The subscript 8 in $L_8(2^7)$ implies that the design matrix has N_f = 8 distinct rows (or distinct FLCs) and thus provides 7 (= 8 – 1) df for studying 7 orthogonal effects, implying that the design matrix can have a maximum of 7 orthogonal columns. Further, since 8 = 2^3, exactly 3 arbitrary columns can be written but the remaining 4 columns must be obtained from the 3 arbitrary columns. We first display the 3 arbitrary columns in Table 7 (using the base-2 elements 0 for low level, 1 for the high level, and later on converting to Taguchi's notation of 1 and 2; (see p. 55 of my Manual).

Table 7. The three arbitrary columns of the Taguchi's L_8 OA in base-2 notation

(1)	(2)	(4)
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

The reader should bear in mind that for Taguchi's OAs, we are using the notation that the numbers inside the parentheses ( ) generally imply columns, i.e., (1) means column 1, (2) means column 2, etc. To generate (3), we simply add columns 1 and 2 (modulus 2), i.e., (3) = (1) + (2), mod 2. To generate column 5 we simply add (1) and (4) mod 2, i.e., (5) = (1) + (4), mod 2; similarly, (6) = (2) + (4), and column (7) = (1) + (2) + (4) mod 2. The entire design matrix using this procedure is provided in Table 7(a). Note that in Table 7(a) it will be totally impossible to generate another column (i.e., an eighth distinct column) which is orthogonal to the other 7 columns of L_8 because the matrix has only 7 df and each column (because of 2 levels) carries exactly one df. To convert Table7(a) to Taguchi's format, we simply place the 3rd arbitrary column in column (4) and the interaction (1)+(2) column in column 3 as shown in Table 7(b). Table 7(b) now shows that the interaction of columns 1 and 2 is column 3, i.e., if factor A is assigned to (1) and B is assigned to column 2, then their interaction $A\times B$ must be

Table 7(a). The seven orthogonal columns of an L_8 OA

(1)	(2)	(4)	(1)+(2)	(1)+(4)	(2)+(4)	(1)+(2)+(4)
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	1	0	0	0	1

assigned to (3) because the contrast function of A$\times$B interaction is $\xi(AB) = x_1 + x_2$ = (1) + (2), mod 2. Similarly, if factor C is assigned to (4), then AC interaction must be assigned to column 5 (if the experimenter desires to study AC interaction). Further, the BC interaction must be assigned to assigned to (6) because (2)+ (4) = (6), mod 2, and ABC interaction must be assigned to (7) because (1) + (2) + (4) = (7), mod 2.

Table 7(b). The seven orthogonal columns of the Taguchi's L_8 OA in base-2 notation

(1)	(2)	(3)	(4)	(1)+(4)	(2)+(4)	(1)+(2)+(4)
0	0	0	0	0	0	0
0	0	0	1	1	1	1
0	1	1	0	0	1	1
0	1	1	1	1	0	0
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1

We next convert to Taguchi's notation by transforming 0 $\rightarrow$ 1 and 1$\rightarrow$ 2 as displayed in Table 7(c). Table 7(c) is identical to the Taguchi's OA on page 55 of the Manual.

So far we have discussed how to construct the Taguchi L_8 OA for a full 2^3 factorial. The next step is how do we construct the 2^{4-1} FFD using Taguchi's L_8 OA. Here we have 4 factors A, B, C, and D that will occupy 4 out of the 7 columns. Although not necessary, it is usually best to assign the main factors to the 3 arbitrary columns, which are (1), (2), and (4). Because the FFD 2^{4-1} has only p =1 generator, it

Table 7(c). The seven columns of the G. Taguchi's L_8 OA in his notation

(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1	1	1	1	1	1
1	1	1	2	2	2	2
1	2	2	1	1	2	2
1	2	2	2	2	1	1
2	1	2	1	2	1	2
2	1	2	2	1	2	1
2	2	1	1	2	2	1
2	2	1	2	1	1	2

is best to maximize resolution by selecting g = ABCD as the design generator. This means that we should assign our factors to the columns of Taguchi's L_8 OA in such a manner as to attain the alias structure A = BCD, B = ACD, C = ABD, D = ABC, AB = CD, AC = BD, and AD = BC. If our design matrix shows that D = ABC, then a resolution R = IV is guaranteed. One way to attain this maximum resolution is to assign A to (1), B to (2), C to (4), and because (1) + (2) + (4) = (7), mod 2, we must assign factor D to column (7); this assignment will ensure a resolution IV design because both effects ABC and D will occupy column 7 and hence will be aliased. A minor deficiency of using Taguchi's L_8 OA is to haphazardly assign the 4 factors in the 2^{4-1} FFD to any column of his L_8 because the experimenter may disregard the two Taguchi's linear graphs (see p. 58 of the Manual) and indeed end up with the inferior resolution III design. If the experimenter follows the column assignments of Taguchi's linear graphs, s/he is assured of a resolution IV design.

A poor choice of column assignments is depicted in Table 7(d) yielding a resolution III design but does not comply with the guidelines set forth by Taguchi's two linear graphs. To illustrate to the reader that the FFD in Table 7(d) is indeed inferior, we revert back to the actual base-2 notation, where 0 represents low and +1 represents the high level of a factor as shown in Table 7(e). Table 7(e) clearly shows that the BCD effect cannot be assessed (or has been sacrificed) to generate the 7 orthogonal columns of the L_8 array, i.e., the generator of the design in Table 7(e) is g = BCD and hence a resolution R = III because g = BCD consists of 3 letters. Further, the contrast

Table 7(d). The inferior assignment of 4 factors to a Taguchi's L_8 OA

A=(1)	B=(2)	AB=(3)	C=(4)	AC=(5)	D=(6)	AD=(7)
1	1	1	1	1	1	1
1	1	1	2	2	2	2
1	2	2	1	1	2	2
1	2	2	2	2	1	1
2	1	2	1	2	1	2
2	1	2	2	1	2	1
2	2	1	1	2	2	1
2	2	1	2	1	1	2

Table 7(e). The L_8 OA in the actual base-2 notation with R = III

A= (1)	B=(2)	AB=(3)	C=(4)	AC=(5)	D=(6)	AD=(7)	BCD
0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0
0	1	1	0	0	1	1	0
0	1	1	1	1	0	0	0
1	0	1	0	1	0	1	0
1	0	1	1	0	1	0	0
1	1	0	0	1	1	0	0
1	1	0	1	0	0	1	0

function for g = BCD is ξ(BCD) = $x_2 + x_3 + x_4$ shows that all 8 FLCs inside the brackets [0000 = (1), 0011 = cd, 0101 = bd, 0110 = bd, 1000 = a, 1011 = acd, 1101 = abd, and 1110 = abc] make the contrast function ξ(BCD) = $x_2 + x_3 + x_4$ equal to zero (mod 2). Hence, the design matrix in Table 7(e), which does not match either of the two Taguchi's linear graph, is the PB of the $\mathbf{2_{III}^{4-1}}$ FFD with the generator g = BCD.

To attain a resolution IV design using Taguchi's L_8 involving 4 factors, we must make the column assignments depicted in Table 7(f), which does follow the column assignments permitted under either of his two linear graphs. The FFD in Table 7(f) has a resolution R = IV because the design generator g = ABCD has 4 letters, i.e., the design matrix in Table 7(f) is the PB of a $\mathbf{2_{IV}^{4-1}}$ FFD. From the above discussions, we conclude that when designing a 2^{4-1} FFD using Taguchi's L_8 OA, the experimenter should follow the column assignment guidelines set forth by either of the two linear

Table 7(f). The Taguchi's L_8 OA in the base-2 notation with R = IV

A=(1)	B=(2)	AB=(3)	C=(4)	AC=(5)	AD=(6)	D=(7)	ABCD
0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0
0	1	1	0	0	1	1	0
0	1	1	1	1	0	0	0
1	0	1	0	1	0	1	0
1	0	1	1	0	1	0	0
1	1	0	0	1	1	0	0
1	1	0	1	0	0	1	0

graphs given in the middle of page 58 of the Manual. Otherwise, s/he may attain a resolution III for the constructed 2^{4-1} design matrix. Before we discuss Taguchi's L_{16} OA, it is paramount to mention the fact that the classical notation for base-2 designs is –1 for the low level, and +1 for the high level of a factor. The use of –1 and +1 is appropriate because when a factor is at 2 levels, only its linear effect (or impact) on the mean of response variable y can be assessed and the contrast coefficients for any 2-level factor, say A, are simply –1 and +1. Thus, contrast(A) = $-1\times A_0 + (1)\times A_1$, where A_0 is the grand subtotal of all responses where factor A is at its low level and A_1 is the grand total of all responses where factor A is at its high level. In classical FFD notation, the columns A, B, C and D of Table 7(f) will take the format presented in Table 7(g). The signs under the generator g = ABCD are obtained by simply multiplying the signs under A, B, C, and D. Because all the signs under the ABCD column is +1, the ABCD effect is also called the identity, I, of the design matrix in Table 7(g) and as a result D = +ABC. Note that the equality D = +ABC can multiplied through by D to obtain D^2 = ABCD, but Table 7(g) shows that if the column D is squared, all its 8 entries will equal to +1, and hence D^2 = I. By the identity element it is meant that any of the columns (1) through (7) of Table 7(g) can be multiplied by the column I = ABCD without changing the column signs of (1) through (7). The identity element for the other ½ fraction (with 8 FLCs) of the $\mathbf{2_{IV}^{4-1}}$ FFD in Table 7(g) is simply I = – ABCD so that the signs under D will be obtained from –ABC. In other words, to determine the 8 FLCs

Table 7(g). The Taguchi's L_8 OA in the classical FFD notation with R = IV

A= (1)	B=(2)	AB=(3)	C=(4)	AC=(5)	AD=(6)	D=(7)	I=ABCD	FLC
– 1	– 1	+1	– 1	+1	+1	– 1	+1	(1)
– 1	–1	+1	+1	– 1	– 1	+1	+1	cd
– 1	+1	– 1	– 1	+1	– 1	+1	+1	bd
– 1	+1	– 1	+1	– 1	+1	– 1	+1	bc
+1	– 1	– 1	– 1	– 1	+1	+1	+1	ad
+1	– 1	– 1	+1	+1	– 1	– 1	+1	ac
+1	+1	+1	– 1	– 1	– 1	– 1	+1	ab
+1	+1	+1	+1	+1	+1	+1	+1	abcd

in the other ½ fraction simply multiply column 7 signs of Table 7(g) by –1 to obtain the FLCs [d, c, b, bcd, a, acd, abd, abc]. Further, the orthogonality of Taguchi's $L_8(2^7)$ design matrix can be verified by the fact that the dot product of any two columns, (1) through (7), of Table 7(g) is zero because each column is simply an 8×1 vector.

The L_8 OA is also very useful for designing a $\mathbf{2_{III}^{5-2}}$, which is a (1/4)th fraction of a 2^5 factorial. Further, it is impossible to generate a resolution IV design with k = 5 factors [similarly it will be impossible to generate a resolution V design with k = 6 or 7 factors in base 2]. For example, if we use g_1 = ABCD and g_2 = BCDE as independent design generators, then our 3rd generator will be (ABCD)×BCDE = AE, which is a resolution II design so that factors A and E will be aliased. To attain an R = III, one possible assignment is A on column 1 (A → (1)) , B → (2), C → (3), D → (4), E → (5); there are now 2 columns left and the experimenter may not arbitrarily select two desired 1st-order interactions to study. The only two-way interactions that can be studied are BD = CE both on (6), and CD = BE both on (7). These lead to one of the three generators g_1 = BCDE. Since both factor C and AB interaction occupy (3), C = AB implies that g_2 = ABC, and hence g_3 = ADE. Note that the two linear graphs provided by Taguchi for the L_8 can be used for designing the $\mathbf{2_{III}^{5-2}}$ FFD but the experimenter is limited to study only two related interactions such as (AB, AC), (AB, BC), (AC, AD), (AC, CD), … , or (CD, DE) and no others. In other words, it will be impossible to embed the seven effects A, B, C, D, E, AB and CD into an L_8 OA without aliasing at least two of these 7 effects because the two interactions AB and CD have no common letters.

The L_8 array can also be used as the $\mathbf{2_{III}^{6-3}}$ and $\mathbf{2_{III}^{7-4}}$ FFDs. In the case of $\mathbf{2_{III}^{6-3}}$ we have 6 factors that will occupy 6 columns of an L_8 array and the one interaction that can be studied should be determined from Taguchi's two linear graphs. The $\mathbf{2_{III}^{7-4}}$ FFD matrix is saturated because every column will be occupied by a separate main factor and thus no two-way interaction can be studied separately from the main factors (hence a resolution R = III design). In summary, a Taguchi L_8 OA can be used to accommodate a 2^3 complete factorial, and any of the four FFDs $\mathbf{2_{IV}^{4-1}}$, $\mathbf{2_{III}^{5-2}}$, $\mathbf{2_{III}^{6-3}}$ and $\mathbf{2_{III}^{7-4}}$.

The Taguchi $L_{16}(2^{15})$ OA

Since this OA has N_f = 16 distinct rows and 16 = 2^4, the exponent 4 shows that the design matrix will have exactly 4 arbitrary columns (see pp. 59-62 of the Manual for the design matrix and the associated linear graphs) and provides 15 df for studying distinct effects. Hence the matrix can have up to and including 15 orthogonal columns. The L_{16} on page 59 clearly shows that columns (1), (2), (4), and (8) have been written completely arbitrarily. As in L_8, the L_{16} can easily be generated by first embedding a complete 2^4 factorial in its 15 columns and using the modulus 2 notation of 0 for the low, and 1 for the high level of a factor. It is paramount that the 4 factors of the full 2^4 factorial be assigned to the 4 arbitrary columns (1), (2), (4), and (8). Without loss of generality, we assign factor A to (1), B to (2), C to (4), and factor D to column (8), as depicted in Table 8. Then the column (3) of Table 8 is generated by adding columns (1) and (2) mod 2; column (11) is generated by adding columns (1), (2) and (8) mod 2 because the contrast function for the ABD effect is given by $\xi(ABD) = x_1 + x_2 + x_4$. In Table 8, if we replace all 0's by 1's and all 1's by 2's, we will obtain the Taguchi L_{16} OA in his own notation. In addition to a 2^4 full factorial, a Taguchi L_{16} can be used to accommodate the FFDs $\mathbf{2_{V}^{5-1}}$, $\mathbf{2_{IV}^{6-2}}$, $\mathbf{2_{IV}^{7-3}}$, $\mathbf{2_{IV}^{8-4}}$, $\mathbf{2_{III}^{9-5}}$, $\mathbf{2_{III}^{10-6}}$, $\mathbf{2_{III}^{11-7}}$, $\mathbf{2_{III}^{12-8}}$,

Table 8. Taguchi's L_{16} OA generated using the base-2 notation of 0 and 1

	A	B	AB	C	AC	BC	ABC	D	AD	BD	ABD	CD	ACD	BCD	ABCD
Run no.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
3	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
4	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
5	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
6	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
7	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
8	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
9	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
10	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
11	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
12	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
13	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
14	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1
15	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1
16	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

2_{III}^{13-9}, 2_{III}^{14-10}, and 2_{III}^{15-11}. This last FFD 2_{III}^{15-11} is saturated because every column of the L_{16} OA is occupied by a main factor and consequently each effect has $2^{11} - 1 = 2047$ aliases (this is due to the fact that there are 2047 blocks each with 16 FLCs that are not studied). The 3 linear graphs (1, a, b, & c) on page 61 of my Manual pertain to a 2_{V}^{5-1} FFD, and Table 8 shows that a resolution V is obtained by simply assigning the 5th factor, E, to column (15) so that E = ABCD $\rightarrow$ g = ABCDE. In this case, the experimenter may also assign the 5 main factors and the 10 two-way interactions according to the linear graphs 1(b & c) to attain a resolution V design. The linear graphs (2a, b, & c) on page 61 of my Manual will generate a 2_{III}^{7-3} FFD, and thus the linear graphs (2a, b & c) are deficient in this case because they yield only a resolution III design. Since the L_{16} meets the Webb's minimum required number of runs of 2k = 14, it may be possible to generate a resolution IV design by selecting the independent generators g_1 = ABCE, g_2 = BCDF, and g_3 = ABFG. Note that this set of generators does yield a resolution IV design because the other 4 generators are g_4 = ADEF, g_5 = CEFG, g_6 = ACDG, and g_7 = BDEG each of which has 4 letters. However, the experimenter will have to follow the guidelines set forth by Bulington et al. (1990) to attain a resolution IV design using Taguchi's L_{16} OA but must assign a 3-way interaction to column (1). Similarly, a $\mathbf{2_{IV}^{6-2}}$ can be embedded into a Taguchi L_{16} OA but the experimenter must assign two 3-way interactions to two of the columns. The linear graphs (3, 4, and 6 a, b, & c) on pages 61-2 of the Manual produce a $\mathbf{2_{III}^{8-4}}$ FFD, but it is possible to generate a resolution IV design using the independent generators g_1 = ABEF, g_2 = ACEG, g_3 = ADEH, and g_4 = ACFH and assigning A to column (1) of an L_{16}, B to (3), C to (5), D to (7), E to (9), F to (11), G to (13), and assigning H to column (15) of an L_{16} Taguchi OA. The linear graphs (5a, b, & c) produce a $\mathbf{2_{III}^{10-6}}$ FFD, and because 2k = 20 > 16, it is impossible to generate a resolution IV design with 16 runs involving 10 factors, i.e., the 5 remaining columns of an L_{16} do not provide sufficient df (or room) for the ${}_{10}C_2$ = 45 interactions to be placed two-at-a-time (up to six-at-a-time) on the remaining 5 columns.

Y. Wu and S. Taguchi [p. iii of introduction to Taguchi and Konishi (1987)] state that the $L_{12}(2^{11})$, $L_{16}(2^{15})$, $L_{18}(2\times3^7)$ and $L_{27}(3^{13})$ are the most commonly-used of G. Taguchi's OAs. We have discussed how to generate an $L_{16}(2^{15})$, and the $L_{12}(2^{11})$ Taguchi OA is a modification of the Plackett-Burman design [for more information see D. C. Montgomery (2001), pp. 343-345] where every pair of columns are orthogonal in the sense that the pairs (1, 1), (1, 2), (2,1), and (2, 2) appear exactly 3 times together in any two columns of the L_{12}. We defer the discussion of the $L_{18}(2\times3^7)$ OA until the last section. Thus, we next discuss Taguchi's L_{27} OA.

The Taguchi $L_{27}(3^{13})$ OA

The $L_{27}(3^{13})$ is an OA of 27 distinct FLCs in base 3; since the N_f = 27 distinct rows provide 26 df for studying different effects and each column of an L_{27} has 2 df (because each has 3 levels), this design matrix can be used to examine a maximum of 26/2 = 13 two-df effects. Thus, the L_{27} can be used to accommodate a full 3^3 factorial and the FFDs $\mathbf{3_{IV}^{4-1}}$, $\mathbf{3_{III}^{5-2}}$, $\mathbf{3_{III}^{6-3}}$, $\mathbf{3_{III}^{7-4}}$, $\mathbf{3_{III}^{8-5}}$, $\mathbf{3_{III}^{9-6}}$, $\mathbf{3_{III}^{10-7}}$, $\mathbf{3_{III}^{11-8}}$, $\mathbf{3_{III}^{12-9}}$, and

$\mathbf{3_{III}^{13-10}}$. The last FFD $\mathbf{3_{III}^{13-10}}$ is saturated in the sense that every 2-df column of the L_{27} array is occupied by a 2-df main factor and each effect will have $3^{10} - 1 = 59048$ aliases. As in the case of L_{16}, the simplest way of generating the L_{27} is to imbed a 3^3 complete factorial design into its 13 columns. The exponent 3 (in $3^3 = 27$) implies that the levels of the 3 factors A, B, and C can be written arbitrarily in 3 columns. The 3 arbitrary columns of the Taguchi L_{27} are columns (1), (2), and (5) (see pp. 67-68 of my Manual). Column (1) is arbitrary because it consists of nine 1's (low level of factor A), followed by nine 2's (the middle level of factor A), and then nine 3's (the high level of factor A). Similarly, column (2) was arbitrarily written as three 1's, three 2's, followed by three 3's, and this pattern repeated twice more, and column (5) is written arbitrarily as levels 1, 2, 3 of factor C and repeated eight more times. Since two-way interactions, such as A×B, have 4 df, then each two-way interaction can be imbedded in two 2-df columns of an L_{27} OA. For example, the A×B effect will occupy columns (3) and (4) of the L_{27}, assuming that A is on column (1) and B on column (2). As was illustrated on pp. 22-24 of my notes, the A×B interaction decomposes into two orthogonal components AB, and AB^2, but Taguchi replaces the component AB^2 by the statistically unconventional component A^2B. Converting to base-3 notation of 0 for low, 1 for middle, and 2 for the high level of a equi-spaced quantitative factor or a qualitative factor, it is noted that the contrast function for A^2B is $\xi(A^2B) = 2x_1 + x_2 = 2\xi(AB^2)$ because in base-3 algebra, 4 = 1 modulus (3), and thus the two components A^2B and AB^2 represent the same effect. Column (3) of L_{27} is occupied by the AB component of A×B, and since the contrast function of AB is $\xi(AB) = x_1 + x_2$, column (3) is generated by adding columns (1) and (2) modulus 3. Similarly, column (4) is generated by adding $2\times(1) + (2)$ (mod 3). We used the Microsoft Excel Mod function to generate the entire L_{27} OA given in Table 9. If we replace 0 by 1, 1 by 2, and 2 by 3 in Table 9, we obtain the Taguchi's L_{27} OA in his own notation as shown on page 67 of the Manual. The table of interaction-between-two-columns (TOIBTCs) on page 68 assisted us in determining which 2nd-order effect would occupy which column of the L_{27} OA. The L_{27} OA can provide a resolution IV design in only one instance, namely for the FFD $\mathbf{3_{IV}^{4-1}}$, and to ensure that an R = IV is attained in this case, all the experimenter has to do is to

Table 9. Generating Taguchi's L_{27} using Microsoft Excel's Mod 3 Function

	A	B	AB	A^2B	C	AC	A^2C	BC	ABC	A^2BC	B^2C	AB^2C	A^2B^2C
Run No.	(1)	(2)	(3)	(4)	(5)	(6)	('7)	(8)	(9)	(10)	(11)	(12)	(13)
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	1	1	1	1	1	1	1	1	1
3	0	0	0	0	2	2	2	2	2	2	2	2	2
4	0	1	1	1	0	0	0	1	1	1	2	2	2
5	0	1	1	1	1	1	1	2	2	2	0	0	0
6	0	1	1	1	2	2	2	0	0	0	1	1	1
7	0	2	2	2	0	0	0	2	2	2	1	1	1
8	0	2	2	2	1	1	1	0	0	0	2	2	2
9	0	2	2	2	2	2	2	1	1	1	0	0	0
10	1	0	1	2	0	1	2	0	1	2	0	1	2
11	1	0	1	2	1	2	0	1	2	0	1	2	0
12	1	0	1	2	2	0	1	2	0	1	2	0	1
13	1	1	2	0	0	1	2	1	2	0	2	0	1
14	1	1	2	0	1	2	0	2	0	1	0	1	2
15	1	1	2	0	2	0	1	0	1	2	1	2	0
16	1	2	0	1	0	1	2	2	0	1	1	2	0
17	1	2	0	1	1	2	0	0	1	2	2	0	1
18	1	2	0	1	2	0	1	1	2	0	0	1	2
19	2	0	2	1	0	2	1	0	2	1	0	2	1
20	2	0	2	1	1	0	2	1	0	2	1	0	2
21	2	0	2	1	2	1	0	2	1	0	2	1	0
22	2	1	0	2	0	2	1	1	0	2	2	1	0
23	2	1	0	2	1	0	2	2	1	0	0	2	1
24	2	1	0	2	2	1	0	0	2	1	1	0	2
25	2	2	1	0	0	2	1	2	1	0	1	0	2
26	2	2	1	0	1	0	2	0	2	1	2	1	0
27	2	2	1	0	2	1	0	1	0	2	0	2	1

assign the 4^{th} factor D to one of the columns (9), (10), (12), or (13) while factors A, B, and C must be imbedded in the 3 arbitrary columns (1), (2), and (5).

As an example, suppose we wish to study 6 factors A, B. C, D, E, and F each at 3 levels and the two-way interactions $C\times D$, $C\times E$, and $D\times E$. These 6 factors and 3 two-way interactions will need $2\times6 + 3\times4 = 24$ df and will occupy $24/2 = 12$ of the 13 columns of an L_{27} OA and the remaining column can be used to study one more effect that can be determined after column assignments are completed. Suppose we assign factor C to (1), D to column (2), then CD $\rightarrow$ (3), C^2D will have to be assigned to column

(4). Assigning E to (5), then Table 9 shows that CE will have to be imbedded onto (6), C^2E onto column(7), DE onto (8) and D^2E must be assigned to column (11). These assignments leave columns (9), (10), (12), and (13) available. Without loss of generality, we may assign factor A to (9), factor B to column (10), and factor F to column (12), which leaves only column (13) empty. The TOIBTCs on page 68 of the Manual now shows that (1)×(12) = (11) & (13), and hence column (13) may be used to study one component of C×F, namely, CF or C^2F. We again used Microsoft Excel to verify that (1) + (12) = (13) mod 3, and hence the effect CF component of C×F can also be studied. The question that now arises is "what is the alias structure of the above $\mathbf{3_{III}^{6-3}}$ FFD"? Since we need 3 independent generators out of the total of $(3^3 - 1)/(3 - 1) = 13$ generators and the Taguchi L_{27} always yields the PB of a FFD, we used the TOIBTCs on page 68 to assist us in identifying the alias structure. This table shows that (13) & (7) = (2)×(10) and thus CF = BD, or $C^3F^3 = BC^2DF^2$, which shows that one of the design generators is $g_1 = BC^2DF^2$ because $C^3F^3 = C^0F^0 = I$. Next, the TOIBTCs on page 68 of my Manual shows that (2)×(6) = (9) & (12). Thus, CDE = A, or $g_2 = AC^2D^2E^2$. Lastly, the same TOIBTCs and our Table 9 show that 2(9)×(10) = (1), which yield $A^2B = C$, or $g_3 = AB^2C$. Thus, the other 10 generators are

$$g_4 = g_1 \times g_2 = (BC^2DF^2) \times (AC^2D^2E^2) = ABCE^2F^2,$$

$$g_5 = \mathbf{g_1^2} \times g_2 = (B^2CD^2F) \times AC^2D^2E^2 = AB^2DE^2F,$$

$$g_6 = g_1 \times g_3 = (BC^2DF^2) \times AB^2C = ADF^2,$$

$$g_7 = \mathbf{g_1^2} \times g_3 = (B^2CD^2F) \times AB^2C = ABC^2D^2F,$$

$$g_8 = g_2 \times g_3 = (AC^2D^2E^2)(AB^2C) = ABDE,$$

$$g_9 = g_2 \times \mathbf{g_3^2} = (AC^2D^2E^2)(A^2BC^2) = BCD^2E^2,$$

$$g_{10} = g_1 \times g_2 \times g_3 = (BC^2DF^2) \times (AC^2D^2E^2) \times (AB^2C) = ACEF,$$

$$g_{11} = \mathbf{g_1^2} \times g_2 \times g_3 = (B^2CD^2F) \times (AC^2D^2E^2) \times (AB^2C) = AB^2C^2D^2EF^2,$$

$$g_{12} = g_1 \times \mathbf{g_2^2} \times g_3 = (BC^2DF^2) \times (A^2CDE) \times (AB^2C) = CD^2EF^2, \text{ and}$$

$$g_{13} = \mathbf{g_1^2} \times \mathbf{g_2^2} \times g_3 = (B^2CD^2F) \times (A^2CDE) \times (AB^2C) = BEF.$$

Note that the minimum length word among the above 13 generators in the defining relation I is 3 and hence a resolution III design. Further, because the FFD 3^{6-3}_{III} is a $(1/27)^{th}$ fraction and only one block out the 27 blocks is studied and 26 blocks are left out of experimentation, each effect must have $3^3 - 1 = 26$ aliases. For example, to obtain the 26 aliases of factor A, we either multiply A by the 13 generators and also multiply A by the 13 generators squared modulus 3. Or, we may multiply A and A^2 by the 13 generators modulus 3 using the statistical convention that the first letter must have an exponent of 1. Following this procedure, the 26 aliases of factor A are A = $ABC^2DF^2 = AB^2CD^2F = ACDE = CDE = ABC^2 = BC^2 = AB^2C^2EF = BCE^2F^2 = ABD^2EF^2 = BD^2EF^2 = AD^2F = DF^2 = AB^2CDF^2 = BC^2D^2F = AB^2D^2E^2 = BDE = ABCD^2E^2 = AB^2C^2DE = AC^2E^2F^2 = CEF = ABCDE^2F = BCDE^2F = ACD^2EF^2 = AC^2DE^2F = ABEF = AB^2E^2F^2$.

The Taguchi $L_{18}(2\times3^7)$ OA

The L_{18} is the most-commonly used mixed-level Taguchi's OA and can accommodate one 2-level factor and a maximum of seven 3-level factors. Since the total number of distinct runs $N_f = 18 = 2^1\times3^2$, summing the exponents 1+2 = 3 implies that exactly 3 columns are written completely arbitrarily [namely columns (1), (2), and (3)]. The design matrix provides 18 –1 = 17 df for studying effects, where the 2-level factor A on column (1) will absorb 1 df, and the seven 3-level factors B, C, D, E, F, G, and H each absorb 2 df. Thus the eight factors altogether will absorb 15 out the possible 17 df that the design matrix provides. This leaves 2 unused df that can be used to study the interaction only between columns (1) and (2). Assuming, without loss of generality, that A is the 2-level factor, the experimenter must imbed the 3-level factor whose interaction with factor A s/he would like to examine in column (2). We will later show in a parameter design example that the only possible interaction that can be studied is (1)×(2) but this interaction cannot be imbedded into the design matrix. The question that now arises is how Dr. Taguchi developed the other 5 columns (4, 5, 6, 7, and 8) of the L_{18} so that the design matrix is orthogonal. As stated earlier, the first 3 columns are written completely arbitrarily. Then, we need to address the construction of columns (4) thru (8) of the L_{18}. The reader must be informed that we are not sure how

G. Taguchi developed his L_{18} OA, and what follows is our explanation. To this end, let us define a group of three 3×1 vectors $G_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ = [1 2 3]′, $\mathbf{G}_2'$ = [2 3 1], and $\mathbf{G}_3'$ = [3 1 2], where prime is used to denote matrix transpose. Note that column (3) of L_{18} is arbitrarily written as $[\mathbf{G}_1' \quad \mathbf{G}_1' \quad \mathbf{G}_1' \quad \mathbf{G}_1' \quad \mathbf{G}_1' \quad \mathbf{G}_1']'$. Next, we translate the above 3 vectors by subtracting 2 from each element so that G_1 transforms to G_4 = [– 1 0 1]′, G_2 transforms to G_5 = [0 1 – 1]′, and G_3 translates to G_6 = [1 – 1 0]′. (Note that G_4 is the linear contrast in base 3 for a quantitative factor.) It is well known that two vectors are orthogonal if and only if their dot product is zero. Clearly, $\mathbf{G}_4' \times G_5$ = [– 1 0 1] [0 1 -1]′ = –1 = $\mathbf{G}_4' \times G_6 = \mathbf{G}_5' \times G_6$, which implies that the vectors G_1 and G_2 are not orthogonal, G_1 and G_3 are not orthogonal, and neither are G_2 and G_3. Further, $\mathbf{G}_i' \times G_i$ = +2 for all i = 4, 5, and 6. A close examination of the 4th column of L_{18} on page 69 of the Manual reveals that column (4) can be written as (4) = $[\mathbf{G}_1' \quad \mathbf{G}_1' \quad \mathbf{G}_2' \quad \mathbf{G}_3' \quad \mathbf{G}_2' \quad \mathbf{G}_3']'$ and column(5) of L_{18} is simply (5) = $[\mathbf{G}_1' \quad \mathbf{G}_2' \quad \mathbf{G}_1' \quad \mathbf{G}_3' \quad \mathbf{G}_3' \quad \mathbf{G}_2']'$. If we now take the dot product of column (4) with (5) after the translation we obtain $[\mathbf{G}_4' \quad \mathbf{G}_4' \quad \mathbf{G}_5' \quad \mathbf{G}_6' \quad \mathbf{G}_5' \quad \mathbf{G}_6'] \times \begin{bmatrix} \mathbf{G}_4 \\ \mathbf{G}_5 \\ \mathbf{G}_4 \\ \mathbf{G}_6 \\ \mathbf{G}_6 \\ \mathbf{G}_5 \end{bmatrix}$ = 2 – 1 – 1 + 2 – 1 – 1 = 0, which shows that columns (4) and (5) of L_{18} are orthogonal. Another pattern that is obvious in the L_{18} is the fact that columns (3) through (8) have their first three rows as the vector G_1. To generate column (6), we have to find another permutation of $[\mathbf{G}_1' \quad \mathbf{G}_1' \quad \mathbf{G}_2' \quad \mathbf{G}_2' \quad \mathbf{G}_3' \quad \mathbf{G}_3']'$, keeping G_1 in the first position, that is orthogonal to both columns (4) and (5). One such permutation is $[\mathbf{G}_1' \quad \mathbf{G}_2'$

$\mathbf{G}'_3 \quad \mathbf{G}'_2 \quad \mathbf{G}'_1 \quad \mathbf{G}'_3]'$ which comprises column (6) of Taguchi's L_{18}. Similarly, two other permutations of $[\mathbf{G}'_1 \quad \mathbf{G}'_1 \quad \mathbf{G}'_2 \quad \mathbf{G}'_2 \quad \mathbf{G}'_3 \quad \mathbf{G}'_3]'$, are columns (7) = $[\mathbf{G}'_1 \quad \mathbf{G}'_3 \quad \mathbf{G}'_2 \quad \mathbf{G}'_2 \quad \mathbf{G}'_3 \quad \mathbf{G}'_1]'$ and (8) = $[\mathbf{G}'_1 \quad \mathbf{G}'_3 \quad \mathbf{G}'_3 \quad \mathbf{G}'_1 \quad \mathbf{G}'_2 \quad \mathbf{G}'_2]'$ of Taguchi's L_{18} OA. The last 5 columns of the L_{18} Taguchi OA in terms of the vectors G_1, G_2, and G_3 are given below and denoted as the matrix G. These developments indicate that the L_{18} is not unique in the sense that there are other permutations of the last 5 rows of the matrix G that yield an L_{18} orthogonal array. We have identified all other G matrices, of which there are eleven, that are somewhat distinct from the above Taguchi's layout but will also yield an L_{18} OA. The other eleven are listed below. Note that the orthogonality of the eleven alternatives to Taguchi's L_{18} was verified by first replacing column (1) of L_{18} by

$$\begin{array}{c} \begin{array}{ccccc} \textbf{(4)} & \textbf{(5)} & \textbf{(6)} & \textbf{(7)} & \textbf{(8)} \end{array} \\ \mathrm{G} = \begin{bmatrix} G_1 & G_1 & G_1 & G_1 & G_1 \\ G_1 & G_2 & G_2 & G_3 & G_3 \\ G_2 & G_1 & G_3 & G_2 & G_3 \\ G_3 & G_3 & G_2 & G_2 & G_1 \\ G_2 & G_3 & G_1 & G_3 & G_2 \\ G_3 & G_2 & G_3 & G_1 & G_2 \end{bmatrix} \end{array}$$

nine –1's followed by nine +1's, then subtracting 2 from every element of the remaining 7 columns so that all 8 columns summed to zero and finally computing
the resulting $X'\times X$ matrix. In all the eleven cases the resulting 8×8 matrix $X'\times X$ was diagonal, and Minitab GLM also verified that all eleven design matrices were orthogonal.

$$\begin{array}{c} \begin{array}{ccccc} (4) & (5) & (6) & (7) & (8) \end{array} \\ \begin{bmatrix} G_1 & G_1 & G_1 & G_1 & G_1 \\ G_2 & G_2 & G_1 & G_3 & G_3 \\ G_2 & G_3 & G_3 & G_2 & G_1 \\ G_3 & G_1 & G_2 & G_2 & G_3 \\ G_1 & G_3 & G_2 & G_3 & G_2 \\ G_3 & G_2 & G_3 & G_1 & G_2 \end{bmatrix} \end{array} , \quad \begin{array}{c} \begin{array}{ccccc} (4) & (5) & (6) & (7) & (8) \end{array} \\ \begin{bmatrix} G_1 & G_1 & G_1 & G_1 & G_1 \\ G_1 & G_2 & G_2 & G_3 & G_3 \\ G_2 & G_3 & G_1 & G_3 & G_2 \\ G_2 & G_1 & G_3 & G_2 & G_3 \\ G_3 & G_2 & G_3 & G_1 & G_2 \\ G_3 & G_3 & G_2 & G_2 & G_1 \end{bmatrix} \end{array} ,$$

$$
\begin{array}{c}
\begin{array}{ccccc}\textbf{(4)} & \textbf{(5)} & \textbf{(6)} & \textbf{(7)} & \textbf{(8)}\end{array}\\
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_1 & G_2 & G_2 & G_3 & G_3\\
G_3 & G_3 & G_2 & G_2 & G_1\\
G_3 & G_2 & G_3 & G_1 & G_2\\
G_2 & G_1 & G_3 & G_2 & G_3\\
G_2 & G_3 & G_1 & G_3 & G_2
\end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc}\textbf{(4)} & \textbf{(5)} & \textbf{(6)} & \textbf{(7)} & \textbf{(8)}\end{array}\\
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_1 & G_2 & G_2 & G_3 & G_3\\
G_3 & G_3 & G_2 & G_2 & G_1\\
G_2 & G_1 & G_3 & G_2 & G_3\\
G_3 & G_2 & G_3 & G_1 & G_2\\
G_2 & G_3 & G_1 & G_3 & G_2
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccc}(4) & (5) & (6) & (7) & (8)\end{array}\\
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_2 & G_1 & G_3 & G_2 & G_3\\
G_1 & G_2 & G_2 & G_3 & G_3\\
G_2 & G_3 & G_1 & G_3 & G_2\\
G_3 & G_3 & G_2 & G_2 & G_1\\
G_3 & G_2 & G_3 & G_1 & G_2
\end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc}(4) & (5) & (6) & (7) & (8)\end{array}\\
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_2 & G_1 & G_3 & G_2 & G_3\\
G_1 & G_2 & G_2 & G_3 & G_3\\
G_3 & G_3 & G_2 & G_2 & G_1\\
G_2 & G_3 & G_1 & G_3 & G_2\\
G_3 & G_2 & G_3 & G_1 & G_2
\end{bmatrix}
\end{array},
$$

$$
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_2 & G_2 & G_3 & G_3 & G_1\\
G_2 & G_3 & G_2 & G_1 & G_3\\
G_1 & G_3 & G_3 & G_2 & G_2\\
G_3 & G_1 & G_2 & G_3 & G_2\\
G_3 & G_2 & G_1 & G_2 & G_3
\end{bmatrix},
\quad
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_1 & G_3 & G_2 & G_3 & G_2\\
G_2 & G_3 & G_1 & G_2 & G_3\\
G_2 & G_2 & G_3 & G_3 & G_1\\
G_3 & G_2 & G_2 & G_1 & G_3\\
G_3 & G_1 & G_3 & G_2 & G_2
\end{bmatrix}
$$

$$
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_1 & G_3 & G_2 & G_2 & G_3\\
G_3 & G_1 & G_2 & G_3 & G_2\\
G_3 & G_2 & G_3 & G_2 & G_1\\
G_2 & G_2 & G_1 & G_3 & G_3\\
G_2 & G_3 & G_3 & G_1 & G_2
\end{bmatrix},
\quad
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_3 & G_2 & G_2 & G_1 & G_3\\
G_3 & G_1 & G_3 & G_2 & G_2\\
G_2 & G_2 & G_3 & G_3 & G_1\\
G_2 & G_3 & G_1 & G_2 & G_3\\
G_1 & G_3 & G_2 & G_3 & G_2
\end{bmatrix},
\begin{array}{c}
\begin{array}{ccccc}\textbf{(4)} & \textbf{(5)} & \textbf{(6)} & \textbf{(7)} & \textbf{(8)}\end{array}\\
\begin{bmatrix}
G_1 & G_1 & G_1 & G_1 & G_1\\
G_2 & G_2 & G_3 & G_1 & G_3\\
G_3 & G_2 & G_1 & G_3 & G_2\\
G_3 & G_1 & G_2 & G_2 & G_3\\
G_2 & G_3 & G_2 & G_3 & G_1\\
G_1 & G_3 & G_3 & G_2 & G_2
\end{bmatrix}
\end{array}
$$

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