## Stress Compatibility Equation in 2D
(Beltrami-Mitchell Equation)

Stress compatibility equations combine all the major equations of stress analysis, strain analysis and stress-strain relations chapters into one compact expression.

For example, consider a 2-D plane strain condition. From Hooke's law we have,

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \quad \text{—(a)}$$

$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right] \quad \text{—(b)}$$

$$\overset{0}{\cancel{\epsilon_{zz}}} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \quad \text{—(c)}$$

and, $$\gamma_{xy} = \frac{1}{G}\sigma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy} \quad \text{—(d)}$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \quad \text{—(e)}$$

Substituting (e) in (a) and (b),

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \\ \epsilon_{yy} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] \end{aligned}\right\} \quad \text{—(1)}$$

Recall, strain compatibility equation in 2-D:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \text{—} (2)$$

substituting (1) in (2) $\Rightarrow$

$$\cancel{\left(\frac{1+\nu}{E}\right)} \frac{\partial^2}{\partial y^2}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] + \cancel{\left(\frac{1+\nu}{E}\right)} \frac{\partial^2}{\partial x^2}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right]$$

$$= 2 \cancel{\left(\frac{1+\nu}{E}\right)} \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad \text{—} (3)$$

To simplify eq (3) further, consider 2-D stress equilibrium conditions $\Rightarrow$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad \text{—} (4)$$

(for no body forces)

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 . \quad \text{—} (5)$$

Differentiate (4) w.r.t. $x$ and (5) w.r.t. $y$ and <u>add</u> the equations $\Rightarrow$

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = -2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad \text{—} (6)$$

substitute (6) in (3) $\Rightarrow$

$$(1-\nu)\left[\frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2}\right] - \nu\left[\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}\right]$$

$$= -\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}\right)$$

simplify this eq. to get,

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\sigma_{xx}+\sigma_{yy}\right) = 0} \quad -(7)$$

or, $\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right) = 0$ where $\nabla^2$ is the Laplacian operator $=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\,\cdot\,)$

It is important to note that (7) incorporates all the major equations of stress, strain and stress-strain analysis. Further eq. (7) is independent of $E$ and $\nu$. Hence, for identical geometry and loading, components made of different materials (say, metal or plastic) produce same stress distributions. This feature is immensely significant when experimental modeling is used to study real structures.

<u>Note</u>: Eq. (7) holds for "plane stress" condition as well.

Also, if body force is non-zero, eq. (7) takes the form

$$\boxed{\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right) = g(\nu)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)} \quad -(8)$$

where $g(\nu) = -\left(\frac{1}{1-\nu}\right)$ for plane strain

$= -(1+\nu)$ for plane stress.

In (8), only the weaker elastic property $\nu$ is present but $E$ is absent.

# Airy's Stress Function

Stress function approach is useful for obtaining stress field solutions. It involves knowing a single function, say $\phi(x,y)$, from which stress field can be obtained.

The so-called Airy's stress function is assumed to be of the form:

$$\sigma_{xx} = \frac{\partial^2 \phi(x,y)}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi(x,y)}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad \text{— (1)}$$

Here $\phi(x,y)$ satisfies the two equilibrium eqs:

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\} \quad \text{— (2)}$$

The Airy's stress function $\phi(x,y)$ also needs to satify the stress compatibility (or, Beltrami-Mitchell) eq:

$$\nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0 \quad \text{— (3)}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

$$\Rightarrow \boxed{\nabla^2(\nabla^2 \phi) = 0} \Rightarrow \boxed{\nabla^4 \phi = 0} \quad \text{— (4)}$$

$$\Rightarrow \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Eq. (4) is called the "biharmonic equation". The function $\phi(x,y)$ is the so-called Airy's stress function. If $\phi(x,y)$ is chosen appropriately to satisfy eq. (4), then solution for a problem can be found since strain compatibility, stress equilibrium and stress-strain relations are embedded in the Beltrami-Mitchell eqs. The only thing one needs to ensure is to satisfy the boundary conditions of the problem.

## Polynomials as stress Functions

One possible way of determing stress function for a given problem is to assume $\phi(x,y)$ to be a polynomial.

Note that eq. (1) involves $2^{nd}$ derivatives of $\phi(x,y)$ and hence the polynomial has to be atleast a quadratic function of $x$ & $y$ to yield a non-trivial solution. That is,

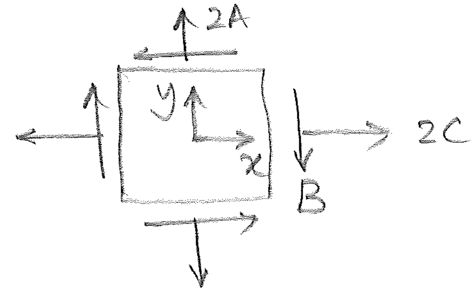
$$\phi(x,y) = Ax^2 + Bxy + Cy^2 \qquad (5)$$

where $A, B, C$ are constant coefficients to be evaluated from the boundary conditions of the problem.

Then, from $\sigma-\phi$ relations (eq. 1),

$$\sigma_{xx} = 2C, \quad \sigma_{yy} = 2A, \quad \sigma_{xy} = -B.$$

Thus, eq. 5 can be used to find or represent a constant state of stress as shown:




Now, consider a cubic function of $x, y \Rightarrow$

$$\phi(x,y) = Dx^3 + Ex^2y + Fxy^2 + Gy^3 \quad — 6$$

Again using $\sigma-\phi$ relations (eq. 1), we get

$$\sigma_{xx} = 2Fx + 6Gy$$
$$\sigma_{yy} = 6Dx + 2Ey$$
$$\sigma_{xy} = -(2Ex + 2Fy)$$

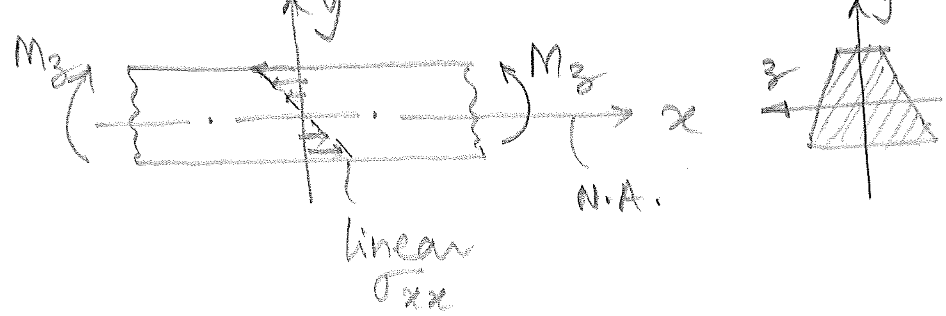
D, E, F, G are constants

Thus, eq. 6 can be used to represent bilinear variation of stresses in a planar object. For example, if $E = F = D = 0$, then

$$\phi(x,y) = Gy^3$$

and, $\sigma_{xx} = 6Gy$, $\sigma_{yy} = 0$, $\sigma_{xy} = 0$

The state of stress is uniaxial and varies linearly along y-axis, thereby fitting well the situation encountered in "pure bending"




[ Note: Both eqs. (5) and (6) satisfy the biharmonic eq. $\nabla^2(\nabla^2\phi) = 0$. (check) ]

In general, a polynomial of the form,

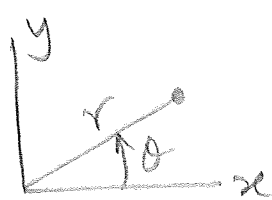
$$\phi(x,y) = Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 + Hx^4 + Ix^3y + Jx^2y^2 + Kxy^3 + Ly^4 + \dots \quad \text{— (7)}$$

can be a general stress function.

<u>Polar coordinates</u>

Eqs. (2) and (3) in polar coordinates are,

$$\nabla^2(\nabla^2\phi(r,\theta)) = 0$$



$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \quad (8)$$

and $\sigma - \phi$ relations $\Rightarrow$

$$\boxed{\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \end{aligned}} \quad (9)$$