

FACTORIAL DESIGNS

An experiment involving 2 or more factors is said to be a (full) factorial design iff at least one observation is taken at every possible factor level combination (FLC) of all the factors involved. A factorial experiment is balanced iff the size of the sample at each FLC (OR CELL) is the same, namely n . If sample sizes for different cells differ, the design is said to be unbalanced. As an example, consider an injection molding process where the output y is the strength of the molded piece measured in psi, and the 2 factors that may have an effect on strength are Temperature (A) and Pressure (B). Factor A has 2 fixed levels (100 °F, and 150 °F), and factor B is at 3 fixed levels (50, 75, 100 psi). This is called a 2×3 fixed factorial design and clearly has 6 FLCs as illustrated in Tables 4(a) & 4(b). In Table 4(a), the cell subtotals are $y_{11} = 79, y_{12} = 55, y_{13} = 133, y_{21} = 82, y_{22} = 113, y_{23} = 132, y_{1..} = 267, y_{.1} = 161,$ and $y_{.2} = 168$.

Table 4(a). An Unbalanced 2×3 Factorial

	B	50 psi	75	100	
A					$y_{i.}$
	100 °F	34, 45	55	48, 40, 45	267
	150	42, 40	55, 58	50, 42, 40	327
	$y_{.j.}$	161	168	265	$y_{...} = 594$

An Unbalanced 2×3 Factorial Experiment, where

$$n_{11} = 2, n_{12} = 1, n_{13} = 3, \dots, n_{23} = 3, N = 13$$

Note that there are $6 = 2 \times 3$ cells or FLCs.

The design in Table 4(a), clearly unbalanced, is CR, i.e., the order of obtaining the 13 observations was completely randomized. Table 4(b) below provides a balanced factorial with $n = 3$ observations per cell. Again the experiment in Table 4(b) is a 2×3 CR factorial, where both factors are quantitative. Equi-spaced and fixed. The values in () of Table 4(b) are cell subtotals $y_{ij.}$. To analyze the data in Table 4(b), as in the case of one-factor experiment, the

Table 4(b) A Balanced 2×3 CRD Factorial with 3 Replicates per FLC

A \ B	50 psi	75	100	$y_{i..}$
100 °F	36, 28, 33 ($y_{11.} = 97$)	55, 60, 59 (174)	47, 39, 38 (124)	395
150	38, 41, 43 (122)	54, 46, 38 (138)	52, 53, 50 ($y_{23.} = 155$)	415
$y_{.j.}$	219	312	279	$y_{...} = 810$

value of USS is $36^2 + 28^2 + \dots + 50^2 = 37,912$ (with 18 *df*); $CF = 810^2/18 = 36,450$ (with 1 *df*) → $SS_T = SS(\text{Total}) = USS - CF = 1462.00$ (with 17 *df*). $SS(\text{Error}) = SS_{11} + SS_{12} + \dots + SS_{23} = (36^2 + 28^2 + 33^2 - 97^2/3) + (55^2 + 60^2 + 59^2 - 174^2/3) + \dots + (52^2 + 53^2 + 50^2 - 155^2/3) = 32.66667 + 14.0 + 48.6 + 12.66667 + 128.0 + 4.66667 = 240.66667$ (with $2 \times 6 = 12$ *df*) → $SS(\text{Model}) = SS_T - SS(\text{Pure Error}) = 1221.33333$ (with 5 *df*). It is paramount to understand that all the calculations that we have performed thus far apply to every factorial design whether balanced or not. As shown above, the model does have 5 *df*, 1 of which is absorbed by factor A, 2 *df* is carried by factor B, and there remains 2 *df* for what is called the interaction effect between factors A and B denoted by $A \times B$, i.e.,

$$SS(\text{Model}) = SS(A) + SS(B) + SS(A \times B) \quad (8)$$

$$5 \text{ df} = 1 + 2 + 2 \text{ df.}$$

Note that the *df* on the LHS of equation (8) must "jive" with those on the RHS of Eq. (8), i.e., $5 = 1 + 2 + 2$. Because the design is balanced we have an orthogonal decomposition of $SS(\text{Model})$ into 3 components. Clearly,

$$SS(A) = \frac{395^2 + 415^2}{9} - \frac{810^2}{18} = 22.2222\bar{2}, \quad SS(B) = \frac{219^2 + 312^2 + 279^2}{6} - \frac{810^2}{18} = 741.00$$

and as a result from Eq. (8), $SS(A \times B) = SS(\text{Model}) - SS(A) - SS(B) = 458.111111$ (with 2 *df*).

Again, it is paramount to be cognizant of the fact that the above procedure for computing SS of different effects (except for Total, Error, and Model) is valid only for Balanced Factorial Experimental Designs. If the design is unbalanced, then the General Linear Model (GLM, or GRST) procedure must be applied in order to compute SS of different effects in the Model.

Unfortunately, GLM procedure is well outside the scope of this course. SAS has the Proc GLM

that gives the Type III SS's in the model, but as expected these adjusted SS's of the model do not add to SS(Model) because of non-orthogonality (or obliquity). Both Minitab and SAS provide Type I SS for model components that are additive.

Before we define the interaction between 2 factors in affecting an output Y, we present the balanced design of Table 4(b) in an orthogonal (OA) array format.

OA			
A	B	Y_{ijk}	$Y_{ij.}$
100	50	36, 28, 33	97
100	75	55, 60, 59	174
100	100	47, 39, 38	124
150	50	38, 41, 43	122
150	75	54, 46, 38	138
150	100	52, 53, 50	155

$y_{...} = 810$

Note that the array on the LHS is orthogonal because every FLC (A_i, B_j) $i = 1, 2, j = 1, 2, 3$ occurs exactly once in the design matrix and is repeated $n = 3$ times.

Hence, $SS_{RES} = SS_{PE}$.

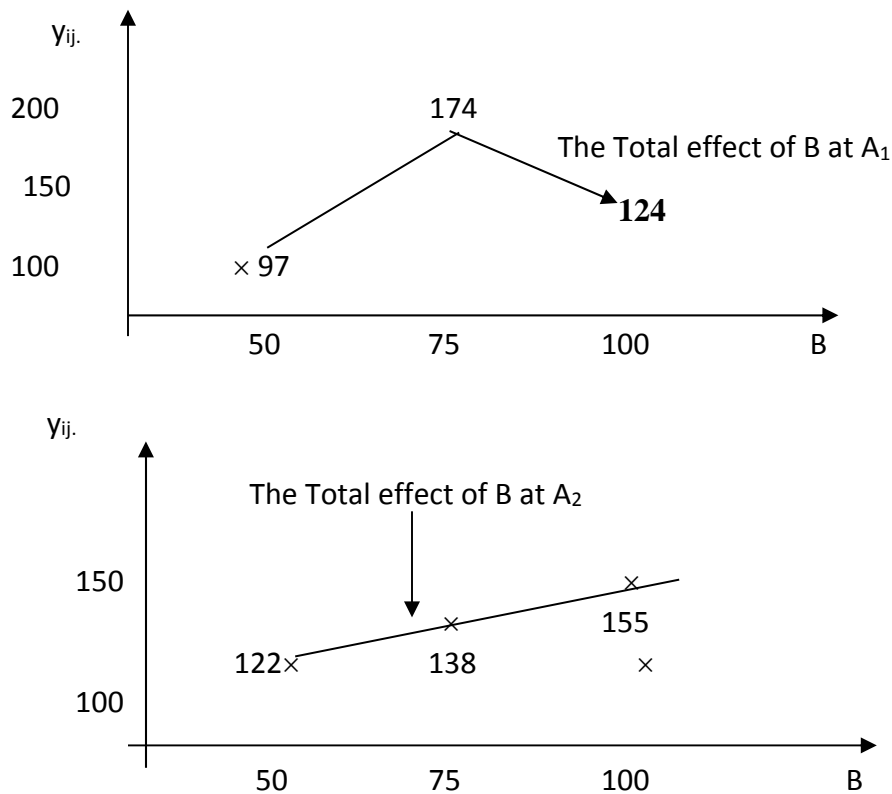
Because the above factorial design has 6 FLCs, then the orthogonal array (OA) has exactly 6 rows. These 6 rows (or FLCs) with cell subtotals $y_{ij.}$ carry exactly 5 df amongst them; since the model also has 5 df, then we can also obtain SS(Model) as follows

$$SS(\text{Model}) = \frac{97^2 + 174^2 + 124^2 + 122^2 + 138^2 + 155^2}{3} - \frac{810^2}{18} = 1221.3333\bar{3}, \text{ as before!}$$

Exercise 14. For the unbalanced design of Table 4(a), compute the SS(Model) in 2 different ways to determine if both methods generally yield the same value of SS(Model)! (b) Then compute SS(A), SS(B), and SS(A×B) as you would compute them for a balanced factorial experiment. (c) Use Minitab's GLM to also verify that the SS of different effects in the model cannot be computed in the same manner that are computed for a balanced design.

Definition. Two factors, A & B, interact as they impact the output Y iff the effect of B at the i^{th} level of A is significantly different from the effect of B at the k^{th} ($k \neq i$) level of A for at least one k. As an example, consider the balanced CRD of Table 4(b), where the effect of factor B at A_1 is negatively quadratic (i.e., concave downward), and the effect of B at A_2 is almost

completely linear as illustrated in Figures 6. Therefore, the factors A and B interact in affecting the response variable Y. The ANOVA is provided in Table 5. Note that when all factors in the experiment are fixed, then all hypotheses are tested against MS(Error). We will learn later that this will not be the case when some factors in the experiment are random! Before developing the identities for breaking down the Total SS of a 2-factor experiment for a balanced design in equations (9) below, you must study section 5.2 on pp. 186-187 of Montgomery(8e) for the 2 advantages of factorials over one-factor-at-a-time experiments!



Figures 6. Depicting A×B

$$y_{ijk} \equiv \mu + [(\mu_{i..} - \mu) + (\mu_{.j} - \mu) + (\mu_{ij.} - \mu_{i..} - \mu_{.j} + \mu)] + (y_{ijk} - \mu_{ij.}) \tag{9a}$$

$$y_{ijk} \equiv \mu + [A_i + B_j + (A \times B)_{ij}] + \varepsilon_{ijk}$$

$$\equiv \mu + [\text{Model effects}] + \varepsilon_{ijk} \tag{9b}$$

Using Montgomery’s(8e) notation at the bottom of page 198, the terms on the RHS of equation (9a), respectively, are $\mu_{i..} - \mu = \tau_i$, $(\mu_{.j} - \mu) = \beta_j$, $(\mu_{ij.} - \mu_{i..} - \mu_{.j} + \mu) = (\tau\beta)_{ij}$; however, I

Table 5 (* = Significant at the 5% level; **= significant at the 1% level)

Source	df	SS	MS	F ₀	P-value
Total	17	1462.00			
Model	5	1221.3333	244.26667	12.1795**	0.00023
A	1	22.22222	22.22222	1.1080	0.31324
B	2	741.00000	370.50000	18.4737**	0.00022
A×B	2	458.11111	229.05556	11.4211**	0.00167
Pure Error	12	240.66667	20.05556		

prefer the notation of $(A \times B)_{ij}$ to Montgomery's $(\tau\beta)_{ij}$ to represent the interaction between the inputs A and B because it is more self-explanatory. Unless all the factors in the experiment are at 2 levels (i.e., a 2-level design), the notation SS_{AB} for interaction SS is not exactly appropriate, although most DOX texts use the notation AB over $A \times B$. Therefore, only for a 2-level design (i.e., all factors at 2 levels), you may use $SS(AB) = SS_{AB} = SS(A \times B)$ to represent interaction SS; otherwise, only the notation $A \times B$ is most descriptive.

The 4th term on the RHS of Eq. (9) represents the interaction effect between A and B, and the last on the RHS gives the error term; further, for fixed-effects every term in (9), except for y_{ijk} and ϵ_{ijk} , is a population parameter. As in the case of single-factor, we replace every parameter in (9a) with its point sample unbiased (LS) estimator to obtain the identity:

$$y_{ijk} \equiv \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.}) \quad (10)$$

$$y_{ijk} \equiv \hat{\mu} + \hat{A}_i + \hat{B}_j + (A \times B)\hat{\tau}_{ij} + \epsilon_{ijk}$$

We now transpose $\bar{y}_{...}$, the 1st term on the RHS of (10), to the LHS.

$$y_{ijk} - \bar{y}_{...} \equiv (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.})$$

$$\text{or } y_{ijk} - \bar{y}_{...} \equiv \hat{A}_i + \hat{B}_j + (A \times B)\hat{\tau}_{ij} + \epsilon_{ijk} \quad (11)$$

Next, we square both sides of (11) and use the fact that the triple sum of all cross-product terms are zero in order to obtain the orthogonal decomposition of Total SS given in Eq. (12).

$$\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{...})^2 \equiv nb \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...})^2 + na \sum_{j=1}^b (\bar{y}_{.j.} - \bar{y}_{...})^2 + n \sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 + \sum \sum \sum (y_{ijk} - \bar{y}_{ij.})^2 \quad (12)$$

In Eq. (12), the 1st term on the RHS represents the SS(A), the 2nd term is equal to SS(B), the 3rd term gives SS(A×B), and the last term on the RHS represents SS(Residuals). Further, "a" represents the number of levels of factor A and "b" represents the no. of factor B levels. Eq. (12) clearly shows that SS(Total) = SS(Model) + SS(Residuals), where the model terms now are A, B, and A×B, and only when the design is balanced do we have an orthogonal partition of SS(Model) as

$$SS(\text{Model}) = SS(A) + SS(B) + SS(A \times B).$$

Exercise 15. Show that for a balanced design

$$SS(A \times B) = n \sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 \equiv \left(\sum_{i=1}^a \sum_{j=1}^b y_{ij.}^2 / n \right) - CF - SS(A) - SS(B).$$

Exercise 16. Work problems 5.4 and 5.5 on page 225 of Montgomery (as part (b) of 5.4, simply compute the Studentized residuals r_{111} and r_{323}). Recall that $V(e_{ijk}) = (n-1)\sigma_{\epsilon}^2/n$. Present the data of Problem 5.4 in an OA format.

DOE INVOLVING MORE THAN TWO FIXED FACTORS

Consider the Example 5.3 on pages 208-211 of Montgomery's 8th edition. There are 3 fixed factors: A (Percent Carbonation) at 3 levels 10, 12, 14%, factor B (Operating Pressure at 25 & 30 psi), and C (Line Speed) at 2 fixed levels 200 & 250. Therefore, the experiment is called a 3×2×2 factorial CRD (completely randomized design) with $n = 2$ observations per cell. Note that 3×2×2 = 12 give rise to 12 FLCs (or cells) each of which corresponds to exactly one row of the OA in Table 6 atop the next page, which has 12 = 3×2×2 rows. Further, because the OA has 12 rows (or FLCs), then the Model will have 11 *df*.

Table 6 (Example 5.3, p. 208 of Montgomery, in OA Format)

A	B	C	y_{ijk_r}	$y_{ijk.}$
10	25	200	-3, -1	-4
10	25	250	-1, 0	-1
10	30	200	-1, 0	-1
10	30	250	1, 1	2
12	25	200	0, 1	1
12	25	250	2, 1	3
12	30	200	2, 3	5
12	30	250	6, 5	11
14	25	200	5, 4	9
14	25	250	7, 6	13
14	30	200	7, 9	16
14	30	250	10, 11	21
			$y_{...} =$	75

As always, the $USS = (-3)^2 + (-1)^2 + (-1)^2 + \dots + (11)^2 = 571$ (with $N = 24$ df) and the $CF = 75^2/24 = 234.375$ (with 1 df); thus $SS(\text{Total}) = SS_T = USS - CF = 336.625$ (with 23 df). Next, we use the observations within each cell to compute the Error SS 's: $SS_{111} = (-3)^2 + (-1)^2 - (-4)^2/2 = 2$ (with 1 df), $SS_{112} = (-1)^2 + 0^2 - (-1)^2/2 = 0.5$ (with 1 df), ..., $SS_{322} = 11^2 + 10^2 - 21^2/2 = 0.5$ (with 1 df) $\rightarrow SS(\text{Pure Error}) = \sum_{i=1}^3 \sum_{j=1}^2 \sum_{k=1}^2 \left[\sum_{r=1}^2 (y_{ijk_r} - \bar{y}_{ijk.})^2 \right] = \sum_{i=1}^3 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{r=1}^2 e_{ijk_r}^2 = 2 + 0.5 + 0.5 + 0 + 0.5 + 0.5 + 0.5 + 0.5 + 0.5 + 0.5 + 0.5 + 2 + 0.5 = 8.5$ (with 12 df). Hence, $SS(\text{Model}) = SS(\text{Total}) - SS(\text{Error}) = 336.625 - 8.5 = 328.125$ (with $23 - 12 = 11$ df).

Again, this last equality shows that Model $df = 23 - 12 = 11$, which, as stated earlier, corresponds to the 12 rows of the above Table 6. Since $y_{1...} = -4$, $y_{2...} = 20$, and $y_{3...} = 59$, and because the design is balanced (and orthogonal), then $SS(A) = (16 + 400 + 59^2)/8 - CF = 252.750$ (with 2 df); similarly, $SS(B) = (21^2 + 54^2)/12 - CF = 45.375$ (with 1 df) and $SS(C) = 22.041667$. To compute $SS(A \times B)$, we cross factors A and B, ignoring factor C and within cell variation altogether, as depicted in the interaction Table 7 atop the next page. The $A \times B$ Table 7 has $3 \times 2 = 6$ cells (or FLCs) that carry 5 df : 2 df belongs to A, 1 df belongs to B, and the remaining 2 df are assigned to $A \times B$. This implies that

Table 7. (Depicting A×B for the Example 5.3 of Montgomery 8e, p.208)

A \ B	25 psi	30	$y_{i..}$
10%	-5	1	-4
12	4	16	20
14	22	37	59
$y_{.j.}$	21	54	$y_{...} = 75$

$$\frac{(-5)^2 + 1^2 + 4^2 + 16^2 + 22^2 + 37^2}{4} - CF = SS(A) + SS(B) + SS(A \times B)$$

$$6 \text{ df} - 1 = 2 + 1 + 2 \text{ df}$$

Therefore, $SS(A \times B) = 2151/4 - 234.375 - 252.750 - 45.375 = 5.25$. Note that the df of $A \times B$ is always $df_A \times df_B$.

Exercise 17. Use the procedures illustrated above to compute the $SS(C)$, $SS(A \times C)$ and $SS(B \times C)$ of Montgomery's Example 5.3 on p. 208.

The only SS left to be computed is $SS(A \times B \times C)$. To obtain this 3-way (or 2nd order) interaction SS , we ignore only the within cell variability, i.e., we must use the cell values of y_{ijk} , which are provided in column 5 of Table 6 on the previous page. Again, we emphasize that the OA in Table 6 above has 12 rows and hence the twelve y_{ijk} values account for 11 df . That is,

$$SS(\text{Model}) = \frac{(-4)^2 + (-1)^2 + (-1)^2 + 2^2 + \dots + 16^2 + 21^2}{2} - \frac{75^2}{24} = 328.125$$

Since $SS(\text{Error}) = SS(\text{Total}) - SS(\text{Model})$ has $23 - 11 = 12 \text{ df}$, then it follows that $SS(\text{Pure Error})$ can also be computed as $SS(\text{Error}) = 336.625 - 328.125 = 8.50$. Finally,

$$SS(\text{Model}) = SS(A) + SS(B) + SS(A \times B) + SS(C) + SS(A \times C) + SS(B \times C) + SS(A \times B \times C) \quad (13)$$

$$11 \text{ df} = 2 + 1 + 2 + 1 + 2 + 1 + 2 \text{ df} \rightarrow SS(A \times B \times C) \text{ must have } 2 \text{ df}.$$

Again note that the df of $A \times B \times C$ is the product of the 3 degrees of freedom of factors A, B, and C. Further, equation (13) shows that

$$\begin{aligned}
 SS(A \times B \times C) &= SS(\text{Model}) - [SS(A) + SS(B) + SS(A \times B) + SS(C) + SS(A \times C) + SS(B \times C)] \\
 &= 328.125 - 327.041666 = 1.08333.
 \end{aligned}$$

The ANOVA Table is given atop page 210 of Montgomery(8e), which shows that only the 3 main factors are significant at the 5% level (the A×B effect is significant at the 5.60% level).

The first four *P-values* confirm the statistical significance of these 4 effects.

For blocking in factorial experiments, study section 5.6 on pp. 219-223 of Montgomery(8e), and work problem 5.21 on page 228 of Montgomery, coding the data by 85.

Exercise 18. Work problem 5.18 on page 227 of Montgomery(8e), coding the data by subtracting 195 from every observation. For part (b), only compute the Studentized residual r_{2121} , where i and $j = 1, 2, 3$ represent the 3 respective levels of % Hardwood (factor A) and Pressure (factor B), and $k = 1, 2$ extends over the levels of Cooking Time (factor C). You should try to verify your answers by means of some software such as Minitab or SAS, etc.

FITTING RESPONSE CURVES TO REGRESS Y ON THE SIGNIFICANT QUANTITATIVE INPUTS IDENTIFIED BY ANOVA

Consider the Example 5.5 on pp. 214-219 of Montgomery(8e), where the ANOVA Table 5.17 for this 3^2 CRD (with 2 replicates per cell) is given on p. 215, and the breakdown of SS's into linear and quadratic components are shown in Table 5.19 on p. 219. Our objective is to obtain a regression model that describes the relationship between the response variable y , the effective tool-life, and the linear and quadratic effects of Tool-angle (factor A = x_1), Cutting-speed (factor B = x_2), and their interactions all of which are significant at the 16.2% level or higher. Note that Table 5.19 atop p. 219 of Montgomery has misprints that I am correcting in my errata at the end of this chapter. The ANOVA Table 5.17 in the middle of page 215 of Montgomery(8e) clearly shows that $SS_{\text{Model}} = 111.0000$, while Table 5.19 atop p. 219 shows that $SS_{\text{Model}} = 49 + 16 + 8 + \dots + 8 = 127.6666666667$, which is impossible.

Following the notation of orthogonal polynomials in Table IX on page 705 of Montgomery(8e), because both factors are at 3 equi-spaced quantitative levels, then the proper quadratic regression model is

$$\hat{Y}_{ijk} = \beta_0 P_0 + \beta_1 P_1(x_1) + \beta_2 P_1(x_2) + \beta_{11} P_2(x_1) + \beta_{22} P_2(x_2) + \beta_{12} P_1(x_1) \times P_1(x_2) + \beta_{122} P_1(x_1) \times P_2(x_2) +$$

$$\beta_{112}P_2(x_1) \times P_1(x_2) + \beta_{1122}P_2(x_1) \times P_2(x_2), \quad (14)$$

where $P_1(x) = \lambda_1 \left(\frac{x - \bar{x}}{d} \right)$, $P_2(x) = \lambda_2 \left[\left(\frac{x - \bar{x}}{d} \right)^2 - \frac{a^2 - 1}{12} \right]$ is a polynomial of order 2, and $P_3(x) = \lambda_3 \left[\left(\frac{x - \bar{x}}{d} \right)^3 - \left(\frac{x - \bar{x}}{d} \right) \left(\frac{3a^2 - 7}{20} \right) \right]$ is a polynomial of order 3, and the letter d stands for the design spacing. Please note that only for the sake of convenience I have not put “hats” on the regression coefficients β 's. To solve for $\hat{\beta}_0$, we triple sum both sides of Eq. (14) from $i = 1$ to 3, $j = 1$ to 3, and $k = 1$ to 2. Note that the design is a balanced 3^2 factorial with $n = 2$ observations per FLC (see Table 5.16 on p. 214 of Montgomery). Since, P_0 is orthogonal to all $P_i(x)$ for $i \neq 0$, then the only non-vanishing term on the RHS of (14) is $N\beta_0$. Thus, $\hat{\beta}_0 = y_{...} / N = 24/18 = 1.33333$. To solve for $\hat{\beta}_1$, we multiply both sides of (14) by the 1st-order polynomial $P_1(x_1)$ and triple sum over the 9 FLCs and the $n = 2$ observations per cell. Again, since all the polynomials on the RHS of Eq. (14) are orthogonal, then the only term that does not vanish is $\beta_1 \sum \sum \sum P_1^2(x_1)$, where the 1st sum extends from $i = 1$ to 3, the 2nd extends from $j = 1$ to 3, and the 3rd sum extends from $k = 1$ to $k = n = 2$. Thus,

$$\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^2 y_{ijk} P_1(x_1) = \sum_{i=1}^3 y_{i..} P_{1i}(x_1) = \beta_1 \sum \sum \sum P_{1i}^2(x_1) \rightarrow$$

$$-1 \times (-1) + 0 \times 16 + 1 \times 9 = \beta_1(3)(2)(1+0+1) \rightarrow \hat{\beta}_1 = 5/6 = 0.8333\bar{3}$$

To solve for $\hat{\beta}_{122}$, we multiply both sides of (14) by $P_1(x_1) \times P_2(x_2)$ and triple sum over i , j , and k . In order to understand the development of the following calculations, you should cross the polynomial coefficients in Table IX on p. 705, which will pave the way for crossing A_L with B_Q , as shown below.

$$-1(-3) + 2(-3) - 1(5) + 0 + 0 + 0 + 1(-1) - 2(11) + 1(-1) = \hat{\beta}_{122} \times 2 \times \sum_{i=1}^3 \sum_{j=1}^3 P_{1i}^2 P_{2j}^2 \rightarrow$$

$$-32 = 2 \times (1 + 4 + 1 + 0 + 0 + 0 + 1 + 4 + 1) \hat{\beta}_{122} \rightarrow \hat{\beta}_{122} = -4/3 = -1.3333\bar{3}.$$

Exercise 19. Obtain the other coefficients in Eq. (14) and simplify the following regression model to verify the requisite regression model given in the middle of my following page.

$$\begin{aligned} \hat{y}_{ijk} = & 4/3 + (5/6)(x_1 - 20)/5 + (4/3)(x_2 - 150)/25 + (-2/3)\lambda_2 \left[\left(\frac{x_1 - 20}{5} \right)^2 - \frac{9-1}{12} \right] + \\ & (-1/3)\lambda_2 \left[\left(\frac{x_2 - 150}{25} \right)^2 - \frac{9-1}{12} \right] - [(x_1 - 20)/5] \times [(x_2 - 150)/25] - \\ & (4/3)[(x_1 - 20)/5] \times \lambda_2 [(x_2 - 150)^2/25^2 - 2/3] - (1/3)\lambda_2 \left[\left(\frac{x_1 - 20}{5} \right)^2 - 2/3 \right] \times (x_2 - 150)/25 - \\ & (1/3)\lambda_2 \left[\left(\frac{x_1 - 20}{5} \right)^2 - 2/3 \right] \times \lambda_2 \left[\left(\frac{x_2 - 150}{25} \right)^2 - \frac{2}{3} \right], \end{aligned}$$

where λ_i ($i = 1, 2$) is computed in such a manner that the polynomials listed in brackets above are the smallest integers as listed in Table IX on page 705, where n should be replaced with a .

Thus, for a 3-level factor, $\lambda_1 = 1$ and $\lambda_2 = 3$. I simplified the above equation for \hat{y}_{ijk} and obtained the following regression model.

$$\begin{aligned} \hat{y}_{ijk} = & -1068 + 136.30x_1 + 14.48x_2 - 4.080x_1^2 - 0.04960x_2^2 - 1.8640x_1x_2 + 0.006400x_1^2x_2^2 + \\ & 0.05600x_1^2x_2 - 0.00019200x_1^2x_2^2. \end{aligned}$$

Note that the above regression function is identical to that of Montgomery's Table 5.19 in the middle of page 197 of his 7th edition, but differs from Table 5.19 at the bottom of p. 218 of his 8th edition. Further, In order to obtain the same regression model using Minitab, I am providing the following Minitab spreadsheet, where A2 represents the quadratic component of factor A, denoted by A_Q , AB represents the interaction $A_L \times B_L$, A^2B represents $A_Q \times B_L$ and similarly for other terms.

Minitab Spreadsheet for the data of Example 5.5 on pp. 214-219 of Montgomery(8e)

A	B	A2	B2	AB	A2B	AB2	A2B2	y
15	125	225	15625	1875	28125	234375	3515625	-2
15	125	225	15625	1875	28125	234375	3515625	-1
15	150	225	22500	2250	33750	337500	5062500	-3
15	150	225	22500	2250	33750	337500	5062500	0
15	175	225	30625	2625	39375	459375	6890625	2
15	175	225	30625	2625	39375	459375	6890625	3

20	125	400	15625	2500	50000	312500	6250000	0
20	125	400	15625	2500	50000	312500	6250000	2
20	150	400	22500	3000	60000	450000	9000000	1
20	150	400	22500	3000	60000	450000	9000000	3
20	175	400	30625	3500	70000	612500	12250000	4
20	175	400	30625	3500	70000	612500	12250000	6
25	125	625	15625	3125	78125	390625	9765625	-1
25	125	625	15625	3125	78125	390625	9765625	0
25	150	625	22500	3750	93750	562500	14062500	5
25	150	625	22500	3750	93750	562500	14062500	6
25	175	625	30625	4375	109375	765625	19140625	0
25	175	625	30625	4375	109375	765625	19140625	-1

I then used Minitab's General Regression to obtain the following model:

Results for: The Example5.5.MTW on pages 214-219 of Montgomery's 8th edition.

Regression Analysis: y versus A, B, A2, B2, AB, A2B, AB2, A2B2

The regression equation is

$$y = -1068 + 136A + 14.5B - 4.08A^2 - 0.0496B^2 - 1.86AB + 0.006400A^2B + 0.0560A^2B - 0.000192A^2B^2$$

Predictor	Coef	se Coef	T	P-value
Constant	-1068.0	702.2	-1.52	0.163
A	136.30	72.61	1.88	0.093
B	14.480	9.503	1.52	0.162
A2	-4.080	1.810	-2.25	0.051
B2	-0.0496	0.03164	-1.57	0.151
AB	-1.8640	0.9827	-1.90	0.090
AB2	0.006400	0.003272	1.96	0.082
A2B	0.0560	0.02450	2.29*	0.048
A2B2	-0.000192	0.00008158	-2.35*	0.043

S = 1.20185 R-Sq = 89.5% R-Sq(adj) = 80.2%

Analysis of Variance

Source	DF	SS	MS	F ₀	P
Regression	8	111.000	13.875	9.61***	0.001
Residual Error	9	13.000	1.444		
Total	17	124.000			

*** = Significance at the 1% level.

Source	DF	Seq SS
A	1	8.333
B	1	21.333
A2	1	16.000
B2	1	4.000
AB	1	8.000
A2B	1	2.667
AB2	1	42.667
A2B2	1	8.000

The above Minitab model is identical to that of Montgomery's given on page 197 of his 7th edition but differs from Table 5.19 atop p. 219 of his 8th edition. Below I illustrate how to compute the $SS(A_Q \times B_Q) = SS(A_2B_2)$ with the aid of Table IX on p. 705 of Montgomery, where $P_2(x) = [1 \quad -2 \quad 1]'$.

$$\text{Contrast}(A_Q \times B_Q) = 1 \times 1(-3) + 1 \times (-2)(-3) + 1 \times 1(5) + -2 \times 1(2) + -2 \times (-2)(4) + -2 \times 1(10) + 1 \times 1(-1) + 1 \times (-2)(11) + 1 \times 1(-1) = -24 \rightarrow$$

$$SS(A_Q \times B_Q) = SS(A_2B_2) = \frac{(-24)^2}{2(36)} = 8, \text{ which is consistent with that given in Table 5.19 on p. 219.}$$

Errata for Chapter 5 of Montgomery's 8th edition

1. Page 214, in the 1st column heading of Table 5.16, change Total Angle to Tool Angle.
2. Pages 218 & 219, replace the entire Table 5.19 on pp. 218-219 with the above Minitab's Output.