INSY 7300-6   Reference : Chapter 9 of Montgomery(8e)   **Maghsoodloo**

For the sake of illustration, consider the Tool Life experiment in Example 5.5 on pages

214-219 of Montgomery(8e). We first present the layout of the experiment below in the OA

format, where A = Tool Angle at 3 levels 0, 1, 2 [A = $(x_1 - 15)/5$], and factor B = Cutting Speed [B

= $(x_2 - 175)/25$], also at 3 levels 0, 1, and 2.   Note that Montgomery has a typo in Table 5.16 on

p. 214; change Total Angle to Tool Angle. Such an experiment is called a 3^2 factorial with n = 2

observations per cell and the order of the N = 2×9 = 18 observations were completely

randomized. Note that 0, 1, & 2 are the elements of base-3 algebra; it is sufficient for a factor

to be either qualitative or quantitative with equi-spacing levels before we can represent its

levels by 0, 1, & 2.   Table 16 below represents the data in the OA (Orthogonal Array) format.

Table 16		AB =J(AB)	AB^2 = I(AB)		
A	B	$x_1 + x_2$	$x_1 + 2x_2$	FLC	y_{ijk}
0	0	0	0	00	−2, −1
0	1	1	2	01	−3, 0
0	2	2	1	02	2, 3
1	0	1	1	10	0, 2
1	1	2	0	11	1, 3
1	2	0	2	12	4, 6
2	0	2	2	20	−1, 0
2	1	0	1	21	5, 6
2	2	1	0	22	0, −1

$$y_{...} = 24.00$$

As always, the USS = $\sum\limits_{i=1}^{3} \sum\limits_{j=1}^{3} \sum\limits_{k=1}^{2} y_{ijk}^2$ = 156.00 (with 18 df), the CF = $y_{...}^2$/N = 24^2/18 =

32.00 (with 1 df) $\rightarrow$ SS$_T$ = SS(Total) = USS − CF = 124.00 (with 17 df).   SS(Pure Experimental

Error) = 0.50 + 4.5 + 0.50 + 2.0 + 2.0 + 2.0 + 0.5 + 0.50 + 0.50 = 13.00 = SS$_{PE}$ (with 9 df) $\rightarrow$

SS(Model) = 124 − 13 = 111 (with 17 − 9 = 8 df).

$$SS(A) = \frac{(-1)^2 + 16^2 + 9^2}{6} - CF = 24.33333 \text{ (with 2 } df\text{). Similarly, SS(B) = 25.33333}$$

(with 2 df) $\rightarrow$ SS(A×B) = SS(Model) − SS$_A$ − SS$_B$ = 61.33333 (with 4 df).

Since the contrast function for the 3rd column of Table 16, that is the AB column, is ξ = A + B = x_1 + x_2 and for all base-3 designs the values of ξ can be only 0, 1, or 2, then we can compute the AB component of A×B interaction as ($AB_0^2 + AB_1^2 + AB_2^2$)/6 − CF, i.e.,

$$SS(AB) = \frac{(-3+10+11)^2 + (-3+2-1)^2 + (5+4-1)^2}{6} - CF = 33.33333 \text{ (with 2 } df\text{), and}$$

similarly SS(AB^2) = (0^2 + 18^2 + 6^2)/6 − CF = 28.00 = SS[I(AB)]   (with 2 df).

Note that AB = J(AB) and AB^2 = I(AB) are 2 orthogonal components of the A×B interaction because SS(AB) + SS(AB^2) = 33.3333 + 28 = 61.3333 = SS(A×B). Further, unlike base-2 designs, all columns in base-3 designs must have 2 df. Since the design matrix in Table 16 has 9 rows (or 9 distinct design points, and therefore 8 df for all effects in the model), then we can write only 4 independent columns, namely A, B, AB, and AB^2 each with 2 df. This implies that A^{2}B^2 = AB and A^{2}B = AB^2. The statistical convention is that the 1st letter should not be squared; thus, the notations AB and AB^2 are conventional while A^{2}B^2 and A^{2}B are not, although these are used by Genichi Taguchi (one of Quality Gurus of the 20th Century).

Before discussing the 3^3 Design, we must assert that unlike base-2 designs the question "about the magnitude of the effect of a 3-level factor A " is ambiguous unless we specify whether we are seeking the linear or quadratic effect of factor A. When the levels of a quantitative factor in base-3 designs are equi-spaced, there are two orthogonal effects of that factor, namely Linear and Quadratic effects. The linear contrast for factor A of Example 5.5 on page 214 of Montgomery(8e) is A$_L$ = − 1×(−1) + 0 ×16 + 1×9 = 10.00, and its quadratic effect is A$_Q$ = 1×(−1) −2(16) + 1×9 = − 24.00. Using the definition of SS(Contrast), we can compute the SS(A$_L$) = 10^2/(6×2) = 8.33333 and SS(A$_Q$) = (−24)^2/(6×6) = 16.00 $\rightarrow$ SS(A$_L$) + SS(A$_Q$) = 24.3333 = SS(A), as expected because the two contrast vectors [−1   0   1]′ and [1   −2   1]′ are indeed orthogonal.

Exercise 31.   For the Example 5.5, compute SS(B$_L$), SS(B$_Q$) and verify that these 2 components are orthogonal. (b) Compute the four orthogonal components of SS(A×B).

The 3³ Design

Consider Montgomery's Example 9.1 on pp. 398-402 of his 8th edition. We have an experiment involving 3 factors each at 3 levels with n = 2 observations per FLC. In order to write the OA for this design, we can write 3 arbitrary columns as depicted in Table 17. The design matrix in Table 17 provides 26 *df* for studying effects (because it has 27 distinct FLCs) and hence it has 13 = (26/2) independent (or orthogonal) columns each with 2 *df*.

Table 17 (The 3³ OA)

A	B	C	AB	AB²	AC	AC²	BC	BC²	ABC	AB²C	ABC²	AB²C²
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	2	1	2	1	1	2	2
0	0	2	0	0	2	1	2	1	2	2	1	1
0	1	0	1	2	0	0	1	1	1	2	1	2
0	1	1	1	2	1	2	2	0	2	0	0	1
0	1	2	1	2	2	1	0	2	0	1	2	0
0	2	0	2	1	0	0	2	2	2	1	2	1
0	2	1	2	1	1	2	0	1	0	2	1	0
0	2	2	2	1	2	1	1	0	1	0	0	2
1	0	0	1	1	1	1	0	0	1	1	1	1
1	0	1	1	1	2	0	1	2	2	2	0	0
1	0	2	1	1	0	2	2	1	0	0	2	2
1	1	0	2	0	1	1	1	1	2	0	2	0
1	1	1	2	0	2	0	2	0	0	1	1	2
1	1	2	2	0	0	2	0	2	1	2	0	1
1	2	0	0	2	1	1	2	2	0	2	0	2
1	2	1	0	2	2	0	0	1	1	0	2	1
1	2	2	0	2	0	2	1	0	2	1	1	0
2	0	0	2	2	2	2	0	0	2	2	2	2
2	0	1	2	2	0	1	1	2	0	0	1	1
2	0	2	2	2	1	0	2	1	1	1	0	0
2	1	0	0	1	2	2	1	1	0	1	0	1
2	1	1	0	1	0	1	2	0	1	2	2	0
2	1	2	0	1	1	0	0	2	2	0	1	2
2	2	0	1	0	2	2	2	2	1	0	1	0
2	2	1	1	0	0	1	0	1	2	1	0	2
2	2	2	1	0	1	0	1	0	0	2	2	1

As an example, I generated the 13th column under AB²C² by using its contrast function ξ(AB²C²)

$= x_1 + 2x_2 + 2x_3$ and by adding the column under A, twice the column under B, and twice the column under factor C. Or, it is even simpler to add the AB column with the BC^2 column (mod 3). To compute the $SS(AB^2C^2)$, we must 1st add all observations in the FLCs (000, 012, 021, 101, 110, 122, 202, 211, 220), where each of these 9 FLCs have $\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3 = 0$ (mod 3); this will yield $(AB^2C^2)_0 = -149$. Second, we must add all y_{ijkr} values obtained under the FLCs (002, 011, 020, 100, 112, 121, 201, 210, 222); this will yield $(AB^2C^2)_1 = 212$. The reader should verify that $\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3 = 1$ for each of these last 9 FLCs in Table 17. Similarly, adding all y_{ijkr} for which $\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3 = 2$ yields $(AB^2C^2)_2 = 102$. Hence,

$$SS(AB^2C^2) = \frac{(-149)^2 + 212^2 + 102^2}{18} - \frac{165^2}{54} = 3804.11111 .$$

Exercise 32. (a) For the Example 9.1 of Montgomery use the above procedure to compute $SS(ABC)$, $SS(AB^2C)$, $SS(ABC^2)$ and then compare your answers against those on pages 401-402 of Montgomery(8e). (b) Use the same procedure to compute $SS(BC^2) = SS[I(BC)]$.

Block Confounding in Base-3 Designs

As an example, suppose it is desired to study the effects of 3 equi-spaced quantitative factors (each at 3 levels) on the mean of density, y, of ceramic tiles, but we can complete only 9 experiments per day. Therefore, days form 3 blocks and we have to be careful in designing the experiment in order not to lose the effects of our main factors A, B, C , and if possible, their three 2-way interactions to blocks. Since the three days will absorb 2 *df*, then we can confound any one of the 4 (ABC, AB^2C, ABC^2, AB^2C^2) orthogonal components of A×B×C with blocks. Note that the number of *df* confounded with blocks must be identical to the number of *df* that blocks possess. We have 4 choices: ABC, AB^2C, ABC^2, and AB^2C^2; because each one of these 4 orthogonal components of A×B×C has 2 *df*, any one of them can be used to generate the three blocks of 9 FLCs each. Let's settle on sacrificing g = ABC^2 as the generator of our blocks. Again we can do this in 2 ways: the hard way, or the easy way thru an OA. Since each block has $3^{3-1} = 9$ FLCs, we can write 2 columns arbitrarily. Then, we will use our contrast function $\xi (ABC^2) = x_1 + x_2 + 2x_3$ in order to generate the PB first, i.e., we require that $x_1 + x_2 + 2x_3 = 0$. Since for the PB, $x_1 + x_2 + 2 x_3 = 0$, then $x_1 + x_2 = -2x_3 = (-2 + 3)x_3 = x_3$ because $3 = 0$

(mod 3). Hence the levels of C are $x_3 = x_1 + x_2$. The PB is given in Table 18. To generate the ξ = 1 block, all we need is one FLC that belongs to this block, say "100". Then simply add this FLC to the last 8 elements of the PB. Thus, the ($\xi = 1$) block consists of the FLCs: 100, 111, 122, 201, 212, 220, 002, 010, 021.

Table 18. (The Principal Block with generator $g = ABC^2$ and $\xi = x_1 + x_2 + 2x_3 = 0$)

A	B	$x_3 = x_1 + x_2 = C$
0	0	0
0	1	1
0	2	2
1	0	1
1	1	2
1	2	0
2	0	2
2	1	0
2	2	1

The ($\xi = 2$) block consists of 200, 211, 222, 001, 012, 020, 102, 110, 121.

Exercise 33. Give the ANOVA outline for the ceramic tile example of the above 2 pages. (b) Work Problem 9.8 on page 445 of Montgomery(8e).

Fractional Factorials for Base-3 Designs

If the experimenter's resources are limited, and therefore, cannot conduct a complete replicate of a 3^k ($k \geq 3$) factorial, then s/he must settle for running a (1/3)rd, (1/9)[th], (1/27)[th], etc., FFD with quantitative factors at 3 equi-spaced levels. For example, a $(1/3)3^3$ FFD consists of dividing the 27 FLCs of the 3^3 design into 3 blocks of 9 FLCs each, selecting one of the 3 blocks at random, and then running the 9 experiments in a completely random order. Such a design is called a 3^{3-1} FF, which will need one generator to be selected from ABC, AB^2C, ABC^2,

and AB^2C^2 in order to maximize resolution. Because there are only 3 letters (A, B, C) in a 3^3 design, it is impossible in this case to generate a FFD with $R >$ III.

As in the case of confounding in blocks, it will be easiest to obtain a 3^{3-1}_{III} FFD by using an L_9 OA and generating the PB first. Suppose we let g = AB^2C^2; for the PB the value of contrast function $\xi = x_1 + 2x_2 + 2x_3$ must be set equal to zero. As before, 2 columns can be written arbitrarily, but the column pertaining to factor C must be obtained from $x_1 + 2x_2 + 2x_3 = 0$. This yields $x_1 + 2x_2 = -2x_3 = (-2 + 3)x_3 = x_3$, i.e., the levels of factor C must be obtained from A + 2B (mod 3). The PB is given in Table 19.

Table 19. (The Principal Block with g = AB^2C^2 & $\xi = x_1 + 2x_2 + 2x_3 = 0$.)

A	B	C = $x_1 + 2x_2 = x_3$
0	0	0
0	1	2
0	2	1
1	0	1
1	1	0
1	2	2
2	0	2
2	1	1
2	2	0

Once the PB is obtained, the ($\xi = 1$) and ($\xi = 2$) blocks can be generated by simply identifying only one FLC in these 2 blocks. The alias structure of the above FFD is given by A = BC = ABC (with 2 df), B = AC^2 = ABC^2 (with 2 df), C = AB^2 = AB^2C (with 2 df), and AB = AC = BC^2 (2 df).

Exercise 34. A company manufacturing engines was concerned about emission characteristic of the engines. To try to minimize various undesirable emission variables, the company proceeded to build and operate experimental engines varying five controllable

factors: throat diameter (A) at 3 levels 0, 1, 2; ignition system (B) at 3 levels 0, 1, 2; temperature (C) at 3 levels 0, 1, 2; velocity of the jet stream (D) at 3 levels, and timing system (E) at 3 levels. Since it was cost prohibitive to build $3^5 = 243$ experimental engines, it was decided to conduct a $(1/9)^{th}$ fraction of all the 243 runs. **(a)** Comment on the deficiency of the FFD with the generators $g_1 = AC^2D$ and $g_2 = ABC^2DE$. **(b)** Obtain the PB for the 3^{5-2} FFD using the generators $g_1 = AB^2C^2$ and $g_2 = ACDE^2$, with factors A and B on arbitrary columns (1) on (2), respectively, and C occupying column 3. If we keep the same $g_1 = AB^2C^2$ generator but select $g_2 = CD^2E^2$ as the 2nd independent generator to generate the 3^{5-2} FFD, would these last two independent generators yield as good a FFD as $g_1 = AB^2C^2$ and $g_2 = ACDE^2$? Then use the above PB (with design generators $g_1 = AB^2C^2$ and $g_2 = ACDE^2$) to generate the $(\xi_1 = 2, \xi_2 = 1)$ block. **(c)** Give the outline of the ANOVA table of your 3^{5-2} FFD with generators $g_1 = AB^2C^2$ and $g_2 = ACDE^2$, listing only the effects thru the 1st order. **(d)** Given the (STB) coded responses y: 15, 19, 30, 7, 14, 2, 8, 15, 20, 23, 35, 10, 14, 3, 11, 9, 5, 13, 19, 28, 20, 25, 15, 33, 16, 14, 22, in order of your OA listing for the PB, obtain the RT, compute each SS, and identify the nearly optimal setting X_O. [Hint: Refer to Table 17 on p. 84 of my notes]. **(e)** Can a 3^{5-3} FFD be constructed with a resolution of at least III? If no, explain exactly why?

     Exercise 35. Study the mixed-level designs on pp. 412-413 of Montgomery(8e). Suppose for the design matrix of Table 9.10 on page 413 the responses in the same run order are 6, 20, 10, 2, 18, 11, 6, and 17. Obtain a complete ANOVA table as in Table 9.11 of Montgomery's 8th edition.

Erratum for Chapter 9 of Montgomery's 8th edition

     Page 402, exchange SS(ABC) and SS(ABC^2), i.e., the correct value of SS(ABC) = 18.77778 and that of SS(ABC^2) = 584.11111.