The Moving Average Control Charts

Suppose that a QCH, X, has a Laplace-Gaussian distribution according to N(μ, σ^2). We consider two possibilities just like the case of EWMA charts. (1) The CNTL is targeted at μ_0 with known process variance σ^2. (2) The CNTL has to be estimated from an initial subgroup of size m, σ^2 is unknown and also has to be estimated from the corresponding moving ranges.

(1) The case of targeted CNTL at μ_0, known σ^2, and n =1

As an example, consider the data on the proportion of un-reacted lime (CaO) given on my website, under the name CaO-MAs, that I borrowed from the text by E. L. Grant & R. S. Leavenworth (1998, 6e, pp. 318-323, McGraw-Hill, ISBN:0-07-024117-1) for m = 30 individual subgroups, and the authors used the most-common moving ranges and averages of span (or width) *W* = 3, while I have also added spans *W* = 2 & 5. Further, the authors state on their page 319 that the targeted μ_0 = 0.170 and $\bar{\mathbf{R}}$ = 0.065 were obtained from the previous two months (July and August) of continuous daily operations, and they list the September and part of October data in their Table 9-5, pp. 320-321 to set up trial control limits, I surmise, for the month of November. Therefore, for the CaO-MAs Example listed on my website, the value of μ_0 = 0.170, and because *W* = 3, $\sigma_0 = 0.065/d_2 = 0.065/1.693 = 0.0383934$. Note that because σ_0 = 0.0383934 is the target, then it will be used as the known value of σ even if $W \neq 3$.

To better understand moving averages, we compute their values using the CaO data at days 7 and 8. The MA of span (or width) *W* = 4 at time t = 7 is defined as $MA_7(W = 4) = \dfrac{x_7 + x_6 + x_5 + x_4}{4}$ = 0.16250, while $MA_8(W = 4) = \dfrac{x_8 + x_7 + x_6 + x_5}{4}$ = 0.15750; clearly, these two consecutive MA's are not independent. I will show how to compute their Covariance on the following page.

In general, a moving average of span *W* at time t, for t ≥ *W*, is defined as

$$\text{MA}_t(W) = \frac{x_t + x_{t-1} + ... + x_{t+1-W}}{W} = \frac{1}{W}\sum_{i=t+1-W}^{t} x_i \tag{22}$$

Note that, unlike Shewhart's 3-sigma charts, for t ≥ W, the points MA_t , MA_{t-1} …, and MA_{t-W+1} on moving range and average charts are correlated, and hence runs of length L, denoted RL, do not have the same statistical significance as they do on 3-Sigma Shewhart charts. For example, using the CaO data, the covariance between MA_9 and MA_6 at the span W = 5 is computed as follows:

$\text{COV}(\text{MA}_9, \text{MA}_6) = \text{COV}(\frac{1}{5}\sum_{i=9+1-5}^{9} x_i , \frac{1}{5}\sum_{i=2}^{6} x_i) = \text{COV}(\frac{1}{5}\sum_{i=5}^{9} x_i , \frac{1}{5}\sum_{i=2}^{6} x_i) = 2\sigma^2/25 =$ 0.00011792416, while the $\text{COV}[\text{MA}_{11}(4), \text{MA}_8(4)] = \text{COV}[\frac{x_{11} + x_{10} + x_9 + x_8}{4}, \frac{x_8 + x_7 + x_6 + x_5}{4}] = \sigma^2/16 = 0.00009212825$, where $\sigma_0^2 = (0.0383934)^2 = 0.001474052$.

When μ is targeted at μ_0 and σ at σ_0, then for any span W, the CNTL is set at μ_0, and to obtain the 3-Sigma control limits, we apply the Variance-Operator to Eq. (22).

$$V[\text{MA}_t(W)] = V(\frac{1}{W}\sum_{i=t+1-W}^{t} x_i) = \frac{1}{W^2}\sum_{i=t+1-W}^{t} V(x_i) = \frac{1}{W^2}(\sum_{i=t+1-W}^{t} \sigma_X^2)$$

$$= \frac{1}{W^2}(W\sigma_X^2) = \sigma^2 / W \rightarrow \text{SE}[\text{MA}_t(W)] = \sigma / \sqrt{W} \tag{23}$$

Using Eq. (23), the value of the correlation coefficient between $\text{MA}_9(5)$ and $\text{MA}_6(5)$ of CaO data at W = 5 is given by $\rho = \frac{2\sigma^2 / 25}{\sigma^2 / W} = 10/25 = 0.40$.

Eq. (23) shows that for a targeted MA control chart of any span W, the lower and upper control limits, for t ≥ W , are given by

$$\text{LCL}_{\text{MA}}(W) = \text{LCL}_{\text{MA}} = \mu_0 - 3\times\sigma_0 / \sqrt{W}, \quad \text{and} \quad \text{UCL}_{\text{MA}} = \mu_0 + 3\times\sigma_0 / \sqrt{W} \tag{24}$$

For the CaO data on my website, I have calculated the process SE's and the control limits for all 3 spans W = 2, 3 and 5 in the indicated columns of the Excel file. At W = 3 and t ≥ 3, the targeted SE is $\sigma_0 / \sqrt{W} = 0.0383934/\sqrt{3} = 0.022166431$, which results in $\text{LCL}_{\text{MA}} = 0.170 - 3\times 0.022166431 = 0.170 - 0.0664993 = 0.103501$, and the UCL_{MA} =

$0.170 + 3\times\sigma_0 / \sqrt{W}$ = 0.2364993, which are consistent with those of the authors' Figure 9-4 and those of Minitab's. Further, at spans W = 2 and 5, I have also assumed that the process standard deviation is known and still targeted at σ_0 = 0.038393, even if this was obtained at W = 3.

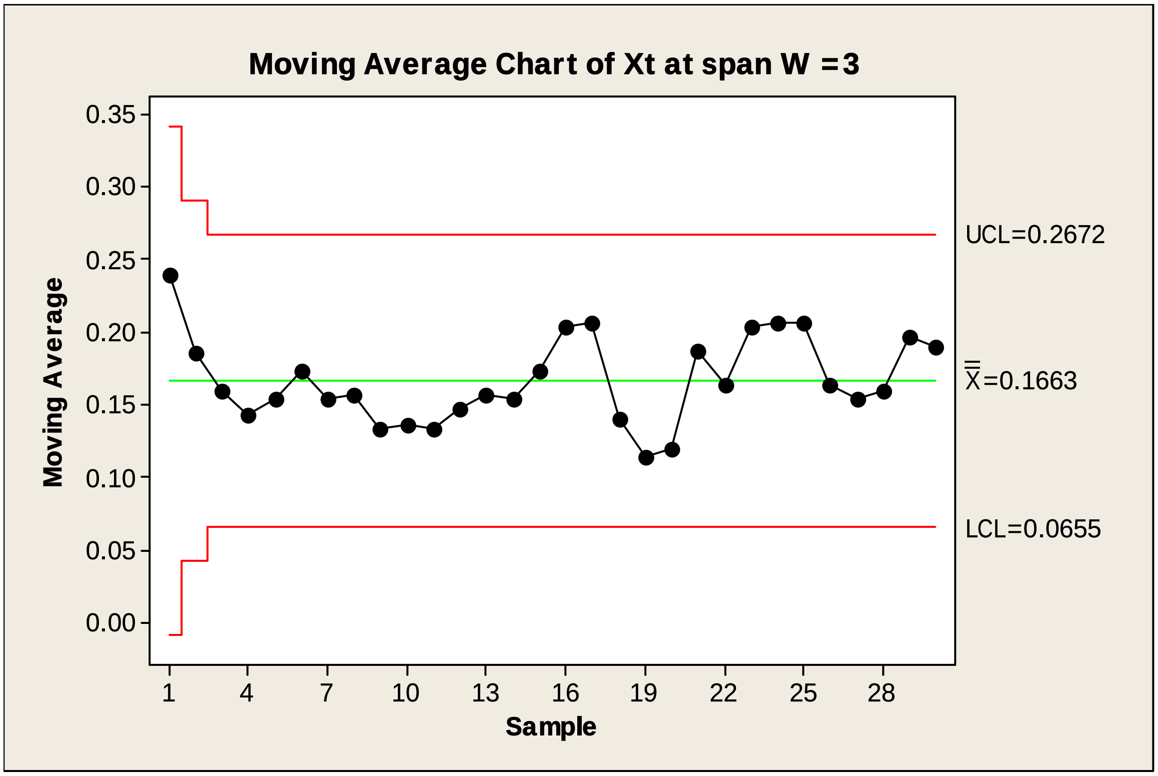
Minitab also provides moving average control limits for 1 ≤ t < W, whose standard errors are given by SE[MA_t(t < W)] = $\sigma / \sqrt{t}$. For example, at time t = 2, the control limits at span three are LCL_{MA}(t =2) = 0.170 − 3×0.0383934/$\sqrt{2}$ = 0.0885553, while the UCL_{MA}(t =2) = 0.170 + 3×0.0383934/$\sqrt{2}$ = 0.25144467. These are in precise agreement with Minitab's output, also posted on my website.

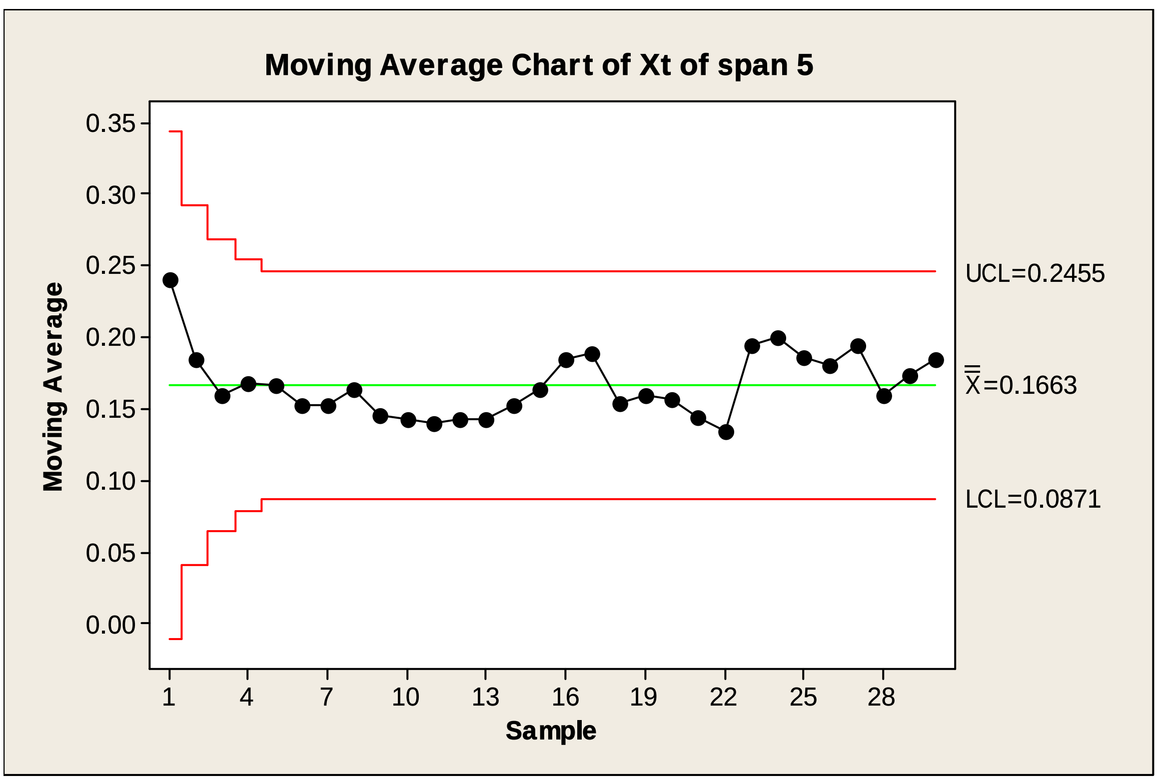
(2) The case of Estimated CNTL at $\bar{X}$, Estimated σ^2, and n =1

Again as an example, consider the data on proportion of un-reacted lime (CaO) given on my website under CaO that I borrowed from the book by E. L. Grant & R. S. Leavenworth (1998, 6e, pp. 318-323, McGraw-Hill, ISBN:0-07-024117-1) for m = 30 subgroups, where the authors provide only the targeted MA-chart of span (or width) W = 3, while now I will also obtain the control limits for the CaO data when the CNTL is set at $\bar{X}$ and σ is estimated from $\overline{MR} / d_2$. From Eq. (23), the estimate of the SE of MA_t at span W is given by

$$se[MA_t(W)] = \hat{\sigma}_x / \sqrt{W} = \overline{MR} / (d_2\sqrt{W}) \tag{25}$$

At the span W = 5, my spreadsheet shows that the estimated SE is given by se[MA_t(5)] = 0.137308/(2.326×$5^{1/2}$) = 0.0590317/$\sqrt{5}$ = 0.0263998, for all t ≥ 5. Recall that the d_2 values for the span W = 2, 3, 4, and 5 are given by 1.128, 1.693, 2.059, and 2.326, respectively. The MA control chart at W = 3 and 5 from Minitab are provided on the next page.


Moving Average Chart of Xt at span W =3
Moving Average
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
UCL=0.2672
X̿=0.1663
LCL=0.0655
1
4
7
10
13
16
19
22
25
28
Sample


Moving Average Chart of Xt of span 5
Moving Average
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
UCL=0.2455
X̿=0.1663
LCL=0.0871
1
4
7
10
13
16
19
22
25
28
Sample