INSY 7300-6 F2012 Reference: Chapter 2 of Montgomery's 8th Edition

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Point Estimation

As an example, consider the Bond-Strength data in Table 2.1, atop page 26 of Montgomery's 8th edition, on Modified Mortar (the experimental group). The most two important sample statistics for the response variable, y_1, of the experimental group are

$$\bar{y}_1 = \sum_{j=1}^{10} y_{1j}/n_1 = 16.764, \quad S_1^2 = \frac{1}{9}\sum_{j=1}^{n_1}(y_{1j}-\bar{y}_1)^2 = 0.10014, \text{ where } n_1 = 10$$

and the unit of measurements is in kgf/cm^2. Before sampling, $\bar{y}_1$ is an unbiased estimator of μ_1, i.e., E($\bar{y}_1$) = μ_1, and S_1^2 is an unbiased estimator of σ_1^2 iff the population is infinite in which case E(S_1^2) = σ_1^2. If the population is finite, E(S_1^2) $\neq \sigma_1^2$. For nearly all underlying distributions, E(*S*) $\neq$ σ, i.e., *S* is a biased estimator of population standard deviation σ. For a normal universe, E(*S*) = $c_4\sigma$, where $0 < c_4 < 1$, and $c_4(n) = \sqrt{\frac{2}{n-1}} \times \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}$.

<u>The operator E is linear because</u>

(1): E(CY) = CE(Y), and (2): E($Y_1 + Y_2$) = E(Y_1) + E(Y_2), where C is any constant.

<u>The operator V is nonlinear because</u>

(1): V(CY) $\neq$ CV(Y). In fact V(CY) = C^2V(Y) and

V($Y_1 \pm Y_2$) = V(Y_1) + V(Y_2) $\pm$ 2COV(Y_1, Y_2). If Y_1 and Y_2 are independent, then

COV(Y_1, Y_2) = σ_{12} = E[(Y_1-μ_1)(Y_2-μ_2)] = 0. The converse of this is not generally true. Now,

consider the numerator of $S^2 = \sum_{j=1}^{n}(y_j - \bar{y})^2/(n-1) = S_{yy}/(n-1) = CSS/(n-1)$:

$$S_{yy} = \sum_{j=1}^{n}(y_j-\bar{y})^2 = \sum_{i=1} y_j^2 - 2\bar{y}\sum y_j + \sum(\bar{y})^2 = \sum y_j^2 - (\sum y_j)^2/n$$

S_{yy} =	CSS	=	USS	–	CF
Degrees of freedom (*df*):	(n –1)	=	n	–	1

In general, if a random variable (rv), Y, has V(Y) = σ_y^2, then E(CSS/ *df*) = σ_y^2.

Interval Estimation

There are 3 types of QCH: Smaller-The-Better (STB), LTB, & Nominal-The-Best (NTB)

STB Examples: Tire eccentricity, Loudness of a compressor, Rate of wear, Braking distance, etc $\rightarrow$ Ideal target = 0 and only a single upper spec-limit, USL= y_u.

LTB (Larger-The-Better) Examples: Welding Strength, TTF (time to failure), Efficiency, Yield, etc. $\rightarrow$ Ideal target = ∞ and a single lower spec-limit, LSL = y_L.

NTB Examples: Clearance, Chemical content level, Output voltage, % Asphalt in a hot-mix asphalt (HMA) which generally ranges within 3.5-8.00%. $\rightarrow$ Ideal target = m and there are always an LSL = m – Δ_1 and an USL = m + Δ_2.

Generally, a CI (Confidence INTERVAL) for an STB parameter should be upper one-sided, for an LTB type parameter should be lower one-sided, and always 2-sided CI for an NTB parameter .

Example 1. A company manufactures ropes for climbing purposes. The consumers' LSL for breaking strength y is y_L = 1200 psi, and y ~ N(μ, 676 psi^2). Obtain a 95 % proper CI for the parameter μ using the average of a random sample of size n = 25, where $\bar{y}$ = 1215 psi.

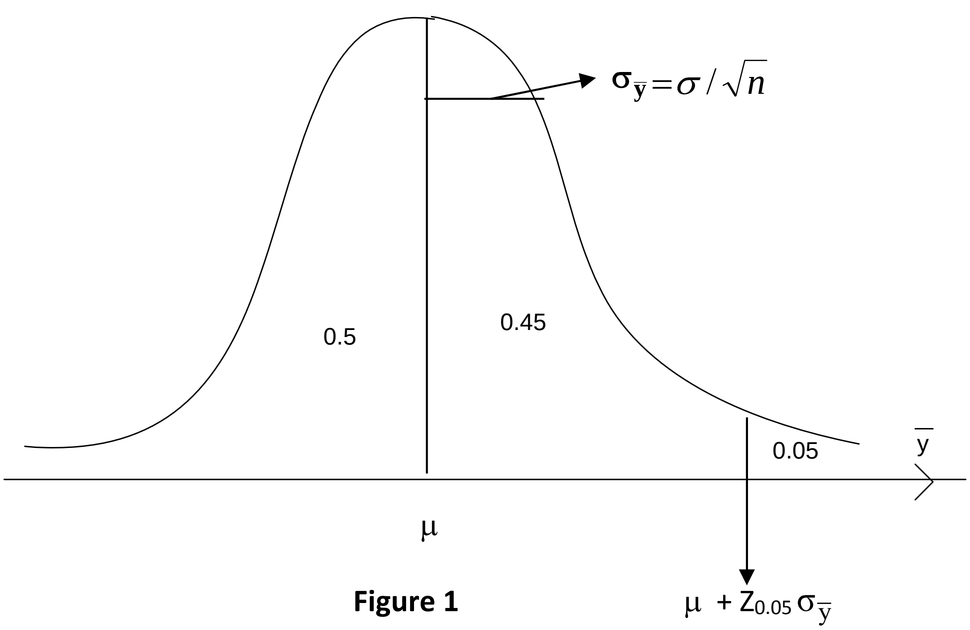


Figure 1

The sampling distribution (SMD) of $\bar{y}$ in Figure 1 shows that the Pr$(\bar{y} \le \mu + 1.645 \times 26/\sqrt{n})$ = 0.95 $\rightarrow 1215 - 1.645 \times 5.2 \le \mu < \infty \rightarrow 1206.446 \le \mu < \infty$, where μ_L = 1206.446, and $Z_{0.05} \cong 1.645$.

If a CI is lower one-sided, then the corresponding test of hypothesis must be right-tailed, i.e., for the above CI we should test H_0: $\mu = \mu_0$ psi versus H_1: $\mu > \mu_0$ (the alternative $H_1 : \mu < \mu_0$ will lead to a contradiction when H_0 is rejected). Typical values of μ_0 = 1205, 1208, 1210, etc.

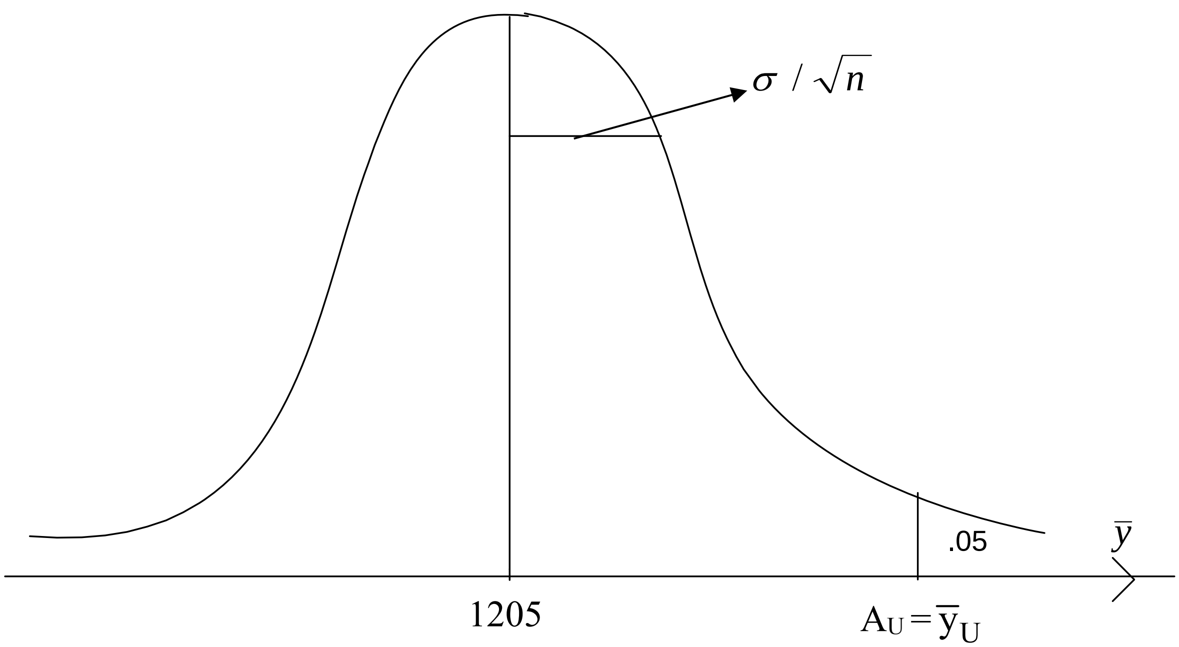


Figure 2. The SMD of $\bar{y}$ given that H_0: μ = 1205 is true

The nominal level of significance is generally set at α = 0.05. A_U = Upper Acceptance Limit = 1205 + $Z_{0.05} \times 26/\sqrt{25}$ = 1213.554 = $\bar{y}_U$ $\rightarrow$ AI (Acceptance Interval) : $0 < \bar{y} \le 1213.554$. The test statistic $\bar{y}$ = 1215 > $\bar{y}_U$ $\rightarrow$ Reject H_0 at the LOS α = 0.05 and conclude that $\mu > 1205$. Note that an upper one-sided CI for the above test is given by $-\infty < \mu \le 1215 + 8.554 = 1223.554$, which includes the hypothesized value of μ = 1205 and hence contradictory to the rejection of H_0.

Assignment 1. Work problem 2.17 on page 60 of your text. ANS: (c) *P-value* = 0.0549, (d) [799.75, 824.25]. (b) Work problem 2.20, p. 61. ANS: n $\ge$ 139. (c) Work problem 2.5. (d) Work problem 2.22 but change part (a) to determining if the population mean repair time is less than 250 hours? Note that in this problem you will have to use the Student's t-distribution.

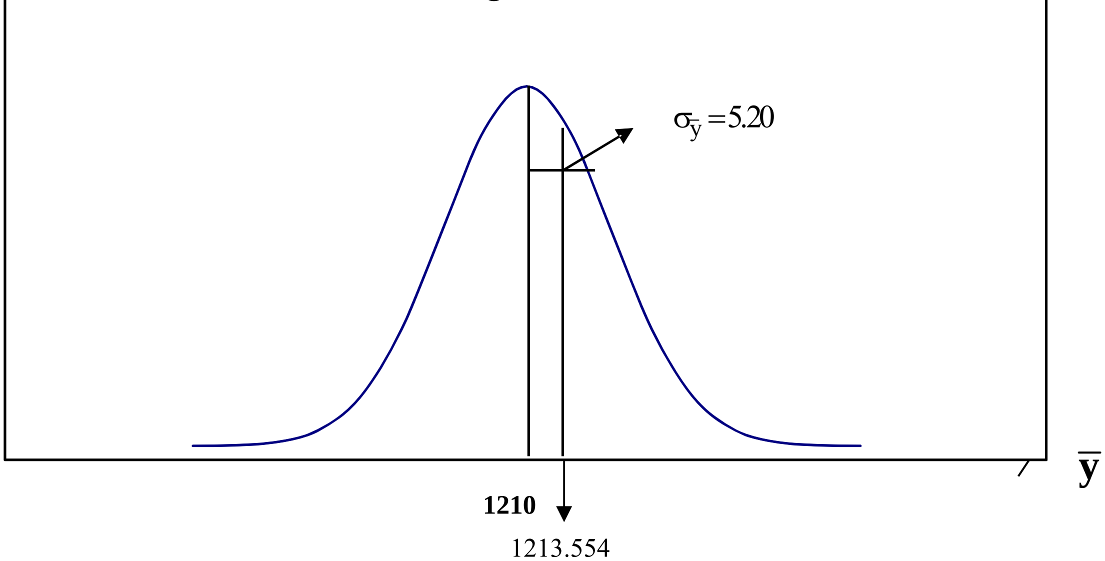
Test of Hypothesis

In conducting any test of hypothesis, only one of the 4 circumstances given in Table 1 will occur, where Pr denotes probability.

Table 1. The four circumstances that may occur when testing H_0

	H_0 is true	**H_0 is false**
Reject H_0	**Type I error (or False Positive)** **Occurrence Pr = α**	**Correct decision (True Positive),** **Occurrence Pr = $1-\beta$ = Power or Sensitivity of the test**
Accept H_0	**Correct decision (True Negative)** **Specificity of the test = Occurrence Pr = $1-\alpha$**	**Type II error (or False Negative)** **Occurrence Pr = β = The Pr of Accepting H_0 at a specified value of the parameter under H_0.**

Figure 3



Note that in order to commit a type II error, H_0 must be false. In reference to Example 1, this implies that μ must differ from 1205 psi, say μ = 1210. From Figure 3, $Z_{1-\beta}$ = (1213.554 – 1210)/5.2 = 0.683462 $\rightarrow$ β(at μ = 1210) = Φ (0.683462) = 0.7528424.

Assignment 2. Compute the values of β for values of μ = 1205, 1213.554, 1215, 1218 and 1225. Graph β as a function of μ. This graph of type II error Pr versus the parameter under H_0 is called the OC (Operating Characteristic) curve.

If the population variance σ^2 is unknown, then statistical inference (i.e., estimation & test of hypothesis) on μ cannot be conducted using the statistic $Z_0 = (\bar{y} - \mu_0)\sqrt{n} / \sigma$, rather resort has to be made to the sampling distribution of the statistic $(\bar{y} - \mu_0)\sqrt{n} / S$ which has (W. S. Gosset's) Student's t-distribution with (n –1) *df*, such as in problem 2.20. (Also see the Example 2.2 on pp. 51-52 of Montgomery's 8th edition, and Problems 2.32, 2.33 & 2.34 all on the paired t-test).

Statistical Inference on σ^2

We use the fact that the SMD (sampling distribution) of the random variable $(n – 1)S^2/\sigma^2$ from a normal (or Laplace-Gaussian) universe follows a Chi-square (χ^2) distribution with (n –1) *df*. As an example, consider the problem 2.31 on page 63 of Montgomery's 8th edition.

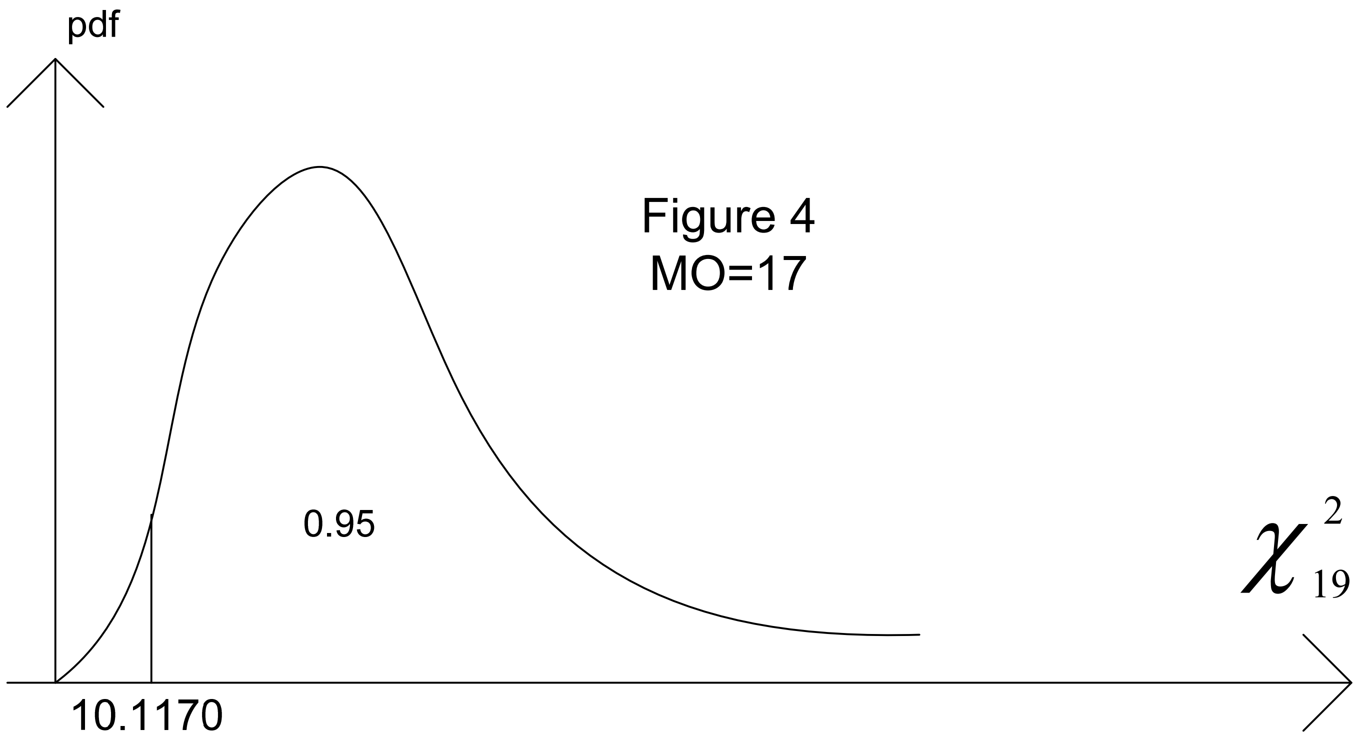
Data Statistics: $\sum_{j=1}^{20} y_j = 116.56,\ \ \bar{y} = 5.828,\ \text{USS} = \sum y_j^2 = 694.3302,\ S^2 = 0.79045$

$\rightarrow$ CSS = 694.3302– 116.56²/20 = 15.01852; Figure 4 atop the next page shows that

$\chi^2_{0.95,19}$ =10.1170 $\rightarrow$ Pr($\chi^2_{19} \geq$ 10.1170) = 0.95 $\rightarrow$ Pr[(n –1)$S^2/\sigma^2 \geq$ 10.1170] = 0.95 $\rightarrow$

$0 < \sigma^2 \leq 19(0.79045)/10.1170 \rightarrow \quad 0 < \sigma^2 \leq 1.48448 \quad \rightarrow 0 < \sigma \leq 1.21839 \rightarrow$

We are 95% confident that the process variance σ^2 lies within the interval (0, 1.4845]. The Pr that this last interval includes σ^2 is 0 or 1. The above CI implies that we cannot reject the null hypothesis H_0: σ^2 = 1.20 versus the alternative H_1: σ^2 < 1.20, i.e., we cannot conclude that σ^2 < 1.20 at the 5% level; however, we can reject the null hypothesis H_0: σ^2 = 1.60 versus the alternative H_1: σ^2 < 1.60 because 1.60 is outside the 95% CI ($0 < \sigma^2 \leq 1.48448$]. Note that 10.1170 represents the 95th percentage point of Chi-square, i.e., 10.1170 = $\chi^2_{0.95,19}$, or its 0.05 quantile.



The Modal point = MO =17

Statistical Inference on Two Normal Population Parameters

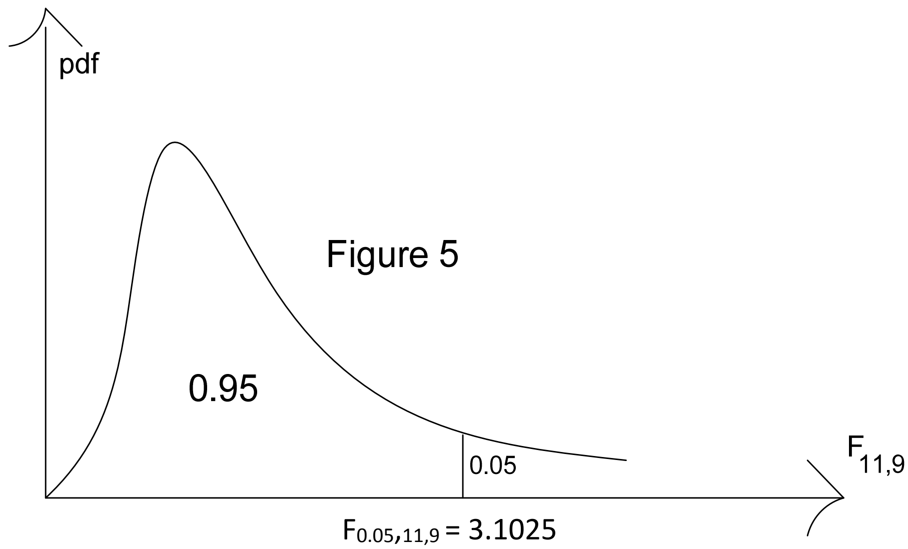
Although Montgomery covers the test of two-variance equality at the end of Chapter 2 (pp. 58-59) and a pretest on $\sigma_1^2 = \sigma_2^2$ is judicious in order to determine whether to pool the variances from 2 independent populations, we will first cover inferences about variances of two normal populations, followed by inferences on two independent population means.

Statistical Inference on Two Variances

Sir Ronald A. Fisher's F statistic describes the sampling distribution of the ratio of 2 scaled χ^2 distributions give below:

$$F = \frac{\chi_1^2/\nu_1}{\chi_2^2/\nu_2} = \frac{[(n_1-1)S_1^2/\sigma_1^2]/(n_1-1)}{[(n_2-1)S_2^2/\sigma_2^2]/(n_2-1)} = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2},$$

where $\nu_1 = n_1 - 1$ is the *df* of the numerator and $\nu_2 = n_2 - 1$ is the *df* of the denominator. The Modal point of an F distribution is roughly ≅ 1 for ν_1 and $\nu_2 > 8$ (but always less than 1). Now, consider the



Example 2.3 on pages 58-59 of Montgomery: $n_1 = 12$, $S_1^2 = 14.5$, $n_2 = 10$, and $S_2^2 = 10.8$. Figure 5 above clearly shows that the $\Pr(F_{11,9} \le 3.1025) = 0.95$, i.e., the cdf of the rv $F_{11,9}$ at $F_{0.05,11,9} = 3.1025$ is equal to 0.95. Put differently, the 5 percentage point of the F-distribution, or the 0.95 quantile, with $\nu_1 = 11$ and $\nu_2 = 9$ is given by $F_{0.05,11,9} = 3.1025$. Consequently, $\Pr(F_{11,9} \le 3.1025) = 0.95 \rightarrow$

$$\Pr\left(\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \le 3.1025\right) = 0.95 \rightarrow \Pr\left(\frac{S_1^2}{3.1025\, S_2^2} \le \sigma_1^2/\sigma_2^2 < \infty\right) = 0.95 \rightarrow$$

$$0.4327 \le \sigma_1^2/\sigma_2^2 < \infty.$$

The above CI is consistent with testing H_0: $\sigma_1^2/\sigma_2^2 = 1$ vs the alternative H_1: $\sigma_1^2/\sigma_2^2 > 1$ because the CI encloses the null-hypothesized value of $\sigma_1^2/\sigma_2^2 = 1$.

However, suppose we had $n_1 = 12$, $S_1^2 = 14.5$, $n_2 = 10$ but $S_2^2 = 4.42$. Now the test statistic $F_0 = 3.2805 > F_{0.05,11,9} = 3.1025$ leads to the rejection of H_0 at the 5% level; however, the 95% 2-sided CI : $0.8386 \le \sigma_1^2/\sigma_2^2 \le 11.7703$, where $0.8386 = \dfrac{S_1^2/S_2^2}{F_{0.025,11,9}}$, $F_{0.025,11,9} = 3.9121$, contains

σ_1^2 / σ_2^2 = 1, which is contradictory to the rejection of H_0! The correct one-sided CI : 1.0574 ≤ $\sigma_1^2 / \sigma_2^2 < \infty$ excludes the null hypothesized value of σ_1^2 / σ_2^2 = 1 as required. Note that for the right-tailed alternative $H_1 : \sigma_1^2 / \sigma_2^2 > 1$, we must obtain a lower one-sided CI because the upper one-sided 95% CI: $0 < \sigma_1^2 / \sigma_2^2 \le \dfrac{S_1^2 / S_2^2}{F_{0.95,11,9}} = \dfrac{3.2805}{0.3453} \rightarrow 0 < \sigma_1^2 / \sigma_2^2 \le 9.5012$ includes the hypothesized value of σ_1^2 / σ_2^2 = 1, which contradicts the rejection of H_0.

Testing the Equality of Two Independent Population Means

(a) Case of H_0: $\sigma_1^2 = \sigma_2^2 = \sigma^2$ not rejected at the 20% level. Consider the 2-sided hypothesis H_0: $\mu_1 - \mu_2 = \delta$ versus $H_1 : \mu_1 - \mu_2 \ne \delta$. Then, the Test Statistic is

$$t_0 = [(\bar{y}_1 - \bar{y}_2) - \delta] / (S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}),$$

where S_p^2 = (CSS₁ + CSS₂)/*df*. Values of $|t_0| > t_{\alpha/2, n_1+n_2-2}$ lead to the rejection of H_0, i.e., the rejection region is $(-\infty, -t_{\alpha/2,n_1+n_2-2}) \cup (t_{\alpha/2,n_1+n_2-2}, \infty)$. See the example on pp. 38-39 of Montgomery, and problems 2.26, 2.27. Note that a pretest on H_0: $\sigma_1^2 = \sigma_2^2$ at the 20% yields $F_0 = S_1^2 / S_2^2$ = 0.10013777/0.06146222 = 1.6293, which yields a *P-value* = 0.4785 > 0.20, and hence the null hypothesis H_0: $\sigma_1^2 = \sigma_2^2$ is tenable. Thus, we may use the pooled t-statistic to test H_0: $\mu_1 - \mu_2 = \delta$, i.e., the *P-value* of the pretest has exceeded 20% providing convincing evidence in favor of pooling variances. Note that H_0 declares that σ_1^2 and σ_2^2 have a common value of σ^2.

Analysis of Data in Table 2.1 on page 26 of Montgomery's 8th Edition

Y = Tension Bond Strength is measured in Kgf/cm^2. Experimental Group: y_{1j} : 16.85, 16.40, 17.21, 16.35,16.52, 17.04, 16.96, 17.15, 16.59, 16.57 → $\bar{y}_1$ = 16.7640; USS_1 = 2811.2182, CF_1 = 167.64^2 / 10 = 2810.31696 ⟶ CSS_1 = 0.90124 → S_1^2 = 0.10013777 → S_1 = 0.31645

Control Group: y_{2j} : 16.62, 16.75, 17.37, 17.12, 16.98, 16.87, 17.34, 17.02, 17.08, 17.27

$\overline{y}_2$ = 17.042, USS_2 = 2904.85080, $CF_2 = 170.42^2/10$ = 2904.2976400 $\rightarrow$ CSS_2 = 0.5531600 $\rightarrow$ S_2^2 = 0.06146222 $\rightarrow$ S_2 = 0.24792 (Control group), and as a result

$$S_p^2 = \frac{\nu_1 S_1^2 + \nu_2 S_2^2}{\nu_1 + \nu_2} = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Only when $n_1 = n_2 = n$, then $S_p^2 = (S_1^2 + S_2^2)/2 = 0.08080$ $\rightarrow$ $S_p = \sqrt{0.0808}$ = 0.2842534.

Assuming that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ is not rejected at the 20% level, , then $t_0 = [(\overline{y}_1 - \overline{y}_2) - \delta]/se$, where $se = se(\overline{y}_1 - \overline{y}_2) = S_p\sqrt{(1/n_1)+(1/n_2)}$ = $0.2842534(0.10 + 0.10)^{0.5}$ = 0.127122 $\rightarrow$ t_0 = [–0.2780 – 0]/0.127122 = –2.1869; $t_{0.025,18}$ = 2.100924 $\rightarrow$ Reject $H_0 : \mu_1 - \mu_2 = 0$ at the LOS α = 0.05 because $|t_0| > t_{0.025,18}$. The *P-value* = $2\times\Pr(T_{18} \le -2.1869)$ = 2×0.021097343 = 0.042194685 < 0.05 because H_0 was rejected at the 5% level. See Table 2.2 on p. 41 of Montgomery.

(2) Case of $\sigma_1^2 \neq \sigma_2^2$

Test statistics: $t_0 = [(\overline{y}_1 - \overline{y}_2) - \delta]/\sqrt{(S_1^2/n_1) + (S_2^2/n_2)}$, but the *df* is given by ν in equation (2.32) on page 48 of Montgomery's 8th edition and generally $\min(n_1, n_2) < \nu \le n_1 + n_2 - 2$. A simplified version of that Eq. (2.32) of Montgomery is given by $\nu = \frac{\nu_1\nu_2[v(\overline{y}_1) + v(\overline{y}_2)]^2}{\nu_2(v(\overline{y}_1))^2 + \nu_1(v(\overline{y}_2))^2} =$

$\frac{\nu_1\nu_2(F_0R_n + 1)^2}{\nu_2(F_0R_n)^2 + \nu_1}$, where $v(\overline{y}_1) = S_1^2/n_1$, $R_n = n_2/n_1$ and $F_0 = S_1^2/S_2^2$. For equal sample sizes, the above formula reduces to $\nu = \frac{(n-1)(F_0+1)^2}{(F_0)^2 + 1}$. For the sake of illustration we assume that $\sigma_1^2 \neq \sigma_2^2$, so that for the data of Montgomery's Table 2.1, the $se(\overline{y}_1 - \overline{y}_2) = [(S_1^2/n_1) + (S_2^2/n_2)]^{0.50}$ = 0.127122, t_0 = –2.1869 , and ν is also given by the formula in Table 2.4 on page 52 of Montgomery's 8th edition. $\nu = \dfrac{9(0.01616)^2}{(0.01001377\overline{7})^2 + (0.00614622\overline{2})^2}$ = 17.025 ⟶

-*value* = 2×Pr($T_{17.025} \le -2.1869$) = 2×0.02149810067712 = 0.0429962 (see Table 2.2, p.41), which is bit more conservative than that of the pooled t-test *P-value*. Note that F_0 = 1.62926 and

$\nu = \dfrac{9(F_0+1)^2}{(F_0)^2+1}$ leads to the same answer of ν = 17.025. Further, Montgomery also covers the paired t-test on pp. 53-57 and Problems 2.32, 2.33 & 2.34. The pertinent hardness-example with data in Table 2.6 will be discussed in class.

The Relative Efficiency in Hypothesis Testing

The relative efficiency (RELEFF) of an α-level statistical test T_1 to an α-level test T_2 is given by n_2/n_1 iff both tests have identical values of type II error probability β. As an example, if T_1 requires a sample of size $n_1 = 20$ and has $\alpha = 0.05$, $\beta = 0.10$, but T_2 requires an $n_2 = 25$ to attain the same α = 0.05 and β = 0.10, then the efficiency of T_1 relative to T_2 is given by 25/20 = 125%, or the RELEFF(of T_2 to T_1) = 20/25 = 80%. On the other hand, if the 5% level tests T_1 and T_2 both use the same random sample of size $n = n_1 = n_2 = 25$, but $\beta(T_1) = 0.10$ while $\beta(T_2) = 0.125$, then the RELEFF of T_1 to T_2 is given by 0.125/0.10 = 125%. Further, suppose the RELEFF(T_1, T_2) = 125%, both having the same α & β, and T_2 has a sample size $n_2 = 30$. Then the sample size for T_1 must be obtained from RELEFF(T_1, T_2) = 1.25 = $n_2/n_1 = 30/n_1 \rightarrow n_1 = 30/1.25 = 24$.

Errata for Chapter 2 of Montgomery's 8th Edition

1. Page 33, in Figure 2.5 change σ^2 to σ.
2. Page 37, atop the page in the description of Figure 2.10, change the terminology critical region to either critical values or critical limits (or possibly rejection thresholds).
3. Page 59, the 2-sided CI on variance ratio σ_1^2/σ_2^2 in Eq. (2.50) should

   appropriately be changed to $0.4331 \le \sigma_1^2/\sigma_2^2 < \infty$, because the test on $\sigma_1^2/\sigma_2^2 = 1$

   is right-tailed.