Fractional Factorials (or Replicates) For Base-2 Designs

As the number of factors in a 2^k factorial experiment increases, the number of runs (or FLCs) required for one complete replicate rapidly outgrows the available resources of most experimenters. One full replicate of a 2^8 factorial requires 256 experimental runs (or experiments). The problem with such a 2^8 full factorial design is 2-fold:

(1) Perhaps too many observations (256 runs) giving rise to much (prohibitive) cost.

(2) Out of the 255 total *df*, only 8 are absorbed by the 8 main factors and only ${}_8C_2$ = 28 *df* are absorbed by the 1st-order interactions AB, AC, AD, ..., GH. The remaining 255 – 36 = 219 *df* (or 85.88%) correspond to the 3-way, 4-way, ..., 8-way interactions. These high- order interactions (HOIs), if found significant, are difficult to interpret and in some cases are assumed nonexistent. Therefore, sometimes unfortunately, their SS's are pooled together and used as the error (or actually residual) term in the ANOVA table.

The above problem worsens more rapidly for the 3^k and 5^k factorials; e.g., one full replicate of a 3^6 factorial requires 729 experiments (or FLCs), where only (72/728)$\times$100 % = 9.89% of the total *df* are absorbed by effects through the 1st order.

For a 2^k factorial, if it can reasonably be assumed that HOIs are not quite as important, then information on the main factors and 1st-order interactions can be gleaned by running only a FRACTION (i.e., 1/2, 1/4th , 1/8th, 1/16th, etc.) of a complete replicate. Such designs are called Fractional Factorials (or Fractional Replicates). Fractional factorial designs (FFDs) are widely used in industrial research for the prime purpose of fine-tuning a process and improving process and product quality. Process and quality engineers who do not understand how to use FFDs, and therefore, do not comprehend the enormous potential of such designs for continuous improvement, are at a terrible disadvantage compared to engineers who do. Manufacturing companies with managers and engineers who are not knowledgeable about the powers of DOE very rarely survive in global competition and are generally doomed to failure.

FFDs for base-2 can also be used as screening experiments where many factors (generally more than 5) are considered with the purpose of identifying those factors that have large linear effects relative to other factors. The influential factors, i.e., those with large effects on the mean of output y, are more completely investigated in subsequent experiments either with a full factorial design or with a FFD in base 3 or 5.

For more advantages of FFDs, study the Sections 8.1 & 8.2 on pages 320-333 of Montgomery's 8th edition.

THE 1/2 FRACTION OF A 2^k FACTORIAL DESIGN

Consider a 2^4 factorial (4 factors each at 2 levels 0 & 1) and suppose that the experimenter has sufficient resources to conduct only 8 experiments. Because 8 = $\frac{1}{2}2^4$, this leads to a 1/2 fraction of a full replicate of a 2^4 factorial, and clearly $(1/2)2^4 = 2^{-1}2^4 = 2^{4-1} = 8$. Therefore, we will have 2 fractions (or blocks) each with 8 FLCs. Just like the case of block confounding, we have to be very careful not to lose the effects of main factors and 1st-order interactions. Further, in order to maximize design resolution (defined later), it is generally best to sacrifice the highest order interactions to generate the two blocks. Since the 2 fractions (or blocks) carry 1 *df* between them, we must sacrifice exactly 1 *df*, namely only one of the 5 interactions ABC, ABD, ACD, BCD, or ABCD. If we use one of the four 3-way interactions as the design generator, g, then we have a resolution III design, denoted by $2^{4-1}_{\mathbf{III}}$. This is due to the fact that any one of the four WORDS (ABC, ABD, ACD, BCD) are represented by 3 letters and hence a resolution III design. To maximize resolution, we decide to use g = ABCD as the generator in order to attain a resolution IV design (i.e., 4 letters in the word ABCD)! Note that the resolution of a FFD is simply the minimum number of letters in the defining relation I, where I consists of all the design generators.

The contrast function (CF) for sacrificing the effect of g = ABCD is $\xi = x_1 + x_2 + x_3 + x_4$, whose values for base-2 designs can be only 0 or 1. The use of this CF (contrast function) and mod 2 algebra leads to the following 2 fractions (or blocks) each with 8 FLCs.

$\boldsymbol{I}$ = Identity element = ABCD

The Principal Block (PB) in which $\xi = 0$:
(1), ab, ac, bc, ad, bd, cd, abcd

Further, the fraction for which $\xi = 0$ is called the principal block for all FFDs, and recall that the notation "*bd*" represents the FLC (0, 1, 0, 1); *abc* = (1, 1, 1, 0), *d* = (0, 0, 0, 1), etc.

I = Identity element= –ABCD

The Fraction with $\xi = 1$: *a, b, c, abc, d, abd, acd, bcd*

Exercise 23. Use the signs under the ABCD column of Table 6-11 at the bottom of page 258 of Montgomery(8e), reproduced on page 52 of my notes, to show that, for the above $(1/2)2^4$ FFD with the generator g = ABCD, the ABCD interaction is also the identity of the design, i.e., *I* = +ABCD for all 8 FLCs in the $\xi = 0$ fraction and I = – ABCD is the identity for all FLCs in the $\xi = 1$ block. Note that Montgomery refers to the block for which I = +ABCD as the principal fraction.

To understand what information is lost when only a (1/2) fraction (say the PB) of a 2^4 factorial is run, consider the Table 6.11 on page 258 of Montgomery, where the 16 FLCs of a complete replicate is listed. Recall that –1 refers to the low and +1 to high level of a factor. We now list all the 8 FLCs that belong to the principal block: *(1), ab, ac, bc, ad, bd, cd, abcd.* Then in Table 13 atop the next page we construct the signs for each of the 15 effects just like in Table 6.11 on page 258 of Montgomery(8e). Table 13 shows that the signs under column A and column BCD are identical, i.e., effects A and BCD are aliased (or confounded together, indistinguishable from each other, or inseparable from each other); therefore, for this (1/2) fraction "the effect or contrast of A" = "the effect of BCD". Similarly, my Table 13 shows that B = ACD, C = ABD, D = ABC, AB = CD, AC = BD, and AD = BC. Further, the effect of ABCD has been sacrificed (or lost altogether) because all the signs under the ABCD column are positive, thus the effect of ABCD = I cannot be studied if only the above 8 FLCs are run. Another way to obtain the alias structure of a FFD is to multiply an effect by its identity elements mod 2 (but use mod 3 for base-3 designs, and mod 5 for base-5 designs, etc.). For our example, A = A$\times I$ = A(ABCD) = $A^2BCD = A^0BCD$ (mod 2) = BCD, and so on!

Exercise 24. Obtain the aliases of the other 12 effects for the above (1/2) fraction in Table 13 using mod 2 algebra. (b) Develop a (1/2) fraction of a 2^4 factorial using ABD as the design generator, i.e., let I = ABD, and then obtain the alias structure of your FFD.

Table 13 I =+ABCD

The FLCs in the PB	A	B	AB	C	AC	BC	ABC	D	AD	BD	ABD	CD	ACD	BCD	I	y_{ijkl}
(1)	–	–	+	–	+	+	–	–	+	+	–	+	–	–	+	45
ab	+	+	+	–	–	–	–	–	–	–	–	+	+	+	+	65
ac	+	–	–	+	+	–	–	–	–	+	+	–	–	+	+	60
bc	–	+	–	+	–	+	–	–	+	–	+	–	+	–	+	80
ad	+	–	–	–	–	+	+	+	+	–	–	–	–	+	+	100
bd	–	+	–	–	+	–	+	+	–	+	–	–	+	–	+	45
cd	–	–	+	+	–	–	+	+	–	–	+	+	–	–	+	75
abcd	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	96

Comment on the deficiency of this design!

We now compute the contrasts for the (1/2) FFD in the above Table 13 using the data on page 257 of Montgomery(8e).

Contrast (A) = –*(1) + ab + ac – bc + ad – bd – cd + abcd* = –45 + 65 + 60 – 80 + 100 – 45 – 75 + 96 = Contrast (BCD) = 76 $\rightarrow$ SS(A) = SS(BCD) = $76^2/8$ = 722.00 & $\bar{\mathbf{A}}$ = 76/4 = 19.00.

<u>Exercise 24 (Continued).</u> (c) Compute the contrast values for the other 12 effects in Table 13 and their corresponding SS's, using the same data on page 257 of Montgomery. Then use the orthogonality property to verify that the values of your SS's are indeed correct!

Montgomery's 8^{th} edition uses the notation [A] to represent the average effect of A, i.e., for the data of Table 13, [A] = Contrast(A)/4 = [BCD] = 19.00, although we can also use $\bar{\mathbf{A}}$ in lieu of [A]. Note that Montgomery uses the notation [A] of average effect for the principal fraction and [A]′ for the other ½ fraction.

Suppose now we had chosen the alternate one-half fraction with $\xi = 1$ consisting of the FLCs: [*a, b, c, d, abc, abd, acd, bcd*]. Although the design generator will stay the same as g = ABCD, but the defining relation or the identity element will be I = –ABCD. This is because the sign of the ABCD interaction for all the 8 runs in the above bracket is –1. Tables 6.10 and 6.11 on pages 257-8 of Montgomery's 8th edition show that for this one-half fraction [A]′ = (*a – b – c + abc – d + abd + acd – bcd*)/4 = 97/4 = 24.25, but [BCD]′ = (*– a + b + c – abc + d – abd – acd + bcd*)/4 = –97/4 = – 24.25 $\rightarrow$ [A]′ = –[BCD]′ .

As Montgomery points out on p.325, any FFD of resolution *R* can be projected into a complete factorial consisting of a subset of influential factors. To understand this concept, study pages 325-327 of Montgomery's 8th edition.

Exercise 25. Work Problem 8.2 page 376 of Montgomery and obtain the optimal condition X_O.

THE (1/4)th FRACTION OF A 2^K FACTORIAL EXPERIMENT

It is often possible that there are too many factors affecting an output of a manufacturing process so that experimenters do not have sufficient resources to conduct even a 1/2 fraction of a complete replicate. In such cases resort has to be made to a smaller fraction (or a smaller number of runs) such as (1/4)th, (1/8)th, (1/16)th, etc. However, an experimenter must be cognizant of the fact that for each block of FLCs that s/he does not conduct, s/he is creating exactly one alias for each effect, i.e., a (1/4)th FFD has 3 aliases for each effect because there are 3 blocks of FLCs that are not studied, a (1/8)th FFD has 7 aliases for each effect because there are 7 blocks of FLCs that are not studied, etc.

Now consider the EXAMPLE 8.4 on pages 336-340 of Montgomery(8e) that pertains to an injection-molding operation involving 6 factors each at 2 levels. In order to reduce a part's shrinkage, a team of QC engineers have decided to use DOX to improve quality, where they have identified 6 controllable factors (or process parameters) that may impact the output y = the amount of shrinkage. Factor A = mold temperature, B = screw speed, C = holding time, D = cycle time, E = gate size, and factor F = holding pressure, again each factor is set at 2 fixed levels (0 or

1). However, due to lack of resources, the QENGRs could not conduct a complete replicate of the 2^6 factorial that would require $2^6 = 64$ experimental runs (or FLCs) and they settled on running only a $(1/4)^{th}$ of the 64 FLCs. In designing any FFD, the objective must be not to lose the effects of the main factors (A, B, C, D, E, F) and 1st-order interactions (AB, AC, AD, ..., EF), and by all means, if at all possible, keep the main factors separate from 2-way interactions, i.e., maximize design resolution such that main factors will not become aliased with 2-way interactions (if possible). We next describe the procedure for obtaining a $(1/4)^{th}$ fraction of a 2^6 factorial design.

Step 1. Since $(1/4)2^6 = 16$, then the 64 FLCs [*(1), a, b, ab, c, ac, …, acdef, bcdef, abcdef*] must be divided up into 4 blocks each containing 16 FLCs, and as before the fraction with FLC (1) = (0, 0, 0, 0, 0, 0) is called the principal block (PB).

Step 2. Since the 4 fractions (or blocks) absorb 3 *df*, then we must sacrifice 3 *df* and the fact that $(1/4)2^6 = 2^{-2}2^6 = 2^{6-2}$, then we need 2 independent generators. Note that this pattern will persist for all FFDs (base-2, 3, 5, etc., as long as the design base is a prime number), i.e., for a 2^{k-p} FFD we need exactly p independent generators. For example, a 2^{7-3} FFD needs 3 independent generators while a 2^{8-4} needs 4 independent generators and so on. Referring back to the example under consideration with $p = 2$ independent generators g_1, g_2 and each generator with 1 *df*, we note that something is not jiving! Recall that the 4 fractional replicates each with 16 FLCs carry 3 *df* amongst them, and therefore, the other one *df* to be sacrificed must be $g_3 = g_1 \times g_2$ (mod 2), where g_3 is called the generalized interaction of g_1 with g_2.

Step 3. Select the p (= 2 for our design) generators in such a manner as to maximize design resolution (which is the minimum no. of letters in all the $\boldsymbol{2^p}-1$ generators). For our example the QENGRs selected g_1 = ABCE, and g_2 = BCDF. Then our 3rd generator is g_3 = (ABCE)×(BCDF) = $AB^2C^2DEF = AB^0C^0DEF$ (mod 2) = ADEF. Since the minimum number of letters in all 3 WORDS (ABCE, BCDF, ADEF) is 4, then our design resolution is R = IV.

Step 4. Appropriately separate the 2^k FLCs into $\boldsymbol{2^p}$ blocks. This can be done the hard way by defining p contrast functions (for our example ξ_1 and ξ_2). Recall that for base-2 designs, the values of ξ can be only 0 or 1. Therefore, for our example, the 4 blocks will pertain to ($\xi_1 = \xi_2 = 0$, the PB), and the ($\xi_1 = 0$, $\xi_2 = 1$), ($\xi_1 = 1$, $\xi_2 = 0$), ($\xi_1 = \xi_2 = 1$) blocks. These contrast functions are $\xi_1 = x_1 + x_2 + x_3 + x_5$ for ABCE, and $\xi_2 = x_2 + x_3 + x_4 + x_6$ for g_2 = BCDF. Next, obtain the PB by computing the values of ξ_1 and ξ_2 for all the 64 FLCs, starting with *(1)* = (0, 0, 0, 0, 0, 0) and through *abcdef* = (1, 1, 1, 1, 1, 1). Clearly $\xi_1((1)) = \xi_2((1)) = 0$.

We now determine the contrast function values for the FLC *ae*: $\xi_1 = 1 + 0 + 0 + 1 = 0$ (mod 2), $\xi_2 = 0 + 0 + 0 + 0 = 0$ $\rightarrow$ The FLC "*ae*" belongs to the PB. Similarly, you may verify that the FLCs *bef, cef*, and *abd* also belong to the PB. By now we have identified sufficient number of independent FLCs that belong to the PB because the remaining 11 FLCs in the PB can be obtained from these 4 independent FLCs (*ae, bef, cef, abd*) by multiplying them (mod 2) two at time, three at a time, and all four at a time. That is to say, the 6th FLC in the PB is (*ae*)(*bef*) = $abe^2f = abe^0f = abf$. The remaining 10 FLCs are: *(ae)(cef) = acf, (ae)(abd) = bde, (bef)(cef) = bc, (bef)(abd) = adef, (cef)(abd) = abcdef, (ae)(bef)(cef)* = $abce^3 = abce$ (mod 2), *(ae)(bef)(abd) = df, (ae)(cef)(abd) = bcdf, (bef)(cef)(abd) = acd*, & *(ae)(bef)(cef)(abd) = cde*. Only the PB, which is summarized below, has this group property (the other fractions generally do not possess this property). Bear in mind that the PB for all FFDs is the one for which all contrast functions (ξ_i, i = 1, 2, …, *p*) have zero values.

The Principal Block (PB) with g_1 = ABCE, and g_2 = BCDF.

The fraction with $\xi_1 = \xi_2 = 0$: *(1), ae, bef, cef, abd, abf, acf, bde,* bc, *adef, abcdef, abce, df, bcdf, acd, cde*

You should check the fact that the values of the contrast functions $\xi_1 = x_1 + x_2 + x_3 + x_5$ and $\xi_2 = x_2 + x_3 + x_4 + x_6$ are indeed zero for every FLC in the above PB.

Step 5. Use the PB to generate the other 3 blocks. Suppose we wish to generate the 16 FLCs in the ($\xi_1 = 0$, $\xi_2 = 1$) fraction; all we need is to identify only one FLC that belongs to the ($\xi_1 = 0$, $\xi_2 = 1$) one-fourth replicate, say the FLC d = (0, 0, 0, 1, 0, 0). Then, we multiply the FLC "d" by the last 15 FLCs of the PB (mod 2) to generate the ($\xi_1 = 0$, $\xi_2 = 1$) fraction. For example, *d(ae) = ade, d(bef) = bdef*, etc. The ($\xi_1 = 0$, $\xi_2 = 1$) block is listed below:

The Block: $\xi_1 = x_1 + x_2 + x_3 + x_5 = 0$ and $\xi_2 = x_2 + x_3 + x_4 + x_6 = 1$

The ($\xi_1 = 0$, $\xi_2 = 1$) fraction: *d, ade, bdef, cdef, ab, abdf, acdf, be, bcd, aef, abcef, abcde, f, bcf, ac, ce*

<u>Exercise 26.</u> Use the above procedure to generate the other 2 one-fourth replicates of the above $2_{\mathbf{IV}}^{6-2}$ FFD, i.e., generate the ($\xi_1 = 1$, $\xi_2 = 0$) and ($\xi_1 =1$, $\xi_2 =1$) blocks.

Fortunately, there is a much simpler method of obtaining the PB of any FFD (in base-2, base-3, base-5, or any other prime base), which we describe below in stepwise fashion.

Step I. For the 2^{k-p} FFD, exactly *k*–*p* columns can always be written arbitrarily. For the FFD in Example 8-4 on pages 336-340, $k - p = 6 - 2 = 4$, which implies we can write 4 arbitrary columns as shown in Table 8.10 on page 336 of Montgomery's 8th edition. However, the assignment of the factors to the 4 columns must be done with extreme care. The first 3 column assignment of factors are again completely arbitrary, but the 4th column (i.e., D) must be done by examining the 3 generators. Recall that g_1 = ABCE so that the levels of E must be obtained from the product E = +A×B×C, and the fact that g_2 = BCDF, then the levels of factor F must be obtained from the levels of factors B, C, and D as in F = +B×C×D. This leaves only factor D that can be assigned to column 4. Had one of the generators been ABCD, then D could not be assigned arbitrarily. If these assignments do produce the PB, then we are almost finished because the other $2^{p} - 1$ blocks can be generated just like the above step 5.

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Step II. If step I above does not produce the PB, then we have to redo the column assignments as described in the following example.

<u>Example .</u> Consider a 2^{5-2} FFD with the 2 independent generators g_1 = ABC and g_2 = BDE. Since there are 4 fractional replicates each containing 8 FLCs, then we need to sacrifice 3 and thus we need one more generator, namely g_3 = (ABC)(BDE) = ACDE. Further, $k - p = 5 - 2 = 3$ implies that we can write 3 columns arbitrarily, bearing in mind that C = A×B because one identity element is I = ABC. The column assignments are shown in Table 14. Clearly the

$(1/4)^{th}$ fraction in Table 14 is not the PB because the FLC (1) = (0, 0, 0, 0, 0) = (-1, -1, -1, -1, -1) is not in the block of Table 14. The Table 14 block is actually the $\xi_1 = x_1 + x_2 + x_3 = 1$ and $\xi_2 = x_2 + x_4 + x_5 = 1$ block. Therefore, we have to redo the C and E columns in such a manner

Table 14. (The Principal Fraction because I = +ABC = +BDE = +ACDE)

A	B	D	C = +A×B	E = + B×D	FLC	y_{ijklm} (LTB)
-	-	-	+	+	*ce*	14
-	-	+	+	-	*cd*	35
-	+	-	-	-	*b*	32
-	+	+	-	+	*bde*	12
+	-	-	-	+	*ae*	9
+	-	+	-	-	*ad*	18
+	+	-	+	-	*abc*	5
+	+	+	+	+	*abcde*	7

that both their signs in the 1st run are –1 instead of +1. This implies that we must let C = –A×B in column 4, and E = – B×D in column 5 in order to obtain the PB. Table 15 shows how to generate the PB for this 2^{5-2}_{III} FFD. Note that the PB, always containing the FLC (1), is different from the principal fraction for this fractional replicate because I = – ABC= – BDE = + ACDE for the PB.

<u>Exercise 27.</u> Use the above procedure to obtain the other 2 blocks of the above 2^{5-2}_{III} FFD and give the corresponding values of their contrast functions. (b) Determine the alias structure of this FFD. Do aliases change from one fraction to the next? (c) Analyze the data in Table 14 in order to obtain the optimal conditions . Hint: Obtain the RT and cross factors that are involved in significant interactions. Answer: X_O = *cd* = (0, 0, 1, 1, 0) or X_O = (1).

Table 15. (The PB with g_1 = ABC, g_2 = BDE, g_3 = ACDE)

A	B	D	C = – A×B	E = – B×D	FLC
-	-	-	-	-	*(1)*
-	-	+	-	+	*de*
-	+	-	+	+	*bce*
-	+	+	+	-	*bcd*
+	-	-	+	-	*ac*
+	-	+	+	+	*acde*
+	+	-	-	+	*abe*
+	+	+	-	-	*abd*

Exercise 28. For the Table 14 above, compute SS(A) and SS(AE) = SS(CD) in 2 different ways. (b) Obtain the ANOVA Table and verify that all your SS's are correct, using orthogonality of the design.

We now refer back to the Example 8.4 on pp. 336-340 of Montgomery, where the complete defining relation is I = ABCE, BCDF, and ADEF. The ANOVA Table 8.11 on page 337 of Montgomery clearly shows that only the effects A, B, A×B = C×E are relatively influential and their levels must be selected in such a manner as to minimize parts' shrinkage. Montgomery states on p. 338 that as long as factor B is set at its low level, virtually any combination of factors A and C will result in low values of average parts shrinkage, and factor C should be set at low level to reduce variability. I further crossed the factor A with B, factor C with E, and even A with C. The 3 interaction tables showed that B must be set at B_0, but if C is set at its low level (to reduce variation), then E must be set at its high level and A must be set at its low level. So, it seems more likely that the optimal condition is $X_O = A_0B_0C_0D_?E_1F_?$.

For a more general FFD, study the $2^{7-3}_{\mathbf{IV}}$ FFD of Examples 8.5 on pages 341-343 of Montgomery. It must be emphasized that all resolution IV designs must have at least 2k runs, where k = number of design factors [Steve Webb (1968), Technometrics, Vol. 10, No.2]. Note that Table 8.15 on page 343 does give the PB because it contains the FLC (1), and since only one fraction out of 8, consisting of 16 FLCs, are run (i.e., 7 blocks of 16 FLCs are not studied), then each effect has 7 aliases. For example, the aliases of factor A are A = BCE = ABCDF = DEF = CDG = ABDEG = ACEFG = BFG.

For blocking in a FFD, study pages 344-349 of Montgomery(8e). Further, all *R* = III designs need at least (k+1) runs, while according to Webb (1968) all *R* = IV designs must have at least 2*k* runs. Resolution *R* = IV designs that have exactly 2*k* runs are called minimal designs (such as $2_{\mathbf{IV}}^{8-4}$ FFD with a set of independent generators such as g_1 = ABCE, g_2 = ABDF, g_3 = BCDG, and g_4 = ACDH).

SEQUENTIAL ASSEMBLY OF FRACTIONS TO SEPARATE EFFECTS By FOLDING OVER THE FIRST (*R* =III) FRACTIONAL REPLICATE

By combining 2 fractional factorial replicates, each with resolution *R* = III, in which certain signs are completely reversed, we can isolate (or dealias) some of the influential effects. As an example, consider the (1/16)th fractional replicate of Example 8.7 whose design layout is listed in Table 8.21 on page 354 of Montgomery(8e). The alias structure of the 7 main factors A, B, C , D, E, F, G through the 1st-order effects are given atop page 355 of Montgomery. Note that each effect has 15 aliases but the alias structure on pages 355 provides the aliases only through 1st-order interactions. Using the data of Table 8.21 on page 354, [B]= (–85.5 –75.1 + 93.2 +145.4 –83.7–77.6 + 95.0 + 141.8)/4 = 38.375 ; similarly, [A] = 20.625, [C] = – 0.275, [D] = 28.875, [E] = –0.275, [F]= – 0.625, and [G] = –2.425. These 7 average effects clearly show that the most influential effects in their order of strength are B, D and A and the other 4 factors have almost trivial effects on the mean of response y = " eye-focus time". In order to dealias all the main factors from their 2-way interactions, we must conduct another block of 8 FLCs with all the signs under all 7 factors reversed (i.e., by folding over the 1st fraction) as shown in Table 8.22 at the bottom of page 355 of Montgomery(8e) with the corresponding observations in the last column. From Table 8.22, simple computations will show that [A]′ = –17.675, [B]′ = 37.725, [C]′ = – 3.325, [D]′ = 29.875, [E]′ = 0.525, [F]′ = 1.625, and [G]′ = 2.675. As Montgomery points out, the defining relation that will involve all the 3- way interactions for the 2nd fractional replicate is I = – ABD, –ACE, –BCF, ABCG, –CDG,

–BEG, – AFG, and – DEF. These show that [A]′ actually estimates A – BD – CE – FG versus aliases on page 355 of Montgomery that show that [A] = A + BD + CE +FG. Thus, to dealias the average effect of A, we compute ([A] + [A]′)/2 = (20.625 – 17.675)/2 = 1.475 = $\bar{\mathbf{A}}$. The remaining dealiased average effects are summarized in the Table atop of page 356 of Montgomery, which clearly shows that the only significant effects are B, D, and B×D.

Since by now in the above design we have conducted 16 experiments, then the original $2_{\mathbf{III}}^{7-4}$ FFD has been folded over onto a 2^{7-3} design whose defining relation, I, and its resolution has yet to be determined. The defining relation for the $2_{\mathbf{III}}^{7-4}$ design in Table 8.21 (p. 354 of Montgomery) is g_1 = ABD, g_2 = ACE, g_3 = BCF, and g_4 = ABCG. The independent generators for the $2_{\mathbf{III}}^{7-4}$ fold-over design in Table 8.22 (p. 355) are g_1 = – ABD, g_2 = –ACE, g_3 = –BCF, and g_4 = ABCG. The combined fractions consisting of 16 FLCs now is a 2^{7-3} FFD and therefore has only 3 independent generators. Although, we have folded the column G in Table 8.21, yet G = ABC stays intact in the fold-over design of Table 8.22, and therefore, one identity element for the combined fractions will stay the same as g_1 = ABCG (see the Excel file on my website under Example8.7). The other 2 generators of the new 2^{7-3} fractional replicate are obtained by multiplying g_1 = ABD, g_2 = ACE, and g_3 = BCF 2-at-a-time and selecting any 2 out of the 3 resulting products as the independent generators. Hence, the independent generators for the $(1/8)^{th}$ fractional replicate is (I = ABCG, BCDE, ACDF), or (I = ABCG, BCDE, ABEF), or (I = ABCG, ACDF, ABEF). Therefore, the combined two fractions yield a resolution *R* = IV design (i.e., a $2_{\mathbf{IV}}^{7-3}$ FFD). The complete defining relation, I, is listed near the bottom of page 356 of Montgomery(8e).

Note that, as Montgomery points out on p. 368, the full fold-over of an *R* = IV design is useless because it does not generate a new fraction and two-way interactions cannot be separated from each other. However, a single-factor fold-over will enable the experimenter to dealias the folded-over factor's two-way interactions from all other two-way interactions.

Further, if our objective in the example 8.7 were to isolate the effects of A from its aliases BD, CE and FG, then we must run another $(1/16)^{th}$ fractional replicate where only the signs under factor A are reversed but all the other signs in the 2^{nd} fraction will stay as those of the 1st fraction (i.e., we would fold over only factor A). The concept of single-factor fold-over is demonstrated on pages 353-354 of Montgomery where the factor D is folded over. The single-factor fold-over has the disadvantage of not improving the design resolution.

Exercise 29. Obtain the aliases of factors A and D for the combined FFDs of Table 8.21 on page 354 of Montgomery plus the fractional replicate with column D of Table 8.21 folded over. Note that you will have to 1st obtain the defining relation, I, for the new 2^{7-3} FFD.

Study pages 357-366 of Montgomery on Plackett-Burman (PL-B) designs which are not of the form 2^k, for some integer *k*, and therefore they possess very complicated alias structure which I have not studied. The reference on p. 723 [Plackett, R. L. & J. P. Burma (1946)] perhaps explains this complex alias structure. The designs themselves are very easy to generate but you may not imbed an interaction in any one of the columns of the PL-B designs. Note that All PL-B designs are OAs.

Exercise 30. Work problem 8.10 pp. 376-7 of Montgomery. As part (d) of 8.7, regress y on the influential effects and use your model to compute e_{52}, where the index i = 1, 2, …, 16. (b) Work problem 8.23 on p. 377 of Montgomery(8e).