BLOCK CONFOUNDING IN 2^k FACTORIALS (*k* factors each at 2 levels)

It is often impossible to run all the 2^k observations in a 2^k factorial design (or a single complete replicate) in one block. For example, consider a 2^3 design where it may be possible to run only 4 experiments in one day, and the other 4 FLCs must be run the next day, while one complete replicate of a 2^3 design has 8 runs to be run in one day. Yet, as another example two batches of raw material may be needed for 8 experimental runs. In such situations, days would form two blocks (or the two batches would form two blocks), and the experimenter has to be careful not to confound a main factor, or a 1st-order interaction with blocks. If the experimenter haphazardly runs the 4 FLCs [*(1), a, c, ac*] of a 2^3 factorial on day one, and the FLCs [*b*, *ab, bc, abc*] on day 2, then in fact s/he has confounded the effect of factor B with blocks. This is because the block contrast will also be computed as [*b + ab + bc + abc*] − [*(1) − a − c − ac*], which is identical to the contrast for factor B. As a result, Days and factor B will be indistinguishable from each other or hopelessly confounded with one another. On the other hand, if the FLCs [*(1), a, bc, abc*] are run on day 1, and [*b, c, ab, ac*] are run on day 2, then the BC interaction is confounded with blocks. For other examples, study pages 304-312 of Montgomery's 8th edition. In general, our objective will be to confound the highest possible order interactions with blocks. So, in our example of the 2^3 experiment which requires to be completed in 2 days, we must design in such a manner as to confound the A×B×C effect with days. This leads to running the FLCs [*(1), ab, ac, bc*] in one day and the FLCs [*a, b, c, abc*] on another day. Which block of FLCs are run on day 1 <u>must be determined at random</u> and also randomization is required within each block (because this is a block design with one-half replicate conducted per day and not a completely randomized design).

To develop a general block confounding scheme for base-2 designs, we must use the algebra in base-2, i.e., we must learn that 2 = 4 = 6 = 8 = ... = 0 (modulus 2), while 3 = 5 = 7 = 9 = ... = 1 (mod 2). Recall that the algebra in base-2 has only 2 elements, namely 0 and 1, and this is why 0 is used to represent the low level of a factor, and 1 represents the high level of a factor for base-2 designs. Further, the base-10 number 2 has no meaning in base 2, and 2 of base 10 is represented by 10 in base 2, i.e., $2_{10} = 10_2$, $3_{10} = 11_2$, etc. Note that each factor has only 2 levels (0 = low level, 1 = high level), not 3 in which case the element 2 would be needed to represent the 3rd level if a factor had 3 equi-spaced quantitative levels, or 3 qualitative levels.

For the sake of illustration, consider the Example 13-10 of Montgomery & Runger (1999, 2nd Ed., pp. 680-683), where the objective is to investigate the effects of 4 factors on the terminal miss distance of a shoulder-fired ground-to-air-missile. The 4 factors are target type (A), seeker type (B), target altitude (C), and target range (D). An optical tracking system will measure the terminal miss, y, to the nearest 10th of a foot, where I have slightly modified the responses. In order to conduct the experiments in less time, 2 gunners (or operators) are used, who may be of different competence levels. Therefore, we have a 2^4 factorial with 2 operators who act as 2 blocks. Since one full replicate of a 2^4 factorial provides 16 distinct FLCs, then such a design would provide a total of 15 *df* in a CRD for studying the 15 orthogonal effects A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, and the 3rd-order interaction A×B×C×D, each having 1 *df*. However, in this example each one of the 2 operators will run 8 of the FLCs, and we must assign the 8 FLCs to each operator in such a manner that none of the lower-order effects (such as main factors and 1st-order interactions) are sacrificed, i.e., it is best to sacrifice the effect of the 4-way (or 3rd-order) interaction ABCD. Put differently, the 2 operators absorb one *df* between them, and ABCD interaction also has one *df*; thus, the assignment of the 8 FLCs to each operator must be carried out in such a manner as to confound the ABCD effect with blocks (or operators). To start our confounding scheme, we define the contrast function for the generator g = ABCD as

$$\xi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$$

where x_1 refers to the levels of factor A, x_2 to the levels of B, etc, and $\alpha_i = 0$ or 1 in base 2. For example, the contrast function for the defining contrast (or the Generator) ABCD is

$$\xi\,(ABCD) = x_1 + x_2 + x_3 + x_4,$$

while the contrast function for confounding the ABD interaction with two blocks is

$$\xi(ABD) = x_1 + x_2 + \quad x_4.$$

Observe that for base-2 designs, the value of the contrast function ξ can be only 0 or 1. Further, for all confounding schemes and fractional replicates, the block with zero contrast function value(s) is called the Principal Block (PB). We now begin the assignment of the 16 FLC's to the 2 blocks, one with ξ(ABCD) = 0 and the other block with ξ(ABCD) =1.

The FLC *(1)*: ξ(ABCD) = 0 + 0 + 0 + 0 = 0; the FLC *a* : ξ = 1 + 0 + 0 + 0 = 1; the FLC *b* : ξ = 0 + 1 + 0 + 0 = 1; the FLC *ab*: ξ = 1 + 1 + 0 + 0 = 2 = 0 (mod 2); the FLC *c*: ξ = 1; the FLC *ac*: ξ = 2 = 0 (mod 2); the FLC *bc*: ξ = 2 = 0 (mod 2); the FLC *abc*: ξ = 1 + 1 + 1 + 0 = 3 = 1 (mod 2), and so on up the FLC *abcd* for which ξ(ABCD) = 1 + 1 + 1 + 1 = 4 = 0 (mod 2). The 2 blocks pertaining to the 2 operators with the corresponding responses, in feet, are shown below. Thus, in the following 2 blocks the effect ABCD is confounded with blocks. Note that the PB always contains the FLC: (1) = (0, 0, 0, …, 0) for all base-b designs (b = 2, 3, 5, 7, …).

The Principal Block (PB), where ABCD is at +1

The Block with ξ = 0: *(1)* = 3.1, *ab* = 6.8, *ac* = 6.3, *bc* = 8.5, *ad* = 10.3, *bd* = 3.7, *cd* = 8.1, *abcd* = 9.2 feet

The ξ = 1 Block, where ABCD is at –1

The Block with ξ = 1: *a* = 7.1, *b* = 4.9, *c* = 6.3, *d* = 4.2, *abc* = 6.2, *abd* = 12.7, *acd* = 9.4, *bcd* = 7.2 feet

Exercise 21. (a) For the above example, sacrifice the effect g = ACD to generate the 2 blocks each containing 8 FLCs. First generate the PB and then use it with one element of the ξ = 1 block to obtain the remaining 7 FLCs of the ξ = 1 block.

The SS's for different effects are easily computed by using their contrast values, ignoring blocking altogether. For example, the contrast for the BD effect is: Contrast(B×D)= +*(1) + a – b – ab + c + ac – bc – abc – d – ad + bd + abd – cd – acd + bcd + abcd* = 3.1 + 7.1 – 4.9 – 6.8 + 6.3 + 6.3 – 8.5 – 6.2 – 4.2 – 10.3 + 3.7 + 12.7 – 8.1 – 9.4 + 7.2 + 9.2 = – 2.8 ⟶

SS(B×D)= $(-2.8)^2/16$ = 0.49. The ANOVA table is summarized in Table 11. The ANOVA Table 11 clearly shows that the most influential (or vital) effects in their order of strength are A, A×C, D, A×D and the other 10 effects are all amongst the trivial many! In fact, the above 4

Table 11 (Confounding an Unreplicated 2^4 Factorial in Two Blocks)

Source	Total	Blocks = ABCD	A	B	C	D	AB	AC	AD
df	15	1	1	1	1	1	1	1	1
SS	97.25	0.25	30.25	1.21	4.41	15.21	0.04	25.0	13.69

Source	BC	BD	CD	ABC	ABD	ACD	BCD
df	1	1	1	1	1	1	1
SS	0.36	0.49	0.36	0.36	4.0	0.81	0.81

vital effects account for 86.53% of total variation in the mean of response y. Before we identify the optimal condition X_O, we note that factor B and all its interactions are relatively weak, and in order to obtain a measure of residuals we may collapse this 2^4 confounded block design with n = 1 into a 2^3 CRD factorial design (ignoring blocking) with n = 2 observations per cell as depicted in the OA below.

The OA for a collapsed 2^4 confounded Factorial

D	C	A	FLCs	y_{ijkr}
-	-	-	*(1), b*	3.1, 4.9
-	-	+	*a, ab*	7.1, 6.8
-	+	-	*c, bc*	6.3, 8.5
-	+	+	*ac, abc*	6.3, 6.2
+	-	-	*d, bd*	4.2, 3.7
+	-	+	*ad, abd*	10.3, 12.7
+	+	-	*cd, bcd*	8.1, 7.2
+	+	+	*acd, abcd*	9.4, 9.2
			$y_{....}$ =	114

The Minitab output listed below gives the ANOVA table for the above OA, which clearly confirms our aforementioned conclusions about the strong effects of A, AC, D and AD on the mean of response y (note that the impact of factor C on the mean of y is almost significant at the 5% level).

ANOVA Table 11: y versus A, C, D

Factor	Type	Levels	Values
A	fixed	2	-1, 1
C	fixed	2	-1, 1
D	fixed	2	-1, 1

Analysis of Variance for y

Source	DF	SS	MS	F_0	*P*
A	1	30.250	30.250	32.18	0.000**
C	1	4.410	4.410	4.69	0.062
D	1	15.210	15.210	16.18	0.004**
A*C	1	25.000	25.000	26.60	0.001**
A*D	1	13.690	13.690	14.56	0.005**
C*D	1	0.360	0.360	0.38	0.553
A*C*D	1	0.810	0.810	0.86	0.380
Error	8	7.520	0.940		
Total	15	97.250			

S = 0.969536   R-Sq = 92.27%   R-Sq(adj) = 85.50%

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Next we cross factor A with C and then A with D in order to obtain the optimal settings of the factors A, C and D.

A × C

A \ C	0	1
0	15.9*	30.1
1	36.9	31.1

A×D

A \ D	0	1
0	22.8*	23.2*
1	26.4	41.6

Assuming that the miss-distance is an STB type QCH, the above interaction tables clearly show that $X_0 = A_0C_0D_0$ or $A_0C_0D_1$

However, the choice $X_O = A_0C_0D_0$ would be better than $X_O = A_0C_0D_1$ if we examine the impact of factor D in isolation (D_0 = 49.2, D_1 = 64.8). Note that if the weak factor B is used to reduce variation (assuming that it impacts σ_y), the above 2^4 factorial should be collapsed in terms of factors A, B and D. It turns out $\overline{R}(B_0)$ = 2.20 and $\overline{R}(B_1)$ = 2.80 so that factor B (seeker type) has minimal impact on σ_y.

COMPLETE CONFOUNDING THE 2^k UNREPLICATED FACTORIAL IN 2^p BLOCKS ($0 \le p \le k-2$, $k \ge 2$)

As an example, suppose in the miss-distance experiment above we had 4 gunners to conduct the 16 experiments, each being assigned 4 different FLCs. How do we make the FLC assignments to the 4 gunners so that we will not sacrifice the main factors and also protect some of the two-way interactions of our choice? Since the 4 blocks carry 3 *df*, then we must sacrifice 3 effects each with 1 *df* to blocks, i.e., we will have to have 2 generators such as g_1 = ABD, g_2 = ACD, and their generalized interaction $g_3 = g_1 \times g_2$ = BC, all 3 effects confounded with blocks. Therefore, the best we can do in this design is to protect our 4 main factors and 5 out of the six 1^{st}–order interactions. Since only two out of the three generators g_1 = ABD, g_2 = ACD and g_3 = BC are independent, then we need only 2 contrast functions (cfs) to allocate the 16 FLCs to the 4 blocks with ($\xi_1 = \xi_2 = 0$), ($\xi_1 = 0$, $\xi_2 = 1$), ($\xi_1 = 1$, $\xi_2 = 0$), and ($\xi_1 = \xi_2 = 1$), where $\xi_1 = x_1 + x_2 + x_4$ and $\xi_2 = x_1 + x_3 + x_4$. Note that $\xi_1 + \xi_2 = x_2 + x_3 = \xi$(BC). We first obtain the principal block (PB), i.e., the block for which $\xi_1 = \xi_2 = 0$ and use it to generate the other 3 blocks. Note that in this design the experimenter has prior knowledge that the BC interaction is either nonexistent or unessential and or both.

The Principal Block (the PB for which $\xi_1 = \xi_2 = 0$) : *(1), abc, ad, bcd*

The ($\xi_1 = 0$, $\xi_2 = 1$) block: *ab, c, bd, acd*

The ($\xi_1 = 1$, $\xi_2 = 0$) block: *ac, b, cd, abd*

The ($\xi_1 = 1$, $\xi_2 = 1$) block: *d, abcd, a, bc*

For another example of confounding in a 2^k factorial in 4 blocks, study the section 7.6 on pp. 313-316 of Montgomery(8e). Again, the reader must observe that the PB always contains the FLC (1), where all factors are at their low levels.

COMPLETE CONFOUNDING THE 2^k REPLICATED FACTORIAL IN 2^p BLOCKS ($p < k$)

Exercise 21 (Continued). (b) Suppose that the miss-distance experiment is conducted in 2 blocks with the generator g = ABCD but replicated 3 times, and in all 3 replicates ABCD is sacrificed as the design generator. Thus, the effect of ABCD will be completely confounded with blocks because ABCD is the design generator in all 3 replicates. Provide the outline of the ANOVA table and show why the error term for the ABCD effect is the interaction effect: Rep×Blocks. Hint: Carefully study the example by Montgomery(8e) on his pp. 309-310, where blocking of ABC has been used to reduce the value of error SS because 6 *df* is absorbed by Replicates and Replicates×Blocks.

Note that in general if at least 2 replicates of a 2^k ($k \geq 2$) factorial are run with complete confounding, then always error SS is reduced by the amount of SS(Replicates) & SS(BLKS×Replicates). Therefore, blocking in general is used as an error variance-reduction technique.

PARTIAL CONFOUNDING A REPLICATED 2^k FACTORIAL DESIGN

Suppose in part (b) of the above Exercise 21 the experimenter decides to confound ACD with blocks in replicate 1, confound ABD with blocks in replicate 2, and confound ABCD with blocks in replicate 3. Therefore, after all the 48 experiments are conducted, 2/3 relative information can be obtained on the confounded effects ACD, ABD and ABCD. Such a design is said to be partially confounded. The layout of the experiment is given atop the next page. (The Block 1 in each replicate is the PB for which $\xi = 0$ for all generators).

The ANOVA outline for this partially confounded design is provided in Table 12. Note that the Residual *df*, which is 27, is actually obtained from the interaction of Replicates with all

Replicate 1 ACD confounded		Replicate 2, where ABD is confounded with blocks		Replicate 3 ABCD confounded	
Block 1	Block 2	Block 1 (ξ = 0)	Block 2 (ξ = 1)	Block 1	Block 2
(1), b, ac, ad, cd, abc, abd, bcd,	*a, ab, c, bc, d, bd acd, abcd*	*(1), ab, c, abc ad, bd , acd bcd*	*a, b, ac, bc, d, abd, cd abcd*	*(1), ab, ac, bc, ad, bd, cd, abcd,*	*a, b, c, abc, d, abd, acd, bcd*

the other effects excluding blocks , i.e., 27 = 2×4 + 2×6 + 1×1 + 2×2 + 1×1 + 1×1 . This is why I would prefer not to refer to this 27-*df* effect as the pure error (PE) term, because there were no exact repetitions at the same FLC. Thus, in essence the Residual SS in this partially confounded design is the unexplained (or unaccounted for) SS. However, I am not exactly accurate in my assessment of no pure experimental error because each FLC was replicated 3 times in the 48 experiments. This implies that the pure error *df* for the above design should be 2×16 = 32, but out of these 32 *df*, two were absorbed by replicates and 3 by blocks so that the left-over (or residual) *df* is equal to 32 – 2 – 3 = 27 as before. Therefore, blocking in this experiment has reduced the SS(Error). However, if SS(PE) were to be computed as though the above were a CRD, then SS(Residual Error) in general would not equal to SS(PE) – SS(Rep) – SS(BLKS within Reps)

Table 12

Source	Total	Replicates	Blocks (within Replicates)	Main Factors A, B, C, D	AB, AC, AD, BC, BD, CD	ABCD (From Rep. 1 & 2)
df	47	2	3	4	6	1
Source	ABC, BCD	ACD (only from 2 & 3)	ABD (From Rep. 1 & 3)	Residuals		
df	2	1	1	27		

because SS(ABCD), SS(ACD) & SS(ABD) of the CRD and the partially blocked design would not be equal to each other. Thus, SS_{RES}(PCFD) = SS_{PE}(from the corresponding CRD) –SS(Rep) – SS(BLKs) +[SS_{CRD} – SS_{PCFD}], where SS_{CRD} represents the SS's of all the partially confounded effects computed as though the design were completely randomized and SS_{PCFD} stands for SS's of all partially Confounded effects.

For a numerical example, study the Example 7.3 on page 318 of Montgomery(8e). Further, you should bear in mind that the PB for all designs, regardless of the base, is always the block containing the FLC (0, 0, 0, …, 0) for which all contrast function values are zero. It is always most convenient 1^{st} to generate the PB and then use it to generate the other blocks using the group property of the PB.

Exercise 22. Work problems 7.1, 7.5, 7.6, 7.15, and 7.16 on page 319.

Errata for Chapter 7 of Montgomery's 8^{th} Edition

Page 318, in Table 7.11, change SS_C to 374850.0625, SS_{AC} to 94402.5625, and SS_{Error} to SS_{RES} =122754.8125.