of system lifetime $T_{Sys} = T_4$ is given by

$$g_{T_4}(t) = \frac{\lambda}{3!}(\lambda t)^3 e^{-\lambda t}, \qquad 0 \le t < \infty$$

Thus, R_{Sys} (500 hours) = P(T_{Sys} > 500) = $\int_{500}^{\infty} \frac{\lambda}{6}(\lambda t)^3 e^{-\lambda t} dt$. Making the transformation x = λt in this last integral yields $R_{Sys}(500) = \int_{0.50}^{\infty} \frac{x^3}{6} e^{-x} dx$. After three integrations by part, we obtain $R_{Sys}(500) = \int_{0.50}^{\infty} \frac{x^3}{6} e^{-x} dx = (0.50^3/6)e^{-0.5} + 0.125\, e^{-0.5} + 0.5e^{-0.5} + e^{-0.5}$ = 0.9982484. Note that this result could also have been obtained by noting that R(500) = P(T_4 > 500 hours) = P[X(500 hours) $\le$ 3 failures] = $\sum_{k=0}^{3} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \sum_{k=0}^{3} \frac{(0.50)^k}{k!} e^{-0.50}$ = 0.9982484, where we have made use of the relationship between the Gamma density and the Poisson pmf, and the rv X(500) represents the number of failures occurring in 500 hours. Further, in this example, if the sensor switch had a constant RE of 0.999 during the interval [0, 500 hours], then system RE would reduce to R_{Sys} (500) = 0.9982484$\times$0.999 = 0.99725013.

Exercise 16. Repeat the above example for n = 3 and R_{SW} = 0.999. ANS: R_{Sys} (500) = 0.999×P(T_3 > 500 hours) = 0.984627.

Now consider an example of a 3-unit standby system where $\lambda_1 = 0.0005$, $\lambda_2 = 0.001$ and $\lambda_3 = 0.001$ so that λ_i's are not equal. Then $TTF_{Sys} = TTF_1 + TTF_2 + TTF_3$ in equation (63) has no longer a gamma pdf with n = 3 and constant FR, and we have to resort to either to the following procedure, or to a Markov-analysis in Chapter 6, in order to compute the system RE for a given t. However, the $MTTF_{Sys} = E(TTF_1 + TTF_2 + TTF_3) = E(TTF_1) + E(TTF_2) + E(TTF_3) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}$ = 4000 hours. Letting λ_{Sys} represent the system's effective failure rate, then λ_{Sys} = 1/4000 = 0.00025 per hour. Note that this does not mean that the system failure intensity (or rate) is a constant! This is due to the fact that $h_{Sys}(t) = f_{Sys}(t)/R_S(t)$.

<u>Mode 1.</u> Unit 1 is put on-line at t = 0 and is reliable for the duration of mission time t = 500 hours, or during the interval [0, 500 hours]. Then

$$\mathbf{R_{Sys}^{(1)}(500) = e^{-\lambda_1 t} = e^{-0.25} = 0.778801.}$$

<u>Mode 2.</u> Unit 1 fails at time t_1 < 500, the switch (assumed to have R_{sw} = 1) works at t_1, and unit 2 is reliable for the duration of 500 – t_1. Hence

$$\mathbf{R_{Sys}^{(2)}(500) = \int_{t_1=0}^{500} \lambda_1 e^{-\lambda_1 t_1} dt_1 R_2(500 - t_1) = e^{-0.50} \int_{t_1=0}^{500} 0.0005 e^{\lambda_1 t_1} dt_1 =}$$

$$\mathbf{e^{-0.50}\left[e^{0.0005 t_1}\right]_0^{500} \rightarrow \qquad R_{Sys}^{(2)}(500) = 0.172270.}$$

Note that $f_1(t_1)dt_1 = \lambda_1 e^{-\lambda_1 t_1} dt_1$ gives the mortality Pr element for unit 1 during the interval (t_1, $t_1 + dt_1$).

<u>Mode 3.</u> Unit 1 fails at t_1, unit 2 fails at t_2 ($0 < t_1 < t_2 < 500$), and unit 3 is reliable from t_2 to 500 hours. Note that all times are measured from 0, as depicted in Figure 5.

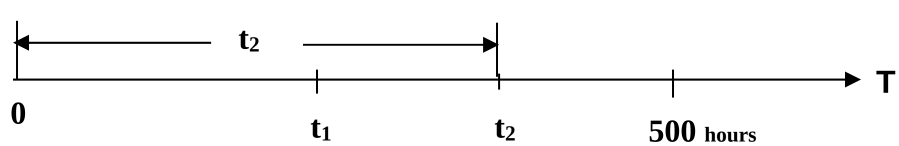


Figure 5.

Recall that $\lambda_1 = 0.0005$, $\lambda_2 = \lambda_3 = 0.001$. Then

$$R_{Sys}^{(3)}(500) = \int_{t_2=0}^{500}\int_{t_1=0}^{t_2} f_1(t_1)\,dt_1\, f_2(t_2-t_1)\,dt_2\, R_3(500-t_2)$$

$$= \int_{t_2=0}^{500}\int_{t_1=0}^{t_2} \lambda_1\lambda_2\, e^{-0.50}\, e^{0.0005t_1}\, dt_1\, dt_2$$

In this last integral integration wrt t_1 must be carried out first followed by t_2. Carrying out the double integral results in $R_{Sys}^{(3)}(500)$ = 0.041274921. Since the above three modes are mutually exclusive, then $R_{Sys}(500)$ = $R_{Sys}^{(1)}(500)$ + $R_{Sys}^{(2)}(500)$ + $R_{Sys}^{(3)}(500)$ = 0.992346. This system RE is a bit larger than a 3-unit standby RE of 0.985612 = $\sum_{k=0}^{2}\frac{(0.50)^k}{k!}e^{-0.50}$, where all 3 failure rates are equal to λ = 0.001; this is due to fact that for this case λ_1 = 0.0005 is equal to half of 0.001. Note that if the quiescent failure rates of the 2 standby units were not close to zero, such as both λ_D = 0.000002 per hour or less, then Mode 2 RE changes as follows:

$$R_{Sys}^{(2)}(500) = \int_{t_1=0}^{500} \lambda_1 e^{-\lambda_1 t_1}dt_1 e^{-\lambda_D t_1} R_2(500-t_1) + \int_{t_1=0}^{500} \lambda_1 e^{-\lambda_1 t_1}dt_1 (1-e^{-\lambda_D t_1})e^{-\lambda_D t_1} R_3(500-t_1),$$

and mode 3 changes to $R_{Sys}^{(3)}(500) = \int_{t_2=0}^{500}\int_{t_1=0}^{t_2} f_1(t_1)dt_1 f_2(t_2-t_1)dt_2 e^{-\lambda_D t_2} R_3(500-t_2)$. At the present, I am not certain about the validity of these last 3 equations.

Exercise 17. (a) Verify the answer $R_{Sys}(500)$ = 0.985612 for the 3-unit standby where each λ_i = 0.001 using the 3 possible modes of success. (b) For mode 3 reliability above, $R_{Sys}^{(3)}(500)$ = 0.04127492, exchange the order of integration and recompute $R_{Sys}^{(3)}(500)$, where integration wrt t_2 is carried out first followed by t_1.

Imperfect Switching

When it is possible for the switch to fail during the mission of length t, which is more realistic, it is generally assumed that it has a constant failure rate λ_{sw} so that its RE function is equal to $e^{-\lambda_{sw}t}$. For example, consider the 3-unit standby in the last example but assume

$R_{sw} < 1$ and $\lambda_{sw} = 0.00001$/hour. Further, $\lambda_1 = 0.0005$, $\lambda_2 = \lambda_3 = 0.001$. How does the imperfect switching affect the values of system REs for modes 2 and 3?

$$\mathbf{R_{Sys}^{(2)}(500) = \int_{t_1=0}^{500} \lambda_1 e^{-\lambda_1 t_1} dt_1 R_{sw}(t_1) R_2(500 - t_1) =}$$

$$\mathbf{= 0.0005\ e^{-0.50} \int_0^{500} e^{0.00049 t_1}\, dt_1 = 0.1718223}$$

which is a bit smaller than $\mathbf{R_{Sys}^{(2)}}(500) = 0.172270$ for the case of perfect switching as expected!

Exercise 18. Compute $\mathbf{R_{Sys}^{(3)}}(500)$ for mode 3 when the switch is imperfect with $R_{sw}(t) = \mathbf{e^{-\lambda_{sw} t}} = \mathbf{e^{-0.00001 t}}$ (ANS < 0.041275). Then compute the overall system RE at t = 500 hours. ANS: $R_{Sys}(500) = 0.9917592063$.

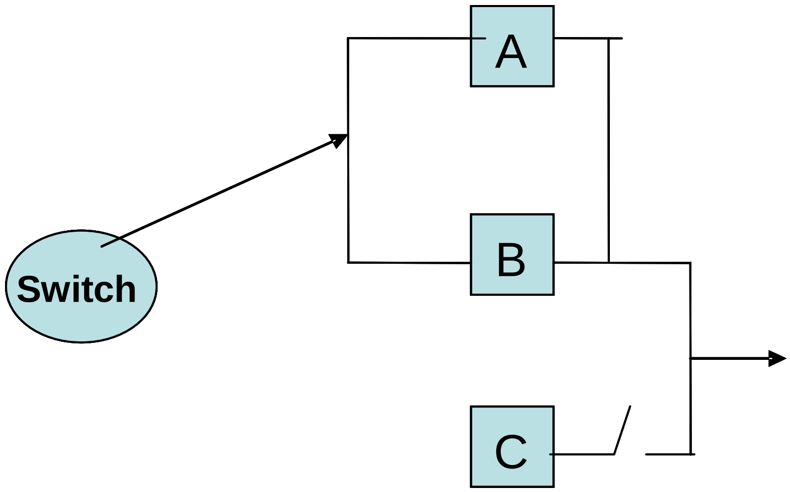
Mixed Parallel and Standby Systems (Hot and Cold Spares)

As an example, consider the following system where units A and B are in pure parallel redundancy (i.e., both energized at t = 0) and unit C is in cold standby (i.e., idle at t = 0). For convenience, let $\lambda = \lambda_A = \lambda_B = \lambda_C = 0.0001$. Mission time t = 1000 hours. We 1st assume that at least one reliable unit is needed for mission success.

Mode 1. At least one of the 2 units, A or B, is reliable for 1000 hours.

$\mathbf{R_{Sys}^{(1)}}(1000) = R_A + R_B - R_A R_B = e^{-0.10} + e^{-0.10} - e^{-0.20} = 0.99094408299394$.

Parameter values are $\lambda_A = \lambda_B = \lambda_C = 0.0001$/hr, and $\lambda_{sw} = 0.00005$/hr.



Mode 2. Unit A fails at $t_1 < 1000$, B is reliable by t_1 but fails at t_2 (or vice versa), switch is reliable at t_2 and C is reliable from t_2 to 1000 hours.

$$\mathbf{R_{Sys}^{(2)}}(1000) = 2\int_{t_2=0}^{1000}\int_{t_1=0}^{t_2}\mathbf{f_A(t_1)dt_1\,R_B(t_1)f_B(t_2-t_1)dt_2\,R_{sw}(t_2)R_C(1000-t_2)} =$$

Note that $R_B(t_1)f_B(t_2 - t_1)dt_2 = f_B(t_2)dt_2 = \lambda\,\mathbf{e}^{-\lambda t_2}\,dt_2$. Hence, the above double integral reduces

to $\mathbf{R_{Sys}^{(2)}}(1000) = 2\int_{t_2=0}^{1000}\int_{t_1=0}^{t_2}\lambda\mathbf{e}^{-\lambda t_1}\,\mathbf{dt_1}\,\lambda\mathbf{e}^{-\lambda t_2}\,\mathbf{dt_2}\,\mathbf{e}^{-\lambda_{sw} t_2}\,\mathbf{e}^{-\lambda(1000-t_2)} =$

$2e^{-0.10}\int_{t_2=0}^{1000}[\int_{t_1=0}^{t_2}\lambda^2\,\mathbf{e}^{-\lambda t_1}\,\mathbf{e}^{-0.00005 t_2}\mathbf{dt_1}]\mathbf{dt_2} = 2\lambda e^{-0.10}$ (46.7980195) = 0.008468919824. Thus,

$R_{Sys}(1000) = \mathbf{R_{Sys}^{(1)}}(1000) + \mathbf{R_{Sys}^{(2)}}(1000) = 0.99094408299394 + 0.008468919824 \rightarrow$

$R_{Sys}(1000) = 0.9994130028$.

Secondly, suppose at least 2 reliable units are needed for mission success, i.e., we now simply have a 2-unit series system with one cold spare in standby redundancy. Then

Mode 1. Both A and B work W/O failure for 1000 hours.

$$\mathbf{R_{Sys}^{(1)}(1000) = e^{-2\lambda t}} = e^{-0.20} = 0.8187307531$$

Mode 2. Either A or B fails at t_1, the other is reliable for 1000 hours, and C is reliable for $1000 - t_1$, while the switch is also reliable at t_1.

$$R_{Sys}^{(2)}(1000) = 2\int_{t_1=0}^{1000} \lambda e^{-\lambda t_1}\, dt_1\, e^{-\lambda t}\, R_{sw}(t_1) R_C(1000 - t_1) =$$

$$= 2\lambda e^{-0.20}\int_0^{1000} e^{-0.00005 t_1}\, dt_1 = 0.15971988002631$$

$\rightarrow$ $R_{Sys}(1000) = R_{Sys}^{(1)}(1000) + R_{Sys}^{(2)}(1000) = 0.97845063313.$

Exercise 19. Consider a mixed system where units A, B, C are in parallel redundancy with $\lambda_A = \lambda_B = \lambda_C = 0.0003$/hour and unit D is in standby redundancy with $\lambda_D = 0.0003$/hr. The switch has a constant failure rate of $\lambda_{sw} = 0.00001$/hr. The switch puts the parallel system on-line at t = 0. Compute the system RE at t = 800 hours if at least 2 reliable units are needed for mission success. For convenience, let $\lambda = 0.0003$/hour for all 4 units.

Shared-Load Parallel Redundancy

Consider 2 components, A and B, in pure parallel redundancy (hot spares). When both units are reliable, their failure rates are at half-load and equal to $\lambda_h = 0.00007$/hr, but ASA one of them fails, the other failure rate increases to full-load at $\lambda_f = 0.00012$/hr. What is this system's reliability for a mission of t = 1000 hours if at least one operational unit is needed for mission success?

Mode 1. Both A and B are reliable at half loads.

$$R_{Sys}^{(1)}(1000) = (e^{-\lambda_h t})^2 = e^{-2\lambda_h t} = e^{-0.14} = 0.86935823539881.$$

Mode 2. Either A or B fails at half load at time t_1 and the other is reliable at half load at t_1 and then is reliable at full load from t_1 to 1000 hours.

$$R_{Sys}^{(2)}(1000) = 2\int_{t_1=0}^{1000} \lambda_h e^{-\lambda_h t_1}\, dt_1\, R_h(t_1) R_f(1000 - t_1) =$$

$$= 2\lambda_h e^{-0.12}\int_{t_1=0}^{1000} e^{-0.00002 t_1}\, dt_1 = 2\lambda_h e^{-0.12}(990.0663346622349) = 0.12293540922846 \rightarrow$$

$R_{Sys}(1000) = 0.99229364462727.$

Exercise 20. Compute $R_{Sys}(1000)$ hours for the system below, where at least one reliable unit is needed for mission success.

The parameter values are $\lambda_h(A) = \lambda_h(B) = 0.00008$/hr, and $\lambda_f(A) = \lambda_f(B) = \lambda_f(C) =$ 0.00014/hour. Mission time = t =1000 hours and $R_{SW} = 0.9990$. The switch is needed only to put unit C on-line. Note that a pure parallel system with 3 units having the same parameters has a RE of 0.9999651 at $\lambda = 0.00008$ and a RE of 0.9977703 at $\lambda = 0.00014$.

ANS: $R_{Sys} = 0.999521912$.