

**References: D. C. Montgomery, Introduction to Statistical Quality control (7<sup>th</sup> edition), John Wiley & sons, Inc.**

**A. J. Duncan, Quality Control & Industrial Statistics (5<sup>th</sup> edition), Irwin.**

## **Statistical Process Control (SPC)**

The objective of SPC is to test the null hypothesis that the value of a process parameter is either at a desired specified value ( $\theta_0$ ), or at a value that has been established from past (long- or short-term) data. This objective is generally carried out through constructing a Shewhart control chart from  $m$  (generally  $m \geq 20$ ) subgroups of data. Further, it is assumed that the underlying distribution is approximately Laplace-Gaussian, and for moderately large sample sizes, it is also assumed that the SMD (sampling distribution) of the statistic used to construct the Shewhart chart is also Laplace-Gaussian. When a sample point goes out of control limits, the process must be stopped in order to look for assignable (or special) causes of variation, and if one is found by the operator, then corrective action must be taken and the corresponding point should be removed from the chart. In case no assignable (or special) causes are found for a point out of control, then the control chart has led to a false alarm (or a type I error) and the corresponding point should be kept on the control chart. Since false alarms are very expensive and disruptive to a manufacturing process, all Shewhart charts are designed in such a manner that the Pr of committing a type I error,  $\alpha$ , is very small. The standard level of significance,  $\alpha$ , of all Shewhart charts, assuming a Gaussian chart ordinate, is set roughly at  $\alpha = 0.0027$  (or 0.27%).

When departures from the underlying assumptions are not grossly violated, then a Shewhart control chart will generally lead an experimenter to 27 false alarms in 10,000 random samples of size  $n$ . Moreover, setting the value of  $\alpha$  at 0.0027, will correspond to three-sigma control limits for a control chart as long as the normality assumption is tenable. Perhaps, the three-sigma control limits by Shewhart was first constituted, and then the 0.0027-level test followed as its result (I am not sure; the chicken& egg problem); in other words, the 3-sigma limit most likely came first and the

value of type I error rate of 27 in 10, 000 followed as a result, assuming normality of the statistic that is being charted. We will discuss only two types of charts: (1) Charts for continuous variables, and (2) Charts for attributes, where the measurement system merely classifies a unit either as conforming to customer specifications or nonconforming to specifications (i.e., Success/Failure, 0/1, Defective/Effective, Pass/Fail, Accept/Reject, etc.), or the measurement system simply counts the number of defects (or nonconformities) per unit.

## Shewhart Control Charts for Variables

Consider the Example 6-3 borrowed from pages 260-267 of D. C. Montgomery's text entitled "*Introduction to Statistical Quality Control*", 7<sup>th</sup> Edition, published by John Wiley & Sons, Inc. (2013), (ISBN: 978-1-118-14681-1) where the objective is to control the dimension of piston ring inside diameters,  $X$ , with design specifications  $X: 74.00 \pm 0.05$  mm. As stated by D. C. Montgomery (2001), the rings are manufactured thru a forging process. Since the random variable  $X$  is continuous, then we need two charts; one to control within-sample process variability (or internal variability measured by  $\sigma_x = \sigma$ ), and a second chart to monitor the between (samples) process variability, or simply the process mean  $\mu$ . If subgroup sample sizes,  $n_i$ , are all equal and lie within  $2 \leq n_i = n \leq 15$ , then an R-chart (i.e., range-chart) should be used to monitor variability, but for  $n > 15$ , an S-chart should be used for control of variation. This is due to the fact that the SMD (Sampling Distribution) of sample range,  $R$ , becomes unstable for moderate to large sample sizes. For sample sizes  $n_i = n = 13, 14$  &  $15$ , it is not clear as to whether the S-chart is preferred to an R-chart. In practice, I would recommend using the one that provides more statistical power to detect sudden shifts in process variation.

To design a trial (or initial) control chart, samples of sizes  $n_i$  ( $i = 1, 2, \dots, m$ ) are taken from a process in the time-order of production, generally at equal intervals of time, (where hourly or daily samples, or samples taken at different shifts, are the most common; further, sampling frequency generally depends on production rate), and the number of initial subgroups  $m$  should generally lie within the interval  $20 < m \leq 50$ . Samples should be taken in such a manner as to minimize the variability within samples ( $\sigma_x$ ) and maximize the variability among (or between) samples ( $\sigma_{\bar{X}}$ ), a

concept that is consistent with Design of Experiments (DOE, or DOX). Such samples are generally referred to as rational subgroups, whose variation is attributable only to a system of constant common causes. Sampling different machines, sampling over extended periods of time, or from combined output of different sources are examples of nonrational sampling (generally leading to stratification) that must be avoided when setting up control charts.

**R and  $\bar{X}$  Control Charts (for  $2 \leq n \leq 15$  and  $n_i = n$  for all  $i = 1, 2, \dots, m$ , i.e., the Case of Balanced Design)**

In practice I recommend that the R-chart should be constructed first in order to bring variability in a state of statistical control, followed by developing the  $\bar{x}$ -chart for the purpose of monitoring the process mean. Although most will construct  $\bar{x}$ -chart first. In order to use the R-chart for monitoring process variation, the subgroup sample sizes  $n_i$  ( $i = 1, 2, \dots, m$ ) must be the same, i.e.,  $n_i = n$  for all  $i$ , or else an R-chart cannot be constructed. All univariate (i.e., a single response variable) control charts consist of a central line, denoted by CNTL, a lower control limit LCL, and an upper control limit UCL. Further, nearly in all cases to ensure  $\alpha \cong 0.0027$ ,  $LCL = CNTL - 3 \times se(\text{sample statistic})$ , and  $UCL = CNTL + 3 \times se(\text{sample statistic})$ , where in the case of the R-chart the sample statistic will be the sample range  $R$ , while for the  $\bar{x}$  chart the sample statistic will be the sample mean  $\bar{x}$ . The pertinent formulas for an R-chart are provided below. (Note that some authors like A. J. Duncan consider  $\bar{x}$ -chart as one word; I will do both in these notes.)

$$CNTL_R = \bar{R} = \frac{1}{m} \sum_{i=1}^m R_i \tag{1}$$

Note that we are taking the liberty to use the terminology standard error,  $se$ , as the estimate of the STDEV of the sample statistic. Thus,  $se(R) = \hat{\sigma}_R = d_3 \bar{R} / d_2$ , where the values of  $d_2 = E(W) = E(R/\sigma)$ , ( $W = \text{Relative Sample Range} = R/\sigma$ ) for a normal universe are given in Table 10 on the next page for  $n = 2, 3, \dots, 15$ . Because  $d_2 = E(W) = E(R) / \sigma_x$ , then  $\sigma_x = E(R) / d_2$ , which implies  $\hat{\sigma}_x = \bar{R} / d_2$ . Further,  $d_3^2 = V(W) = V(R/\sigma) = V(R) / \sigma^2$  implies that  $V(R) = d_3^2 \times \sigma_x^2 \rightarrow \hat{V}(R) = d_3^2 \times \hat{\sigma}_x^2 = d_3^2 \times (\bar{R} / d_2)^2 \rightarrow$

$se(R) = d_3 \times (\bar{R} / d_2) = \bar{R} \times d_3 / d_2$ , or  $\hat{\sigma}_R = \bar{R} \times d_3 / d_2$ , and the values of  $d_3$  for a normal universe are given in Table 11. Since the most common of all sample sizes for constructing an R- and  $\bar{x}$ - chart is  $n = 5$ , for illustrative purposes we compute  $\hat{\sigma}_R$  only for  $n = 5$ . From Tables 10 & 11 (due to E. S. Pearson), the  $se(R) = d_3 \bar{R} / d_2 = 0.8641 \times \bar{R} / 2.326 = 0.37145 \times \bar{R}$ . In general, the  $LCL_R = \bar{R} - 3 \times d_3 \bar{R} / d_2 = (1 - 3 \times d_3 / d_2) \bar{R} = D_3 \bar{R}$ , where the universal QC constant  $D_3 = 1 - 3 \times d_3 / d_2$ .

Table 10. The Expected-Value,  $d_2$ , of Relative Range ( $W=R/\sigma$ ) for a  $N(\mu, \sigma^2)$

n	2	3	4	5	6	7	8	9	10
$d_2$	1.128	1.693	2.059	2.326	2.534	2.704	2.847	2.970	3.078
n	11	12	13	14	15				
$d_2$	3.173	3.258	3.336	3.407	3.472				

Table 11. The SE of Relative range  $W = R/\sigma$ ,  $d_3$ , for a Normal Universe

n	2	3	4	5	6	7	8	9
$d_3$	0.8525	0.8884	0.8798	0.8641	0.8480	0.8330	0.8200	0.8080
n	10	11	12	13	14	15		
$d_3$	0.797	0.787	0.778	0.770	0.763	0.756		

E. S. Pearson, *Biometrika* 32 (1941-42), pp. 301-308.

Thus, for  $n = 5$ , the  $LCL_R = \bar{R} - 3 \times 0.37145 \times \bar{R} = \bar{R} (1 - 1.1144) \rightarrow LCL_R = 0$ , and  $UCL_R = (1 + 3 \times d_3 / d_2) \bar{R} = D_4 \bar{R} = 2.11444 \times \bar{R}$ . In fact, it can be shown that for a balanced design the value of  $D_3 = 0$  and  $LCL_R = 0$  for all sample sizes in the range  $2 \leq n \leq 6$ , but  $D_3 > 0$  for all  $n > 6$ .

If the process standard deviation is targeted at  $\sigma_0$ , then because  $E(R/\sigma) = d_2$ , the CNTL for the R-chart becomes  $d_2 \sigma_0$ . Further, because the  $V(R/\sigma) = d_3^2$ , then the 3-sigma limits for the targeted R chart are given by  $LCL_R = d_2 \sigma_0 - 3d_3 \sigma_0 = (d_2 - 3d_3) \sigma_0 = D_1 \sigma_0$ , and similarly,  $UCL_R = d_2 \sigma_0 + 3d_3 \sigma_0 = (d_2 + 3d_3) \sigma_0 = D_2 \sigma_0$ . Thus, the QC constants

$D_1 = d_2 - 3 \times d_3$  and  $D_2 = d_2 + 3 \times d_3$ .

The central line (CNTL) for an  $\bar{x}$ -chart, in general, is given by

$$\text{CNTL}_{\bar{x}} = \bar{\bar{x}} = \frac{\sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij}}{\sum_{i=1}^m n_i} = \frac{1}{N} \sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij} \quad (2)$$

where  $N = \sum_{i=1}^m n_i$  is the grand total number of observations. Note that only for the

case of equal sample sizes (balanced sampling design  $n_i = n$  for all  $i$  the  $\text{CNTL}_{\bar{x}}$  is

$$\text{given by } \bar{\bar{x}} = \frac{\sum_{i=1}^m \sum_{j=1}^n x_{ij}}{mn} = \frac{\sum_{i=1}^m \sum_{j=1}^n x_{ij} / n}{m} = \frac{1}{m} \sum_{i=1}^m \bar{x}_i, \text{ and the } \text{SE}(\bar{x}) = \frac{\sigma}{\sqrt{n}}. \text{ When}$$

the sample sizes within subgroups differ, then  $\text{SE}(\bar{x}_i) = \frac{\sigma}{\sqrt{n_i}}$ . Since a point estimate

of  $\sigma$  is given by  $\hat{\sigma}_x = \bar{R}/d_2$ , then for the balanced sampling scheme ( $n_i = n$ ),  $\hat{\sigma}_{\bar{x}} =$

$$\frac{\bar{R}}{d_2 \sqrt{n}}, \text{ as a result the } \text{LCL}_{\bar{x}} = \bar{\bar{x}} - 3 \times \text{se}(\bar{x}) = \bar{\bar{x}} - \frac{3\bar{R}}{d_2 \sqrt{n}} = \bar{\bar{x}} - A_2 \bar{R}, \text{ and } \text{UCL}_{\bar{x}} =$$

$$\bar{\bar{x}} + \frac{3\bar{R}}{d_2 \sqrt{n}} = \bar{\bar{x}} + A_2 \bar{R}, \text{ where the QC constant } A_2 = \frac{3}{d_2 \sqrt{n}}. \text{ For samples of size } n = 5$$

per subgroup,  $A_2 = \frac{3}{d_2 \sqrt{n}} = \frac{3}{2.326\sqrt{5}} = 0.5768$ , and these last two control limits

reduce to ( $\text{LCL}_{\bar{x}} = \bar{\bar{x}} - 0.5768\bar{R}$ ,  $\text{UCL}_{\bar{x}} = \bar{\bar{x}} + 0.5768\bar{R}$ ).

If the process mean and STDEV are targeted at  $\mu_0$  and  $\sigma_0$  (i.e., these two parameters are the specified standard values), then the  $\text{LCL}_{\bar{x}} = \mu_0 - 3\sigma_0/\sqrt{n} = \mu_0 -$

$A\sigma_0$  and  $\text{UCL}_{\bar{x}} = \mu_0 + A\sigma_0$ , where the QC constant  $A = 3/\sqrt{n}$ . For  $n = 5$ ,  $A = 1.3416$ .

The data for the inside diameter of Piston Rings of the Example 6-3 of D. C.

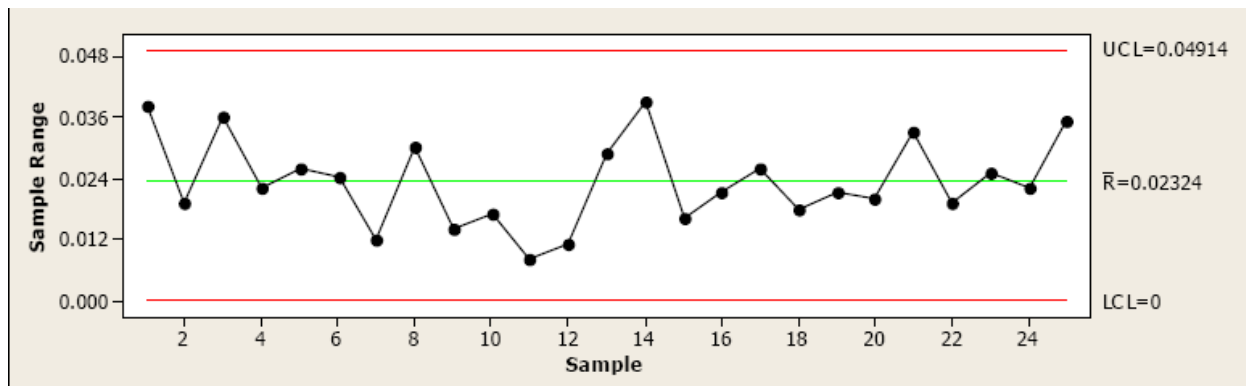
Montgomery are given his Table 6-3 on page 260, where  $m = 25$  subgroups were taken in order to set up trial control limits. For your convenience, I have provided the data on a spreadsheet, named Table 6-3 on my website. From D. C. Montgomery's Table 6-3

we obtain  $\sum_{i=1}^{25} R_i = 0.5810$  and  $\sum_{i=1}^{25} \sum_{j=1}^5 x_{ij} = 9250.1470$ , which lead to  $\bar{R} = \text{CNTL}_R =$

$0.5810/25 = 0.023240$ , and  $\text{CNTL}_{\bar{x}} = \bar{\bar{x}} = 9250.1470/(5 \times 25) = 74.001176$ . Since for samples of size  $n = 5$  the  $se(R) = 0.37145 \times \bar{R} = 0.37145 \times 0.023240 = 0.0086325$ , then  $LCL_R = \bar{R} - 3 \times 0.0086325 = 0.023240 - 0.0258975 \rightarrow LCL_R = 0.0000$ , and the  $UCL_R = 0.023240 + 0.0258975 = 0.0491375$  mm. The spreadsheet (Table6-3DCM) on my site now shows that the minimum sample range occurs on the 11<sup>th</sup> subgroup (i.e.,  $R_{11} = 0.0080$ ) and the maximum sample range occurs at the 14<sup>th</sup> subgroup ( $R_{14} = 0.0390$ ) and hence all the 25 sample ranges lie within the interval  $0.0080 \leq R_i \leq 0.0390$ .

Therefore, no sample range is outside trial control limits (0, 0.0491375 mm), leading us to the conclusion that within sample variability is in a state of excellent statistical control. I used both Minitab and MS Excel to obtain the R-chart for the Piston-Ring diameters of D. C. Montgomery's 7<sup>th</sup> edition, which are given below. In general, we should not proceed to construct an  $\bar{x}$ -chart unless the R-chart shows that within-sample (or internal) variability of the process is in a state of statistical control. If a point on the R chart is out of control, Dr. Shewhart recommends that the corresponding assignable cause(s) for that point must be searched for, and if found, those assignable

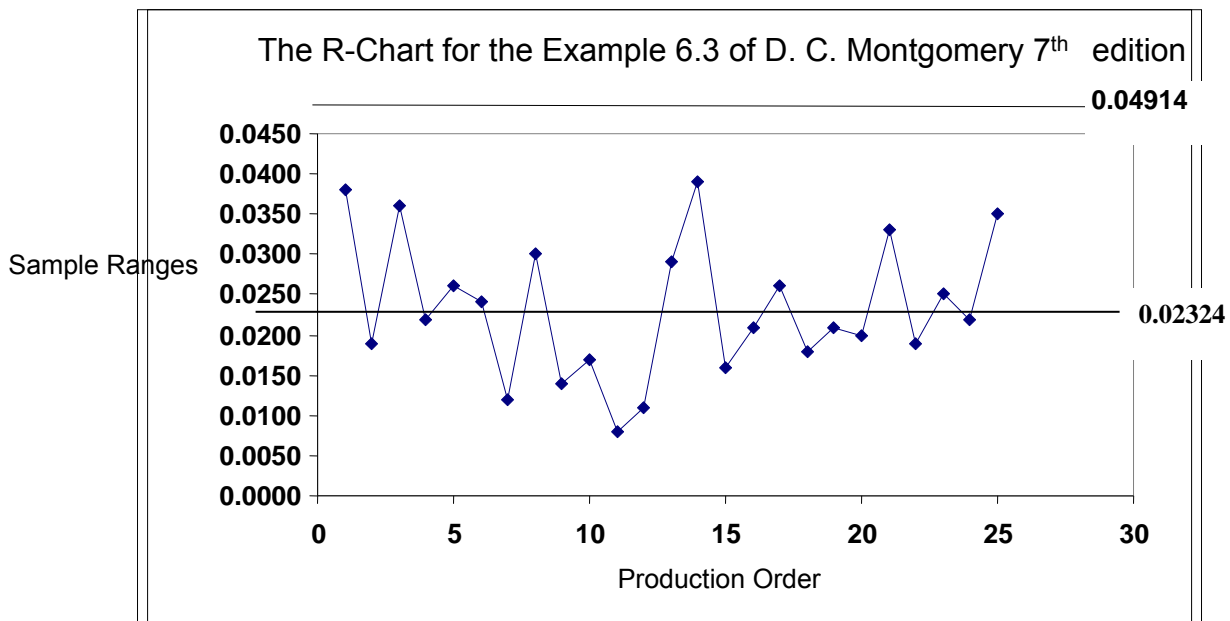
The R-Chart for the Example 6.3 on p. 262 of D. C. Montgomery  
7<sup>th</sup> Edition



causes of variation must be removed from the process and the limits on the R-chart

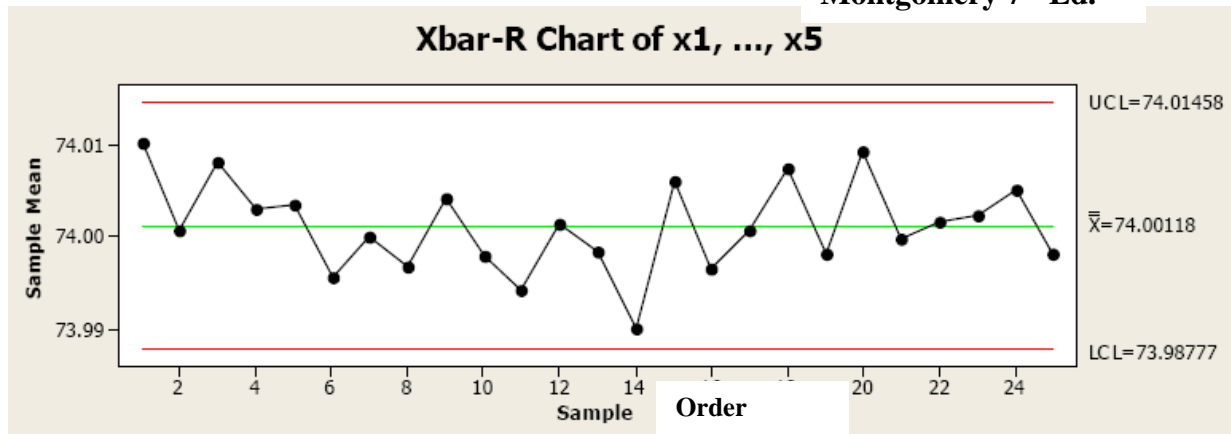
must be revised W/O the out-of-control point. Since, the R-chart below has no point outside of control limits and does exhibit random pattern, then within (or internal) variability of the process seems to be in a good statistical control, and hence we may proceed to set up an  $\bar{x}$  chart.

Next we obtain the control limits for the  $\bar{x}$ -chart to examine the between-sample variability. Since  $\hat{\sigma}_x = \bar{R}/d_2 = 0.023240/2.326 = 0.00999140155$ , the  $se(\bar{x}) = \hat{\sigma}_x/\sqrt{5} = 0.0044683$ ,  $3 \times se(\bar{x}) = 0.013405 \rightarrow LCL_{\bar{x}} = 74.001176 - 0.013405 = 73.98777113$ , and  $UCL_{\bar{x}} = 74.001176 + 0.013405 = 74.014581$  mm. The  $\bar{x}$ -chart from Minitab is given below. The following  $\bar{x}$ -chart clearly shows that every  $\bar{x}_i$  is well inside control limits, and further, the chart pattern seems completely random, and thus the between-sample variation is also in good statistical control. The above control



limits may be used to monitor future production, as long as process FNC is tolerable. However, a periodic review (or revision) of control limits is highly recommended on a monthly (if a process is very stable) or weekly basis.

So far we have established that the data of Example 6.3 of D. C. Montgomery



do not exhibit any points out of control and they also exhibit random pattern, and thus we may conclude that there are no special (or local) causes of variation in the process such as broken tool, contaminated raw material, wrong machine settings, and personnel with insufficient training, etc. Note that local (or sporadic) problems can generally be corrected simply by operators and/or immediate supervisors. Further, if there are no special or assignable causes of variation, then the process is governed only by a system of common causes of variation.

On the other hand, system problems can be corrected only by teams of Plant, QC, and Production managers. Typical system problems are (1) Poor workstation design, (2) Poor lighting, (3) Poor training and supervision, (4) Poor Methods, and (5) Poorly maintained machines or machines W/O sufficiently tight tolerances. At this point, because Dr. W. Edwards Deming was the “Father” of institutionalization of SPC throughout the world (starting in Japan after World War II), we will state Deming’s 14 Points that management must institute for the prime purpose of never-ending quality and productivity improvement.

- (1) Innovate and allocate resources in such a manner as to fulfill the long-term needs of the company and its customers, rather than short-term profitability.
- (2) Discard the old philosophy of tolerating and accepting defective products.
- (3) Eliminate dependence on mass inspection for quality control.
- (4) Reduce the number of multiple source suppliers.
- (5) Use statistical methods to identify the two sources of waste: Local Faults (between 15% to 20%), and System Faults (about 80 to 85%).
- (6) Institute more thorough and better job training.



- (7) Provide supervision that includes statistical training.
- (8) Reduce fear throughout the organization by encouraging open, two-way, non-punitive communications.
- (9) Reduce waste by encouraging design and research engineers to learn more about production problems.
- (10) Eliminate goals and slogans to encourage productivity unless training and top management support is also provided.
- (11) Closely examine the impact of work standards. Do they consider quality or help anyone do a better job?
- (12) Institute a broad, basic and elementary statistical training on a company-wide scale.
- (13) Institute a vigorous program for retraining people in new skills.
- (14) Make maximum use of statistical knowledge and talent in one's company.

Before closing this section, it is almost impossible to mention Deming's name as the key quality Guru of the twentieth century, and not to mention Dr. Joseph M. Juran's contributions to the field of statistical QC, along with many other notables such as Armand V. Feigenbaum (TQC, 1951) and Philip Crosby. Dr. Juran worked with Dr. Walter A. Shewhart, the originator of control charts, at the AT&T Bell labs during the late 1940's and 1950's. Dr. Juran's philosophy is more focused on managerial aspects of quality improvement (QI) and shares Dr. Deming's view that at least 80% of quality problems are system type and that only top management can address such problems. The overall Philosophy of Deming and Juran can be summarized in an acronym called "TQM", which stands for Total Quality Management. Dr. Juran refers to Deming's system faults as chronic problems, and he refers to Deming's special faults as sporadic problems. On the other hand, Dr. Crosby is probably the most successful quality consultant and he does have his own 14 points, which are not far different from those of Deming's. Crosby does have four Absolutes of Quality: (1) Quality is conformance to customer requirements, and therefore, there is absolutely no reason to sell faulty products. (2) A quality system must be based on prevention rather than detection of nonconforming units. (3) The performance standard must be zero defects. (4) The

measure of quality is the price of nonconformance. Crosby states that the cost of quality is divided in two areas: the price of nonconformance (PONC) and the price of conformance (POC). PONC is the price of not doing it right the first time. POC is the sum of all the costs associated with quality efforts, such as prevention measures and education (SPC & QI, By J. M. Smith, 4<sup>th</sup> Ed., 2001, Prentice Hall, pp. 28-32).

All three (Deming, Juran, and Crosby) emphasize total management commitment to quality through a prevention (rather than detection) system. There is a bit of philosophical difference between Deming and Crosby in that Deming opposes slogans while Crosby likes posters and zero-defect concept.

### Out-of-Control Criteria and Patterns

- (1) One point outside the 3-Sigma control limits (for sudden shifts)
- (2) A run of at least 7 successive points below or above the CNTL
- (3) A run-down or run-up of length at least 6 indicates a very high probability of a downward or upward trend, respectively, on a control chart
- (4) Two successive points or 2-out-of-3 points in the region of ( - 3-Sigma, - 2-Sigma) or (+2-Sigma, +3-Sigma) signals a high probability of an out-of-control process.

If a control chart is revised once for assignable causes, but then now there is one or two points which were originally in control go out-of-control, then the experimenter must check the once-revised chart by the above Criteria (2), (3) and (4) for lack of randomness.

### **How Do We Ascertain that a Process Suffers From System (or Chronic) Problems?**

This is a difficult question to answer, and I am not certain that my answer to such a question is totally accurate. My recommendation is as follows. After the use of control charts identifies all local (or special) faults and all assignable causes of variation are removed from a process by operators and/or immediate supervisors,

then process variability should be governed only by system problems that are the common sources of variation. Then as a second step, we must do a process capability study from our control chart data to ascertain if the process yield, defined as  $(1 - p)$ , meets company-wide standards. For example, if tolerable process FNC company-wide is  $\alpha = 100$  ppm (parts per million), but our system problems are producing a FNC =  $p = 500$  ppm, then there are system problems, and only management can institute further QI for the process. I am not certain that my partial answer to the above question is adequate, but to illustrate this concept I will do a process capability analysis of the Example 6-3 of D. C. Montgomery's 7<sup>th</sup> edition, assuming that the random variable piston inside diameter is  $N(74.001176, 0.00999140155^2)$  and tolerable FNC =  $\alpha = 100$  ppm (i.e., 0.01% FNC). We simply estimate  $p$  by computing  $\hat{Z}_L = (LSL - 74.001176)/0.00999140155 = -5.12200413 \rightarrow \hat{p}_L = 0.0^{615115263434}$ ;  $\hat{Z}_U = (USL - 74.001176)/0.00999140155 = 4.8866017201 \rightarrow \hat{p}_U = 0.0^{651295667011} \rightarrow \hat{p} = 0.0^{664109304451} \rightarrow \hat{p} = 0.66411$  ppm  $\ll \alpha = 100$ . Thus, it seems that there are no system problems because  $\hat{p} \ll \alpha = 100$  ppm. Note that my numerical answers differ a bit from those of Montgomery's (4<sup>th</sup> edition) listed in the middle of his page 216 because I carried more decimals in my computations. He gives  $\hat{p}$  roughly equal to 20 ppm while my answer is approximately 1 ppm. Therefore, the capability of the above process is estimated at  $(USL - LSL)/\hat{\sigma}_x = 0.10/0.00999140155 = 10.00860585$  sigma, or the process capability ratio is estimated to be  $PCR_{hat} = \hat{C}_p = 10.00860585 / 6 = 1.668101$ , where PCR stands for Process Capability Ratio). Since the above process is a bit off-centered based on the sample result, it is best to measure process capability from the capability index (or process performance index), which takes this into account to some extent, as computed below.

$$\hat{C}_{pk} = \frac{1}{3} \text{Min}(|\hat{Z}_L|, \hat{Z}_U) = \frac{1}{3} \text{Min}\left(\frac{|LSL - \bar{\bar{x}}|}{\hat{\sigma}_x}, \frac{USL - \bar{\bar{x}}}{\hat{\sigma}_x}\right)$$

$$= \frac{1}{3} \text{Min}\left(\left|\frac{73.95 - 74.001176}{0.0099914}\right|, \frac{74.05 - 74.001176}{0.0099914}\right) = \frac{4.8866017201}{3} =$$

1.62886724  $\leq \hat{C}_p$ . Note that  $\hat{C}_p$  disregards the position of the process mean relative to the ideal target  $m$ , while  $\hat{C}_{pk}$  to some extent takes into account the off-centering of the mean from the ideal target, which in this example is  $m = 74.000$  mm. To fully take the off-centering of the process into account, it is best to use the Taguchi quality concept to define the process capability index shown below.

$$C_{pm} = \frac{\text{Tolerance Range}}{6\sqrt{\sigma^2 + (\mu - m)^2}} = \frac{USL - LSL = 2\Delta}{6\sqrt{E(\text{Taguchi's QLF}/k)}} \quad (3)$$

For the example 6-3 of Montgomery, from equation (3) the estimated value of  $C_{pm}$  is given by

$$\hat{C}_{pm} = \frac{2 \times 0.05}{6\sqrt{0.0099914^2 + (74.001176 - 74)^2}} = 1.6566651 \leq \hat{C}_p.$$

## Motorola's Definition of 6-Sigma Process Quality

Motorola, due to global competition, instituted Six-Sigma Quality Program throughout the company for the quality of individual components (or individual parts) of a complex system in order to reduce rework, scrap and field failure. The six-sigma concept revolves around improving (or tightening) machine tolerances to the point that the design specifications (LSL and USL) are at least 6 standard deviations from the process mean  $\mu$  (i.e., a PCR of exactly equal to 2). I have not seen the original document from Motorola, and therefore, I am not certain if they defined 6- $\sigma$  quality as the capability of a machine that can maintain at least one of the two specification limits at six STDEVs from the process mean  $\mu$  or from the ideal target  $m$ . If I had to venture a guess, I would say 6- $\sigma$  quality implies  $LSL = \mu - 6\sigma$ , and  $USL = \mu + 6\sigma$ . If a process is centered, then the point I am raising is totally moot and irrelevant because  $\mu = m$ , but if a process is way off-centered, say by two STDEVs, then it does make a difference how 6- $\sigma$  quality is defined. If a process is centered and is normally distributed and operates at 6- $\sigma$  quality (or at a process capability of 12-sigma), then on

the average of 0.0019731752901 ppm are nonconforming to design specifications. However, if a process is off-centered to the right such that  $\mu = m + 2\sigma$ , then a 6- $\sigma$  quality process on the average puts out 31.67124184 ppm NCUs (nonconforming units). If we try to compute this last 31.67124184 ppm NCUs using the 6- $\sigma$  concept where  $LSL = \mu - 6\sigma$  and  $USL = \mu + 6\sigma$ , then we would have to change our design specifications to  $LSL = (m + 2\sigma) - 6\sigma$  and  $USL = (m + 2\sigma) + 6\sigma$  as the process mean shifts, which does not make sense because design specifications are fixed and are generally determined by product developers/ engineers and customer requirements. So, it seems that Motorola should have used Taguchi's quality concept and defined "Six-Sigma Quality" as spec limits at a distance of  $6\sigma$  from the ideal target  $m$  (not  $\mu$ ).

**Exercise 17.** The data on my website give 28 subgroups each of size  $n = 4$ , where the response variable  $X$  represents shaft diameters with an ideal target of 0.500 inches and a tolerance range of 0.495-0.505. For your convenience I am providing the data on a spreadsheet on my website under Ex17. Obtain the R- &  $\bar{x}$ - charts, assuming assignable causes for any point out of control. Then, perform a complete process capability analysis, assuming  $\alpha = 0.002$ . You must draw the two control charts, only after removing points out-of-control, using Minitab and Excel and determine if there are system problems.

### The Average Run Length (ARL) for an $\bar{x}$ – Chart

Suppose the mean of a process is in statistical control on the Shewhart control chart at  $\mu = \bar{\bar{x}}$ , then the Pr of a false alarm on each sample of size  $n$  (most often  $n = 5$ ), assuming normality of  $\bar{x}$ , is equal to  $\Pr(|Z| > 3) = 2 \times 0.00135 = 0.0027$ , where  $Z \sim N(0, 1)$ . Therefore, as we take samples of size 5 from a normal process, which is in a state of statistical control, then on the average the out-of-control signal (being a Geometric process) will occur every  $1/0.0027 = 370.398347345$  sample, i.e., the average number of samples before a sample point  $\bar{x}$  gives a false alarm is equal to 370.40. The quantity  $1/0.0027 = 370.40$  is called the ARL at  $\mu_0$  and is denoted by  $ARL_0 = 1/0.0027 = 370.40$ . Note that the formula  $ARL_0 = 1/\alpha$  can easily be obtained from the fact that as we sample the process, say on an hourly basis, then we are going through

a geometric process where the Pr of a false alarm at each trial is  $\alpha$ , and hence the average number of Bernoulli trials required for the occurrence of the 1<sup>st</sup> false alarm is

$$\text{given by } \alpha + 2(1 - \alpha)\alpha + 3(1 - \alpha)^2\alpha + 4(1 - \alpha)^3\alpha + \dots = \alpha \sum_{i=1}^{\infty} i(1 - \alpha)^{i-1} = 1/\alpha.$$

Suppose now there is a shift in the process equaling to  $1.5\sigma_{\bar{x}} = 1.5\sigma_x/\sqrt{n}$ , then the Pr of catching this shift on the 1<sup>st</sup> sample after the shift has occurred is equal to  $1-\beta$ , which is computed as depicted in Figure 4 below. Figure 4 shows that the Pr of catching a  $1.5\sigma_{\bar{x}}$  shift (or  $z = 1.5$  standardized units) on the  $\bar{x}$ -chart is given

$$1 - \beta \cong \Pr(\bar{x} > \text{UCL}_{\bar{x}}) = \Pr\left(\frac{\bar{x} - \mu}{\sigma_{\bar{x}}} > \frac{\text{UCL}_{\bar{x}} - (\bar{\bar{x}} + 1.5\sigma_{\bar{x}})}{\sigma_{\bar{x}}}\right) = \Pr(Z > 1.5) = 0.06681,$$

and hence the value of the ARL at  $\mu = \bar{\bar{x}} + 1.5\sigma_{\bar{x}}$  is equal to  $\text{ARL}_1 = 1/0.06681 = 14.96844623 \cong 15$ , i.e., if there is a shift equal to  $1.5\sigma_{\bar{x}}$  in the process mean, then a Shewhart 3-sigma chart will on the average require 15 samples each of size 5 before a correct alarm is sounded on the  $\bar{x}$ -chart. Further, the Pr of catching the shift on the 5<sup>th</sup> sample after the  $1.5\sigma_{\bar{x}}$  shift has occurred is  $(0.93319)^4 \times 0.06681 = 0.05067$ , and

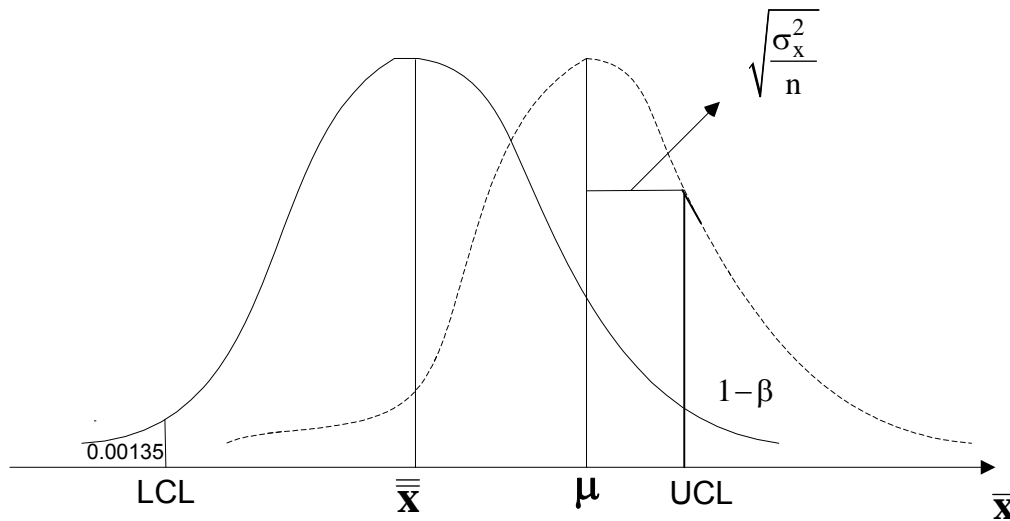


Figure 4

the Pr of catching the shift after the 5<sup>th</sup> sample is given by  $(0.93319)^5 = 0.70771$ .

Note that if a process is in a state of statistical control, then a run of length 2 within two-sigma (or warning) and 3-sigma limits has roughly an occurrence Pr of  $(0.02275 - 0.00135)^2 = 0.00045796 \ll 0.00135$ . Therefore, such an event on a Shewhart control chart would be highly significant because its occurrence Pr is less than 0.00135 and will require work stoppage in order to look for special causes of variation.

Exercise 18. (a) Compute the values ARL at a shift from  $\bar{\bar{X}}$  to  $\bar{\bar{X}} + (0.3 \sigma_{\bar{X}}, + 0.5 \sigma_{\bar{X}}, + 0.80 \sigma_{\bar{X}}, + 1 \sigma_{\bar{X}}, 1.4 \sigma_{\bar{X}}, 1.6 \sigma_{\bar{X}}, 2.0 \sigma_{\bar{X}}, 2.5 \sigma_{\bar{X}}, 3.00 \sigma_{\bar{X}})$  and graph ARL versus the amount of shift. (b) Determine what run length within either the interval ( - 1-Sigma, - 2-Sigma) limits or within (1-Sigma, 2-Sigma) limits on Shewhart chart would be significant at the 0.0027 level.

## S- and $\bar{X}$ -Charts

### (a) The case of Balanced Sampling Scheme

It can be shown (using the properties of  $\chi^2$ ) that for one random sample of size n from a normal universe the  $E(S) = (c_{4,n}) \times \sigma$  (this proof is a bonus problem for you worth 15

points), where the QC constant  $c_{4,n} = \sqrt{\frac{2}{n-1}} \times \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}$  lies in the interval

$[0.7978845608, 1)$  for all  $n \geq 2$  and the limit of  $c_{4,n}$  as  $n \rightarrow \infty$  is equal to 1. Further, this author has shown that for  $n \geq 20$ , the value of  $c_{4,n}$  can be approximated, to 5 decimals, by

$c_{4,n} \cong \frac{4n^2 - 8n + 3.876}{(4n - 3)(n - 1)}$ . These discussions imply that, in the long-run, the statistic S

underestimates the population standard deviation  $\sigma$ , and hence an unbiased estimator of  $\sigma_X$  for a normal universe is given by  $\hat{\sigma}_X = S/c_{4,n}$  [this is due to the fact that  $E(S) = c_{4,n} \times \sigma \rightarrow E(S/c_{4,n}) = \sigma_X$ ]. The reader must bear in mind that if  $E(S)$  were equal to  $\sigma$ , then  $V(S) \equiv 0$  and hence S would not be a random variable.

### (i) The Case of Targeted $\sigma$ and Balanced Design

If the desired value of  $\sigma$  is targeted at  $\sigma_0$  for which the sampling scheme must be balanced, then  $E(S) = c_{4,n}\sigma$  shows that the  $CNTL_S = c_4\sigma_0$ ; further, because  $V(S) = E(S^2) - [E(S)]^2 = \sigma^2 - (c_{4,n}\sigma)^2 = [1 - c_4^2(\text{at } n)]\sigma^2$ , then the  $SE(S) = \sigma_x \sqrt{1 - c_{4,n}^2}$ , where  $c_{4,n} = c_4(\text{at } n)$

$$= \sqrt{\frac{2}{n-1}} \times \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}$$

Hence, for the targeted  $\sigma$ , the  $SE(S) = \sigma_0 \sqrt{1 - c_{4,n}^2}$  and

$$SE(\bar{x}) = \sigma_0 / \sqrt{n}.$$

### (ii) The Case of Unknown and Untargeted $\sigma$ but Balanced

If all sample sizes are all equal ( $n_i = n$  for all  $i$ , or  $n > 15$ ), and the S-chart is being used to monitor process variability, then it is common to use

$$CNTL_S = \bar{S} = \frac{1}{m} \sum_{i=1}^m S_i$$

because both  $\bar{S}$  and  $S$  are biased estimator of  $\sigma$  with the same amount of bias due to the fact that all  $n_i$ 's are all equal; bear in mind what are being charted on an S-chart are  $S_i$  values each of which is a biased estimator, and hence a biased  $CNTL_S$ .

Furthermore, it can easily be shown that for the case of balanced design ( $n_i = n$  for all  $i$ )

$S_p^2 > (\bar{S})^2$ , where  $S_p$  is defined on the next page. This claim follows from the fact that

for the balanced case  $\sum_{i=1}^m S_i^2$  clearly must exceed  $\frac{1}{m} (\sum_{i=1}^m S_i)^2$  due to the fact that

$$\sum_{i=1}^m S_i^2 - \frac{1}{m} (\sum_{i=1}^m S_i)^2 = \sum_{i=1}^m S_i^2 - \bar{S} \sum_{i=1}^m S_i = \sum_{i=1}^m [S_i (S_i - \bar{S})] = \sum_{i=1}^m [S_i (S_i - \bar{S})] -$$

$$\bar{S} \sum_{i=1}^m (S_i - \bar{S}) = \sum_{i=1}^m [(S_i - \bar{S}) (S_i - \bar{S})] = \sum_{i=1}^m (S_i - \bar{S})^2 > 0.$$

Before computing the  $SE(S)$ , we must state that for  $m$  random subgroups each of identical size  $n$  from a normal universe, we may easily show that  $E(\bar{S}) = c_{4,n}\sigma$  so that an unbiased estimator of  $\sigma_x$  is given by  $\bar{S}/c_4$ . Further, if you are using Excel to compute the value of  $c_{4,n}$  for a given  $n$ , then  $\Gamma(n) = \exp(\text{gammaln}(n))$ . Because  $SE(S) = \sigma_x \sqrt{1 - c_{4,n}^2}$ ,



then the sample  $se(S) = (1 - c_{4,n}^2)^{1/2} \times \hat{\sigma}_x = (1 - c_{4,n}^2)^{1/2} \times (S/c_{4,n}) = \bar{S} \sqrt{(c_{4,n})^{-2} - 1}$ . Thus

the control limits for the case  $n_i = n$  for all  $i = 1, 2, \dots, m$  are given by

$$LCL_S = \bar{S} - 3\bar{S} \sqrt{(c_{4,n})^{-2} - 1} = [1 - 3\sqrt{(c_{4,n})^{-2} - 1}] \bar{S} = B_3 \bar{S} \quad (4a)$$

$$UCL_S = \bar{S} + 3\bar{S} \sqrt{(c_{4,n})^{-2} - 1} = [1 + 3\sqrt{(c_{4,n})^{-2} - 1}] \bar{S} = B_4 \bar{S} \quad (4b)$$

Note that the  $LCL_S = 0$  when  $2 \leq n \leq 5$ , but  $LCL_S > 0$  when  $n > 5$ . The quantity  $1 -$

$3\sqrt{(c_{4,n})^{-2} - 1}$  on the RHS of (4a) is denoted by  $B_3$  in QC literature and the QC

constant  $B_4 = 1 + 3\sqrt{(c_{4,n})^{-2} - 1}$ . Only for the balanced case

$$\begin{aligned} \text{CNTL}_{\bar{X}} &= \bar{\bar{X}} = \sum_{i=1}^m \bar{x}_i / m \\ \text{LCL}_{\bar{X}} &= \bar{\bar{X}} - 3 \frac{\bar{S}}{c_{4,n} \sqrt{n}}, \quad \text{and} \quad \text{UCL}_{\bar{X}} = \bar{\bar{X}} + 3 \frac{\bar{S}}{c_{4,n} \sqrt{n}}. \end{aligned} \quad (5a)$$

## (2) The Case of Unknown and Untargeted $\sigma$ but Unbalanced

If subgroup sample sizes differ and /or  $n > 15$ , then process variation must be monitored by an S-Chart. The most common occurrence of an S-Chart is when the sampling design is not balanced, i.e.,  $n_i$ 's ( $i = 1, 2, \dots, m$ ) are not the same, then the experimenter has no option but to use an S-Chart for the control and monitoring of variation. The central line for the unbalanced case is given by

$$\text{CNTL}_S = S_p = \left[ \frac{\sum_{i=1}^m (n_i - 1) S_i^2}{\sum_{i=1}^m (n_i - 1)} \right]^{1/2} = \left[ \sum_{i=1}^m \text{CSS}_i(W) / (N - m) \right]^{1/2},$$

where  $N = \sum_{i=1}^m n_i$  is the grand total number of random observations, and the quantity

$(n_i - 1) S_i^2 = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 = \text{CSS}_i(W)$  is called the corrected sum of squares within the

$i^{\text{th}}$  subgroup. The reader must be cognizant of the fact that when the sampling design

is unbalanced, then the value of  $S_p$ , which is biased, must be used to represent the CNTLs. When sampling design is unbalanced, then each point on the S-chart has its own control limit, which is approximately given by

$$LCL(S_i) = S_p - 3S_p \sqrt{(c_{4,n_i})^{-2} - 1} = [1 - 3\sqrt{(c_{4,n_i})^{-2} - 1}] \times S_p \quad (6a)$$

$$UCL(S_i) = S_p + 3S_p \sqrt{(c_{4,n_i})^{-2} - 1} = [1 + 3\sqrt{(c_{4,n_i})^{-2} - 1}] \times S_p. \quad (6b)$$

Once variability is in a state of statistical control (i.e., all sample  $S_i$ 's lie within their own control limits), then an  $\bar{x}$ -chart is developed to monitor the process mean. The central line of an  $\bar{x}$ -chart is given by

$$CNTL_{\bar{x}} = \bar{\bar{x}} = \frac{\sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij}}{N} = \frac{\sum_{i=1}^m n_i \bar{x}_i}{N}$$

where  $N = \sum_{i=1}^m n_i$ . For the case of differing sample sizes, since the  $V(\bar{x}_i) = \sigma^2/n_i$ ,  $se(\bar{x}_i)$

$= \hat{\sigma}_x / \sqrt{n_i} = S_p / \sqrt{n_i}$ , and as a result for the  $i^{\text{th}}$  subgroup

$$LCL(\bar{x}_i) = \bar{\bar{x}} - 3 \frac{S_p}{\sqrt{n_i}}, \quad \text{and} \quad UCL(\bar{x}_i) = \bar{\bar{x}} + 3 \frac{S_p}{\sqrt{n_i}}. \quad (5b)$$

Note that if the sampling design is unbalanced, then all points on the S- and  $\bar{x}$ -charts have the same CNTL but every point on both charts has its own control limits due to differing sample sizes. This implies that in the formulas (6) and (5b) the control limits vary according to the size of the sample,  $n_i$ , in the  $i^{\text{th}}$  subgroup because the value of  $c_{4,n_i}$  depends on  $n_i$ . It should be clear that the subgroups with larger sample sizes have tighter control limits.

## Shewhart Control Chart for Fraction Nonconforming (the p-Chart)

As an example, consider an injection molding process that produces instrument panels for an automobile. The occurrence of splay, voids, or short shots will make the panel defective. Thus, we have a binomial process where each panel is classified as defective (i.e., 1) or as conforming (i.e., 0). The binomial rv,  $X$ , represents the number

of nonconforming panels in a random sample of size  $n_i$ , where it is best to have at least  $m > 20$  subgroups in order to construct the (preliminary or trial) p-chart, where  $p$  is the FNC of the process. The sample FNC is given by  $\hat{p} = X/n$ , and if  $n > 30$  and  $np$  and  $nq > 10$ , then the SMD of  $\hat{p}$  is approximately normal with mean  $p$  and  $SE(\hat{p}) = \sqrt{\frac{pq}{n}}$ , where  $q = (1 - p)$  is the process fraction conforming (or process yield). The central line is given by

$$CNTL_p = \frac{\sum_{i=1}^m X_i}{\sum_{i=1}^m n_i} = \frac{\sum_{i=1}^m n_i \hat{p}_i}{N} = \bar{p} \quad (7)$$

where  $N = \sum_{i=1}^m n_i$  is the total number of units inspected by attributes in all  $m$  samples,

$X_i$  represents the number of NC units in the  $i^{\text{th}}$  subgroup, and  $\hat{p}_i = X_i / n_i$  is the sample

FNC of the  $i^{\text{th}}$  subgroup. Since the estimate of the  $se(\hat{p}_i) = \sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}}$ , then the

control limits for the  $i^{\text{th}}$  subgroup is given by  $LCL_i(\hat{p}) = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}}$ , and  $UCL_i(\hat{p})$

$= \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}}$ . Note that, when subgroup sizes differ on a Shewhart p-chart, then

every sample FNC,  $\hat{p}_i$ , has its own control limit. If the difference between maximum and minimum sample sizes do not exceed 10 units, then a p-chart based on average sample size should be constructed for monitoring process FNC. In all cases the central line stays the same, but the average control limits simplify to  $LCL(\hat{p}) = \bar{p} - 3 \times$

$\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}}$ , and  $UCL(\hat{p}) = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}}$ , where  $\bar{n} = \frac{1}{m} \sum_{i=1}^m n_i$ . The reader is

cautioned to the fact that if a p-chart based on an average sample size is used to monitor process FNC, then all points (i.e., all sample FNCs) that are close to their average control limits (whether in or out of control) must be checked against their own

limits to ascertain their control nature.

## Shewhart Control Chart for Number of Nonconformities per Unit (the u-Chart)

Since the construction of the u-chart is not as straight forward as the others discussed thus far, we will describe the methodology through an example. In practice, it is best to have at least  $m = 20$  subgroups to construct the trial control limits, but herein for simplicity we will use  $m = 10$  samples of differing sizes. Consider a textile process that produces oilcloth in lots of differing sizes (borrowed from *A. J. Duncan, 5<sup>th</sup> edition, pp. 471-473, Irwin Press*) measured in square meters. An inspector selects  $m = 10$  lots at random and counts the number of defects,  $c_i$ , in each lot (or sample). Duncan's data are displayed in Table 12. Note that in Table 12, because of different square meters, we have arbitrarily let 100 square meters equal to one unit, although 50, or, 10, or any other convenient square meters would work just as well. Further,  $u_i = c_i / n_i$  represents the average number of defects per unit. Note that because of differing ample sizes, it would be erroneous to compute the average number of defects per unit from  $\sum_{i=1}^m u_i / m = 7.0289$ , where this last formula would work only if all  $n_i$ 's were identical. The correct formula for the central line is given by

Table 12

Sample Number	i = 1	2	3	4	i = 5	6	7	8	9	10
Square Meters	180	150	120	90	150	160	120	140	130	175
$c_i$	9	15	6	5	16	10	4	12	14	9
$n_i$	1.8	1.5	1.2	0.90	1.5	1.6	1.2	1.4	1.3	1.75
$u_i$	5.0	10.00	5.00	5.556	10.667	6.25	3.333	8.571	10.769	5.143
$LCL_i$	1.123	0.5554	0.00	0.00	0.555	0.762	0.00	0.327	0.0724	1.038
$UCL_i$	13.012	13.579	14.348	15.474	13.579	13.372	14.348	13.807	14.062	13.096

$$\text{CNTL}_u = \bar{u} = \frac{\sum_{i=1}^m c_i}{\sum_{i=1}^m n_i} = \frac{\sum_{i=1}^m n_i u_i}{\sum_{i=1}^m n_i} = \frac{100}{14.15} = 7.0671$$

It is well known that the random variable number of defects per unit,  $C$ , follows a Poisson distribution, and hence its variance is also given by  $E(C)$ , i.e.,  $V(C)$  can be approximated by  $\bar{u}$ . Unfortunately, the terminology and notation for a  $u$ -chart has been somewhat confusing in statistical and QC literatures and we anticipate no change. Therefore, herein we attempt to remedy the notational problem to some extent. First of all, the 5<sup>th</sup> row of Table 12 actually provides the average number of defects per unit for the  $i^{\text{th}}$  sample, and hence the proper notation for the 5<sup>th</sup> row should be  $\bar{u}_i$  (not  $u_i$  as is used in QC literature) because a bar is generally placed on averages in the field of Statistics. This implies that a  $u$ -chart is actually a  $\bar{u}$ -chart because it is the average number of defects per unit that is plotted on this chart. Secondly, the central line should be called  $\bar{\bar{u}}$  because the CNTL gives the weighted grand average of all average number of defects per unit. These discussions lead to the fact that firstly  $V(\bar{u}) = V(C)/n$ , and secondly  $V(\bar{u})$  can be estimated by  $\bar{\bar{u}}/n$ . Since we do not wish to deviate from QC literature terminology, we will stay with the existing notation and let  $u_i$  represent the average number of defects per unit with the CTL as  $\bar{u}$  and the  $se(u) = \sqrt{\bar{u}/n_i}$ . Thus, the  $LCL_i(u) = \bar{u} - 3\sqrt{\bar{u}/n_i}$ , and  $UCL_i(u) = \bar{u} + 3\sqrt{\bar{u}/n_i}$ . The values of control limits for all the  $m = 10$  samples are provided in the last two rows of Table 12. Table 12 clearly shows that each  $u_i$  is well within its own control limits, implying that the process is in a state of excellent statistical control. Further, in all cases when the value of LCL became negative, a zero LCL was assigned in row 6 of Table 12.

This example provides a good illustration of a process that is in an excellent state of statistical control, but one that is in all  $Pr$  not capable of meeting customer specifications due to the fact that  $\bar{u} = 7.0671$  is too large and customers in today's global market will generally demand lower average number of defects per unit. If this manufacturer's management does not improve its process capability through QI methods by removing some of the system problems, it may not survive very long in global competition.

Finally, SPC is not a QI tool but simply an on-line procedure to monitor process quality and to identify where the quality problems lie. After problems are identified, then off-line methods (DOE or Taguchi Methods) can be applied to further fine-tune a process.

**Exercise 19.** The metal body of a spark plug is made by a combination of cold extrusion and machining. The occurrence of surface cracking following the extrusion process has been shown by a Pareto diagram to be responsible for producing over 90% of all the defective parts. During one shift, 25 subgroups each of differing sizes were collected, and the corresponding data are on my website on an Excel file under Exercise19. Conduct a complete p-chart analysis, assuming Shewhart assignable causes for all points out of control. (b) Obtain an S-Chart for the data of Exercise 17 and ascertain if every  $S_i$  is in a state of statistical control.