

Factorial Experiments with Random Factors

So far emphasis has been placed on factorial experiments where all factors are at “*a*”, “*b*”, “*c*”, ... fixed levels (i.e., their levels were not selected at random). In such experiments, conclusions must be restricted to the region of factor space spanned by the ranges of the factors. Then, regression may be used to predict the response, *y*, within the region of factor spaces. In general, we may claim that almost all quantitative factors (such as temperature, pressure, feed rate, etc.) considered in the design of an experiment are fixed while majority of qualitative factors are random. By a random factor we mean that its levels in the design phase of the experiment are actually selected at random from a (finite or infinite) population of levels. I would venture to say that 90% of all random factors in all designs of experiments in the universe are qualitative factors! Examples of qualitative factors are operators, parts, days, material type, batches of raw material, nozzle type, tool shape or geometry, etc., et cetera. When all factors in an experiment are fixed, the corresponding model is called fixed-effects; when all factors in an experiment are random, the corresponding model is called random-effects or the components of variance model; when some factors are fixed and the rest are at random levels, then the corresponding statistical model is called mixed effects. The SS's computations for the ANOVA remain exactly the same as before for all three models, but obtaining the F-statistic, the conclusions, and POST-ANOVA procedures are different for the three models.

BAD NEWS

So far everything has been “gravy” because we have dealt only with fixed-level factors so that the denominators of all F statistics, i.e., the denominators of F_0 values, have always been the MS(Error) or MS(Residual Error). Unfortunately, if at least one factor in experiments has random levels, giving rise to a random- or mixed-effects model, no longer the denominator of all F_0 values will be the MS(Error), but rather, each denominator has to be determined separately based on its expected mean square (EMS). For example, if we have a $3 \times 4 \times 2$

factorial design with A and B at 3 and 4 fixed levels, respectively, factor C at 2 random levels and $n = 2$ random observations per cell, I would not know (with my 40 years of experience in the DOE) what the denominator of each F statistic will be until I develop the EMS column for this $3 \times 4 \times 2$ experiment. In order for us to learn how to do this systematically, we only need to learn an extra concept in designing experiments, namely nesting (which will be covered in Chapter 14). Factor B is said to be nested under factor A iff the levels of B cannot be crossed with those of A. As examples, see Figures 14.1 & 14.2 on page 605 of Montgomery's 8th edition, where Batches are nested within (or under) Suppliers, and in Figure 14.5 on page 614 of Montgomery(8e), where Ingots are nested under Heats and Heats are nested within Alloy formulation (at 2 levels). Although we have not yet covered nested factorials in this course, we need this concept because repeat observations (or replications within the same FLC) are indeed nested within each cell (or FLC), i.e., for the above $3 \times 4 \times 2$ design with $n = 2$ observations per cell, the $\epsilon_{ijk r}$ term should actually be written as $\epsilon_{r(ijk)}$ or $\epsilon_{(ijk)r}$ because the index r is nested within the (ijk) cell. In obtaining the EMS (Expected Mean Squares) column, every index inside parentheses () will be called dead and those outside () will be called live. We now state the rules for the Bennett & Franklin (I am not certain of this reference) algorithm to obtain the EMS column for all the effects in a mixed-effects factorial experiment.

Rule 1. In each row, write "1" if the dead subscripts in the row matches column subscripts.

Rule 2. For the case of restricted model, in each row, write 0 for fixed columns and write 1 for random columns wherever live subscripts match. For the unrestricted model, write 1 wherever live subscripts match.

Rule 3. In the remaining empty row positions, write the number of levels shown in the column headings.

We now use the above 3 rules to obtain the following Table 20 needed to derive the EMS (expected mean squares) column for the CR (completely randomized) $3 \times 4 \times 2$ factorial with 2 repeat observations per cell and then will describe the last rule after constructing the Table.

In Table 20, $\Phi_A = \sum_{i=1}^3 A_i^2 / 2$, $\Phi_B = \sum_{j=1}^4 B_j^2 / 3$, and $\Phi_{A \times B} = \sum_{i=1}^3 \sum_{j=1}^4 (A \times B)_{ij}^2 / 6$, where $(A \times B)_{ij} = \mu_{ij} -$

Table 20. (Rand = R = Random; F = Fixed; the case of restricted model)

Model Terms	Fixed 3 i	F 4 j	Rand 2 k	R n = 2 r	EMS	df
A_i	0	4	2	2	$16\Phi_A + 8\sigma^2_{A \times C} + \sigma^2_{\epsilon}$	2
B_j	3	0	2	2	$12\Phi_B + 6\sigma^2_{B \times C} + \sigma^2_{\epsilon}$	3
$(A \times B)_{ij}$	0	0	2	2	$4\Phi_{A \times B} + 2\sigma^2_{A \times B \times C} + \sigma^2_{\epsilon}$	6
C_k	3	4	1	2	$24\sigma^2_C + \sigma^2_{\epsilon}$	1
$(A \times C)_{ik}$	0	4	1	2	$8\sigma^2_{A \times C} + \sigma^2_{\epsilon}$	2
$(B \times C)_{jk}$	3	0	1	2	$6\sigma^2_{B \times C} + \sigma^2_{\epsilon}$	3
$(A \times B \times C)_{ijk}$	0	0	1	2	$2\sigma^2_{A \times B \times C} + \sigma^2_{\epsilon}$	6
$\epsilon_{(ijk)r}$	1	1	1	1	σ^2_{ϵ}	24

$\mu_i - \mu_j + \mu$; further, the $E(MS_A)$ was obtained by 1st covering the entire column i, second using only the rows whose effects in the 1st column has the index i and then multiplying the visible numbers by the corresponding variance component. The relevant rows for $E(MS_A)$ are A_i , $(A \times B)_{ij}$, $(A \times C)_{ik}$, $(A \times B \times C)_{ijk}$ and $\epsilon_{(ijk)r}$ because these rows all contain the index i. Table 20 clearly shows that only the effects C, A×C, B×C, and A×B×C must be tested against MS(Error), i.e., the divisor of their F statistics must be MS(Error). However, Factor A must be tested against MS(A×C); Factor B must be tested against MS(B×C), and the statistic for testing the null hypothesis $H_0 : (A \times B)_{ij} = 0$ is $F_0 = MS(A \times B) / MS(A \times B \times C)$. Unfortunately, the above design is inadequate because the denominator of the F statistics for the 2 fixed factors A and B have only 2 and 3 df, respectively. Generally, the denominator of the F test should have at least 5 df (preferably 6 df) for the F test to be fairly stable and to possess sufficient power to reject H_0 if H_0 is false. Therefore, we have to redesign the above experiment by using at least 4 levels of the random factor C.

Exercise 36. Work problem 5.19 on pages 227-228 of Montgomery(8e) but omit the last (i.e., the 3rd) observation in every cell so that you have a 3×2×3 factorial with n = 2 observations per cell. Further, assume that only A (Cycle Time indexed by i) and B (Temperature indexed by j) are fixed but the three operators were randomly selected from all the operators in the plant. Code the data by subtracting 23 and comment on the adequacy of the design.

The 2-Factor Factorial Design with Both Factors at Random Levels

Consider the experiment in Example 13.1 of Montgomery on pp. 575-578, where factor P (Part Number) is at 20 random levels and factor O (Operators) is at 3 random levels with n = 2 observations per FLC. Therefore, the statistical model $y_{ijk} = \mu + P_i + O_j + (P \times O)_{ij} + \epsilon_{ijk}$, $i = 1, 2, \dots, 20$; $j = 1, 2, 3$, and $k = 1, 2$ is called the components of variance (or random-effects) model because both P and O were selected at random. The assumptions of this model are as follows:

(1) P_i , O_j , $(P \times O)_{ij}$ and ϵ_{ijk} are NID distributed with variances

$$V(P_i) = \sigma_P^2, \quad V(O_j) = \sigma_O^2, \quad V[(P \times O)_{ij}] = \sigma_{P \times O}^2, \quad \text{and} \quad V(\epsilon_{ijk}) = \sigma_\epsilon^2.$$

(2) Because of the independence assumption of the rv's P_i , O_j , $(P \times O)_{ij}$ and ϵ_{ijk} , the total variance is $V(y_{ijk}) = \sigma_P^2 + \sigma_O^2 + \sigma_{P \times O}^2 + \sigma_\epsilon^2$. Thus, there are 4 components of variance in the model $y_{ijk} = \mu + P_i + O_j + (P \times O)_{ij} + \epsilon_{ijk}$, where σ_ϵ^2 measures gauge repeatability, while $(\sigma_O^2 + \sigma_{P \times O}^2)$ measures gauge reproducibility, i.e., $V(\text{Gauge}) = (\sigma_O^2 + \sigma_{P \times O}^2) + \sigma_\epsilon^2$.

(3) There are three null hypotheses to be tested from the ANOVA, which are $H_0: \sigma_P^2 = 0$, $H_0: \sigma_O^2 = 0$, and $H_0: \sigma_{P \times O}^2 = 0$ VS $H_1: \sigma_{\text{Eff}}^2 > 0$, where Eff = P, O, or P×O.

Since there are only random factors involved in the experiment, the denominator of all F statistics will no longer be MS(Error) and have to be determined through the algorithm (Bennett and Franklin 1954) described earlier under Bad News above. Applying that algorithm to the experiment of Montgomery's Example 13.1 leads to the EMS column given in Table 21. Table 21 clearly shows that both random factors P and O must be tested against MS(P×O), but the denominator of the F statistic for the P×O effect must be MS(Error). The ANOVA is provided on

Table 21. (The EMS for the Example 13.1 on pp. 575-8 of Montgomery(8e); R→ Random)

Source	R	R	R	df	EMS = Expected Mean Square
	20	3	2		
	i	j	k		
P_i	1	3	2	19	$6\sigma_P^2 + 2\sigma_{P \times O}^2 + \sigma^2$ →
O_j	20	1	2	2	$40\sigma_O^2 + 2\sigma_{P \times O}^2 + \sigma^2$
$(P \times O)_{ij}$	1	1	2	38	$2\sigma_{P \times O}^2 + \sigma^2$ ←
$\epsilon_{(ij)k}$	1	1	1	60	$\sigma^2 = \sigma_\epsilon^2$

page 577 of Montgomery(8e) in Table 13.2, which shows that only Part-to-Part variation is highly significant; there is a slight variation due to operators (P -value = 0.173), but the null hypothesis $H_0: \sigma_{P \times O}^2 = 0$ VS the alternative $H_1: \sigma_{P \times O}^2 > 0$ cannot at all be rejected (P -value = 0.8614). As post-ANOVA, we first obtain point moment estimates of variance components using the corresponding EMS from the above Table 21. The EMS column clearly shows that $E(MS_P) = 6\sigma_P^2 + 2\sigma_{P \times O}^2 + \sigma_\epsilon^2$. Note that every term in this equation is a parameter and has to be estimated from the ANOVA Table. In order to estimate σ_P^2 , we replace every term in this last parametric equation with their corresponding moment estimator, i.e., $MS_P \cong 6\hat{\sigma}_P^2 + MS(P \times O) \rightarrow \hat{\sigma}_P^2 = \frac{MS_P - MS(P \times O)}{6} = \frac{62.3908 - 0.71184}{6} = 10.280$; similarly, $MS(O) \cong 40\hat{\sigma}_O^2 + MS(P \times O) \rightarrow \hat{\sigma}_O^2 = (1.3083\bar{3} - 0.71184)/40 = 0.01491$. However, an attempt to estimate $\sigma_{P \times O}^2$ in the same manner will lead to a negative value of $\hat{\sigma}_{P \times O}^2$, which is contradictory to the fact that variances can never be non-positive. Hence, we have to assume that $\hat{\sigma}_{P \times O}^2 = 0$. As Montgomery(8e) points out on his page 575, that this is a gauge R&R study (i.e., gauge repeatability and reproducibility study) where repeatability is estimated by $MS(\text{Error}) = 0.99166$ and reproducibility is measured by $\sigma_O^2 + \sigma_{P \times O}^2$, which in this case is

estimated by $\hat{\sigma}_O^2 + \hat{\sigma}_{P \times O}^2 = 0.01491$. Therefore, $\hat{\sigma}_{\text{gauge}}^2 = 0.99166 + 0.01491 = 1.006570$.

Since $V(y) = V(\text{Parts}) + V(\text{Gauge})$, then 91.082% ($= 10.280/11.28657$) of the total variability is due to Parts, and hence, the gauge is fairly capable of measuring different parts dimensions! The reader should study the ANOVA of the reduced model on page 578 of Montgomery(8e).

Exercise 37. Work Problem 13.1 on page 601 of Montgomery(8e). You must code the data by subtracting 45 from all $N = 10 \times 2 \times 3 = 60$ observations, which were CR. Compute the % variation due to the measuring instrument.

The Two-Factor Mixed-Effects Model

Suppose in the Example 13.1 of Montgomery there are only three operators in the plant that make use of the single gauge to make measurements of the parts, i.e., the entire population levels of factor O were used in the experiment so that now O (= Operator) is a factor that is fixed at 3 levels, i.e., our conclusions will pertain only to the three operators used in the experiment. Montgomery presents two mixed models but we will discuss and require the knowledge of the most commonly encountered one in these notes. If operators are

assumed fixed, then the restrictions that will be placed in the model are $\sum_{j=1}^3 O_j = 0$,

$\sum_{j=1}^3 (P \times O)_{ij} = 0$; further, P_i 's are rvs distributed as $N(0, \sigma_P^2)$ and the rvs $(P \times O)_{ij}$ are assumed

$N(0, \sigma_{P \times O}^2)$ and stochastically independent from P_i 's. If the restriction $\sum_{j=1}^3 (P \times O)_{ij} = 0$ is

removed, then the model is called the unrestricted mixed model, with different divisors for the F-statistics. To obtain the EMS column that paves the way to the proper F tests, we modify

Table 21 above as in Table 22. Table 22 atop the next page, where $\Phi_O = \sum_{j=1}^3 O_j^2 / 2$, clearly

shows that now only the fixed factor (O) must be tested against MS(P×O) but the two random effects, P and P×O, must be tested against the MS(Error) in the ANOVA table. Further, $V(y) =$

Table 22. ($\sigma^2 = \sigma_{\epsilon}^2$; F \rightarrow Fixed-levels factor; R = Random-levels factor)

Source	R 20 i	F 3 j	R 2 k	df	EMS
P _i	1	3	2	19	$6\sigma_P^2 + \sigma^2$
O _j	20	0	2	2	$40\sigma_O^2 + 2\sigma_{P \times O}^2 + \sigma^2$
(P \times O) _{ij}	1	0	2	38	$2\sigma_{P \times O}^2 + \sigma^2$
$\epsilon_{(ij)k}$	1	1	1	60	σ^2

$\sigma_P^2 + \sigma_{P \times O}^2 + \sigma_{\epsilon}^2$, and therefore, reproducibility is measured only by $\sigma_{P \times O}^2$. Since Operators

are fixed, then operator effects can be estimated as $\hat{O}_j = \bar{y}_{.j.} - \bar{y}_{...}$ with $se(\bar{y}_{.j.}) =$

$\sqrt{MS_{P \times O} / (2 \times 20)} = \sqrt{MS_{P \times O} / 40}$ because $F_0(O) = MS_O / MS_{P \times O}$. See the Example 13.2 on pp. 582-583 of Montgomery(8e).

Mixed-Model Factorials With at Least 2 Random Factors

Consider a mixed factorial experiment with 3 or more factors where at least 2 of the factors are at random levels. Factorial designs where there is only one random factor pose no problem from the standpoint of forming a direct F-statistic to test $H_0: A_i = 0$ and $H_0: B_j = 0$, et cetera. However, in such a case the designer has to be careful to randomly select sufficient levels of the random factor so that the denominator of all F statistics have at least, say, 5 df (I would prefer 6 df if at all possible). Now, consider a 3-factor CR experiment where factor A is at 2 fixed levels, factor B has 3 random levels, factor C is at c random levels, the value of c to be determined after the EMS column has been developed, and $n = 2$ observations at each FLC. Table 23 shows the EMS (Expected Mean Squares) column for this balanced $2 \times 3 \times c$ factorial with $n = 2$ observations per cell. Table 23, atop the next page, clearly shows that there are direct F tests for every effect except for factor A, whose levels are fixed. Table 23 clearly shows that the random factors B and C must be tested against $MS(B \times C)$; the $A \times B$ and $A \times C$ effects must be tested against $MS(A \times B \times C)$, and the null hypotheses $H_0: \sigma_{B \times C}^2 = 0$ VS $H_1: \sigma_{B \times C}^2 > 0$ and $H_0: \sigma_{A \times B \times C}^2 = 0$ versus

Table 23. (The EMS for a 3-factor CR experiment with 2 random factors B & C)

Model Terms	F 2 i	R 3 j	R c k	R n=2 r	EMS	df
A_i	0	3	c	2	$6c\Phi_A + 2c\sigma_{A \times B}^2 + 6\sigma_{A \times C}^2 + 2\sigma^2(A \times B \times C) + \sigma_\epsilon^2$	1
B_j	2	1	c	2	$4c\sigma_B^2 + 4\sigma_{B \times C}^2 + \sigma_\epsilon^2$	2
$(A \times B)_{ij}$	0	1	c	2	$2c\sigma_{A \times B}^2 + 2\sigma_{A \times B \times C}^2 + \sigma_\epsilon^2$	2
C_k	2	3	1	2	$12\sigma_C^2 + 4\sigma_{B \times C}^2 + \sigma_\epsilon^2$	c-1
$(A \times C)_{ik}$	0	3	1	2	$6\sigma_{A \times C}^2 + 2\sigma_{A \times B \times C}^2 + \sigma_\epsilon^2$	c-1
$(B \times C)_{jk}$	2	1	1	2	$4\sigma_{B \times C}^2 + \sigma_\epsilon^2$	2(c-1)
$(A \times B \times C)_{ijk}$	0	1	1	2	$2\sigma_{A \times B \times C}^2 + \sigma_\epsilon^2$	2(c-1)
$\epsilon_{(ijk)r}$	1	1	1	1	σ_ϵ^2	6c

$H_1: \sigma_{A \times B \times C}^2 > 0$ must be tested against MS(Error). Since MS(B×C) appears in the denominator of two of the F tests, then $c > 3$ so that $2(c-1) \geq 6$, or $c \geq 4$. Table 23 also shows that, unfortunately, there is no direct F statistic for testing $H_0: A_i = 0$, where A is a fixed factor. However, the Table shows that $E(MS_{A \times B} + MS_{A \times C} - MS_{A \times B \times C}) = 2c\sigma_{A \times B}^2 + 6\sigma_{A \times C}^2 + 2\sigma_{A \times B \times C}^2 + \sigma_\epsilon^2$. This implies that if we let $MS' = MS_A$ and $MS'' = MS_{A \times B} + MS_{A \times C} - MS_{A \times B \times C}$, then the statistic $F_0 = MS'/MS''$ has an approximate (or pseudo) F distribution with 1 df for the numerator and q df for the denominator, where q is given in Eq. (13.21) on page 593 of Montgomery's 8th edition. The expression for q, assuming $MS' = MS_A$ and $MS'' = MS_{A \times B} + MS_{A \times C} - MS_{A \times B \times C}$ and $c = 4$, is given by

$$q = \frac{(MS'')^2}{MS_{A \times B}^2 / 2 + MS_{A \times C}^2 / 3 + (-MS_{A \times B \times C})^2 / 6}.$$

Note that Montgomery defines MS' differently near the middle of his page 593, which also provides an appropriate $F_0 = MS'/MS''$ for testing $H_0: A_i = 0$. I would use Montgomery's expressions for MS' and MS'' on p. 593 only in if $MS_{A \times B \times C} \geq MS_{A \times B} + MS_{A \times C}$. Otherwise, my definition provides a more direct pseudo F-test for $H_0: A_i = 0$.

Exercise 38. Study the Example 13.6 on pages 593-597 of Montgomery(8e) but

conduct the pseudo F test for the fixed factor A as I have outlined above.

Approximate Confidence Intervals for Variance Components

Reference to my Chapter 2 Notes on pages 5 & 6 indicates that for a normal parent universe, the statistic $(n - 1)S^2/\sigma^2$ has a χ^2 distribution with $v = n - 1$ df. Using this and the fact that $E(S^2) = \sigma^2$ for any infinite universe, it was illustrated on pp. 5 & 6 of my notes how to construct a 95% CI for the parameter σ^2 . For the Example 13.1 of Montgomery, this leads to $0.71431 \leq \sigma_{\epsilon}^2 \leq 1.46980$, where $\chi_{0.975,60}^2 = 40.481748$ and $\chi_{0.025,60}^2 = 83.297675$.

Unfortunately, as stated by Montgomery(8e) p. 598, in experiments involving several fixed and random factors, $E(MS)$ will generally not equal to one of the variance components, and hence, it will not be possible to use the χ^2 distribution directly to obtain a CI for, say, σ_B^2 , where B is one of the random factors in the experiment with $v_B = b - 1$. What we mean is that $E(MS_B)$ will, in general, not equal to σ_B^2 in a mixed-model factorial experiment. As an example, for the R&R study of Example 13.1 on pages 575-578 of Montgomery(8e), if we let $B = P$, then $E(MS_P) = 6\sigma_P^2 + 2\sigma_{P \times O}^2 + \sigma^2$, which is quite different from σ_P^2 .

From the above discussion, it is clear that we have to develop a linear combination of MS's that is a point unbiased estimate of σ_B^2 , where B is a random factor. For the factor P (= Parts) of Montgomery's Example 13.1 because $E(MS_P) = 6\sigma_P^2 + E(MS_{P \times O})$, then $E(MS_P) - E(MS_{P \times O}) = 6\sigma_P^2 \rightarrow E[(MS_P - MS_{P \times O})/6] = \sigma_P^2$. Thus, let this point estimator be denoted by $\hat{\sigma}_P^2 =$

$\frac{MS_P - MS(P \times O)}{6} = (1/6)MS_P - (1/6)MS(P \times O)$, which is a linear combination. Note that $E(\hat{\sigma}_P^2) = \sigma_P^2$; further, $19MS_P/\sigma_P^2$ and $38MS(P \times O)/\sigma_{P \times O}^2$ do not seem to have χ^2 distributions with df equal to 19 and 38, respectively, because $E(MS_P) \neq \sigma_P^2$ and $E[MS(P \times O)] \neq \sigma_{P \times O}^2$. This last statement is not consistent with that of Montgomery's near the bottom of page 598 right after his Eq. (13.24). Consequently, Satterthwaite's (1946) pseudo F procedure can be used to

deduce that $(r \hat{\sigma}_p^2) / \sigma_p^2$ has an approximate χ^2 distribution with r *df*, whose value is given in Eq. (13.25) at the bottom of page 598 and r is generally close to *df* of the random factor, i.e., in the Example 13.1 the value of r should be close to 19 *df* (because there were 20 randomly selected parts).

We now use the above developments to obtain an approximate 95% CI for the variance component σ_p^2 of Example 13.1 on pages 575-578 of Montgomery(8e). To this end, we must first compute the *df* of the χ_r^2 , from the Eq. (13.25) on p. 598, to be used in obtaining the desired CI (recall that $\hat{\sigma}_p^2 = 10.27983$, $MS_P = 62.3908$, and $MS_{P \times O} = 0.71184$.)

$$r = \frac{(\hat{\sigma}_p^2)^2}{\sum_{i=1}^m MS_i^2 / (k^2 \times f_i)} \quad (\text{Eq. 13.25 p. 598 of Montgomery 8e})$$

$$r = \frac{(10.280)^2}{[(62.3908)^2 / (36 \times 19)] + [(0.71184)^2 / (36 \times 38)]} = \frac{(MS_P - MS_{P \times O})^2}{MS_P^2 / 19 + MS_{P \times O}^2 / 38} = 18.56771$$

where for the Example 13.1 of Montgomery and its variance component σ_p^2 , $m = 2$, $k = 6$, $f_1 = 19$, and $f_2 = 38$. To obtain a 95% CI for σ_p^2 , we use the fact that $(r \hat{\sigma}_p^2) / \sigma_p^2$ has an approximate χ^2 distribution with $r = 18.56771$ degrees of freedom. Therefore, the $\Pr(8.6133 \leq r \hat{\sigma}_p^2 / \sigma_p^2 \leq 32.2802) \cong 0.95$. This last confidence probability statement leads to the desired approximate 95% CI : $5.9130 \leq \sigma_p^2 \leq 22.1602$.

Exercise 39. Work problems 13.7, and 13.28 on pp. 602-603 of Montgomery(8e).