5 Laminar External Flow

5.1 The boundary layer

A boundary layer is an easy physical concept to grasp; it is the result of the no–slip boundary condition as fluid flows over the surface of an object. Because of viscosity, the fluid directly adjacent to the object is sheared, and this effect is confined to a relatively thin layer parallel to the surface, i.e., the boundary layer.

A boundary layer is also a mathematical concept, and "boundary layer" behavior can occur whenever the higher–order derivatives in a differential equation are multiplied by relatively small parameters. This statement is more arcane, yet it is important in terms of gaining insight into why physical boundary layers are "thin", and what solution methods can be used to solve for the transport of momentum and heat across the boundary layers.

A simple example – both physical and mathematical – of boundary layer behavior can be seen in a 1–D convective–diffusive problem. Say a uniform, 1–D flow enters a control volume with temperature \( T_0 \). At some distance \( L \) downstream, the flow is brought to a new temperature \( T_1 \). All the while the density \( \rho \) and the velocity \( u \) stay constant. The boundary at \( x = L \) might represent, for example, a porous surface which is maintained at \( T_1 \).

The steady 1–D energy equation for this situation is

\[
\rho UC \frac{dT}{dx} = k \frac{d^2T}{dx^2}
\]  

(1)

Denote non–dimensional variables as

\[
\bar{T} = \frac{T - T_1}{T_0 - T_1}, \quad \bar{\tau} = \frac{x}{L}
\]

(2)

and the DE becomes

\[
\frac{dT}{d\bar{\tau}} = \frac{1}{Pe_L} \frac{d^2 T}{d\bar{\tau}^2}
\]

(3)

\[ T(0) = 1, \quad T(1) = 0 \]

(4)

with

\[
Pe_L = \frac{u L}{\alpha} = \frac{\rho C u L}{k}
\]

(5)

The solution to this problem is

\[
T(\bar{\tau}, Pe_L) = \frac{e^{Pe_L} - e^{Pe_L \bar{\tau}}}{e^{Pe_L} - 1}
\]

(6)

A plot of the results is shown below.

Note that as \( Pe_L \) increases, the temperature distribution goes from being linear to increasingly "piled up" into a narrow region adjacent to the \( x = L \) boundary. Indeed, for \( Pe_L = 100 \) the temperature is basically...
uniform at $T = 1$ – except for a small region of steep gradient at the outflow surface. This region of steep gradient would be considered a boundary layer. The 2nd-order derivative, for this case, is multiplied by the small parameter $1/100 = 0.01$. This might lead one to assume that the 2nd derivative term could be neglected from the DE – and it could, through most of the flow region. Getting rid of this term would result in
\[ \frac{dT}{dx} = 0 \quad \rightarrow \quad T = \text{constant} = 1 \]  
(7)
where the constant is evaluated from the first BC. However, the problem has two boundary conditions, and the zero–temperature condition must be maintained at the outflow wall. A first–order DE cannot accommodate two boundary conditions, so somewhere the second–order derivative term must become significant. It will become significant in the neighborhood of the $T = 0$ surface.

The thickness of the boundary layer, for this simple problem, can be determined by reformulating the problem. Reverse the coordinate direction, so that $x$ runs from the $T = 0$ surface, and define a scaled (or stretched) coordinate $\tilde{x}$ via

\[ \tilde{x} = \pi Pe L \]  
(8)
The scaled DE becomes
\[ -\frac{dT}{d\tilde{x}} = \frac{d^2T}{d\tilde{x}^2} \]  
(9)

\[ T(0) = 0, \quad T(\tilde{x} \to \infty) \to 1 \]  
(10)

In this view the far surface (with $T = 1$) now exists at $\tilde{x} \to \infty$: the model describes only the behavior within the boundary layer. The solution is now

\[ T(\tilde{x}) = 1 - e^{-\tilde{x}} \]  
(11)
and this indicates that the temperature reaches $T = 0.99$ for $\tilde{x} \approx 4.61$. The thickness of the boundary layer is therefore

\[ \delta_{bl} \approx \frac{4.61 L}{Pe L} = \frac{4.61 \alpha}{U} \]  
(12)

Some points to make regarding the boundary layer effect are:

1. The boundary layer represents the region in which all derivatives in the governing DE contribute significantly to the transport process. Outside of this region transport is dominated by a single mechanism (convection, in this case).

2. The details of the boundary layer (i.e., the thickness) can be neglected if one is interested in predicting only the bulk characteristics of the flow. For the simple model examined here, the bulk (or freestream) characteristics would be predicted by Eq. (7), and this result gives the conditions at the edge of the boundary layer.

3. On the other hand, the details of the boundary layer are absolutely critical if one wishes to predict the rate of transport to/from a surface that is exposed to a flow. Note that Eq. (7) cannot predict the rate of heat transfer to the $T = 0$ surface.

4. On an order–of–magnitude level, the rate of transport across the boundary layer will be

\[ q'' \approx k \frac{T_\infty - T_s}{\delta_{bl}} \]  
(13)

This approximation does not have much relevance to the simple 1–D model, because the heat flux must be $q'' = \rho u C(T_1 - T_0)$ by virtue of the first law for the system (and this is precisely what the above equation would give). It will have more bearing in 2–D flow situations.
5.2 Momentum transfer on a flat plate

Consider a situation in which a flat plate is exposed to a flow that runs parallel to its surface. Far from the surface of the plate the velocity is $u_\infty$. A boundary layer will form at the leading edge of the plate, and the thickness of the layer will grow with distance downstream. Everywhere the flow remains laminar (the conditions for laminar BL flow will be discussed below). The objective is to predict the velocity profile in the boundary layer.

Assuming incompressible flow, the continuity and momentum equations for the situation are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{14}
\]

\[
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{15}
\]

where $x, y$ are the streamwise and normal coordinates, and $u, v$ are the associated components of velocity. Assumptions are 1) there is negligible pressure gradient in the normal direction, so that $P = P(x)$, 2) streamwise diffusion of momentum is negligible relative to streamwise convection.

Outside the boundary layer the flow is 1-D in the $x$ direction, so $u = u_\infty = \text{constant}$ and $v = 0$. The momentum equation, applied in this region, would show that $dP/dx = 0$. The pressure gradient term will be neglected from here on out, yet it will become relevant (i.e., non-zero) whenever the external flow is accelerating or decelerating, as would be the case for a surface that is tilted with respect to the flow.

5.2.1 Order–of–magnitude analysis

It is possible to use Eqs. (14) and (15) to estimate how the boundary layer thickness changes with position $x$ from the leading edge. At some point $x$, at which the boundary layer is $\delta$ thick, the change in $u$ is, at most, $u_\infty$ over the distance $x$. The normal component of velocity goes from 0 to some characteristic value $v$ over the normal distance $y = \delta$. An order of magnitude analysis of the continuity equation would then suggest that

$$\frac{u_\infty}{x}, \frac{v}{\delta}$$

are both of the same magnitude. Therefore, the characteristic scale of the normal velocity will be

$$v \sim \frac{u_\infty}{x} \frac{\delta}{x}$$ \tag{16}

This scale can be used in the momentum equation. The conclusion would be that

$$u_\infty \frac{u_\infty}{x}, \nu \frac{u_\infty}{\delta^2}$$

are both of the same magnitude. This indicates that the boundary layer thickness will scale as

$$\delta \sim \left(\frac{\nu x}{u_\infty}\right)^{1/2} \sim x (Re_x)^{-1/2}$$ \tag{17}

The condition of a thin boundary layer, i.e., $\delta/x \ll 1$, therefore implies that $1/Re_x^{1/2} \ll 1$, or $Re_x$ is large.

An estimation of the shear stress acting on the surface can be obtained from

$$\tau_s = \rho \nu \left. \frac{du}{dy} \right|_{y=0} \sim \rho \nu \frac{u_\infty}{\delta} \sim \frac{\rho \nu}{x} Re_x^{1/2}$$ \tag{18}

It turns out that the exact result will be identical to the above formula, multiplied by an order–unity constant. The procedure to the exact solution is described in the next section.
5.2.2 Similarity solution

A practical way of developing a solution for two dimensional fluid flow problems is to define a stream function \( \Psi \), so that

\[
\begin{align*}
u &= \frac{\partial \Psi}{\partial y}, \\
v &= -\frac{\partial \Psi}{\partial x}
\end{align*}
\]

The stream function thus automatically satisfies the continuity equation, Eq. (14), and the problem reduces to solving a single nonlinear PDE – the momentum equation – for \( \Psi \). This problem can be further simplified by making use of a similarity transform. The idea here is to assume that the normalized velocity profile \( u/u_\infty \) is dependent on the single variable \( \eta = y/\delta(x) = y [u_\infty/(\nu x)]^{1/2} \). This assumption would imply that

\[
\Psi(x, y) = u_\infty \delta(x) f(y/\delta(x))
\]

because, from Eq. (19), the velocity \( u \) would be

\[
u(x, y) = \frac{\partial \Psi}{\partial y} = u_\infty \delta(x) \frac{df}{dy}(\eta) = u_\infty f'(\eta)
\]

which fits the similarity assumption. The normal component of velocity would now be

\[
v = -\frac{\partial \Psi}{\partial x} = u_\infty \delta'(x) (\eta f'(\eta) - f(\eta))
\]

and the momentum equation would transform to

\[
2f'' + f'f = 0
\]

Both components of velocity are zero at the surface, and the velocity reaches the free stream value outside of the boundary layer. The boundary conditions on \( f \) are therefore

\[
\begin{align*}
f(0) &= 0 \\
f'(0) &= 0 \\
f'(\eta \to \infty) &\to 1
\end{align*}
\]

Equations (23–26) represent a third–order nonlinear, two–point boundary value problem. That is, the solution must satisfy BCs at the surface (\( \eta = 0 \)) and in the freestream (\( \eta \to \infty \)). The fact that the transformed equation is solely a function of \( \eta \) confirms the similarity assumption. There is no analytical solution to this problem and numerical methods must be used to obtain a solution, yet computing a solution to a nonlinear ODE is considerably more easy than that for a nonlinear PDE.

The standard numerical strategy to solving an ODE of the form in Eq. (23) is to split the equation into a set of three coupled first–order DEs for the functions \( f \), \( f_1 \), and \( f_2 \), via,

\[
\begin{align*}
f'(\eta) &= f_1(\eta) \\
f'_1(\eta) &= f_2(\eta) \quad (= f''(\eta)) \\
f'_2(\eta) &= -\frac{1}{2} f_2(\eta) f(\eta) \quad (= -\frac{1}{2} f''(\eta) f(\eta))
\end{align*}
\]

These equations can then be numerically integrated from \( \eta = 0 \) out to a sufficiently large value of \( \eta \) so that the solution reaches its asymptotic limit. A simple method would be to approximate the derivatives as forward differences, i.e.,

\[
f'_1(\eta) \approx \frac{f(\eta + \Delta\eta) - f(\eta)}{\Delta\eta}
\]

and to then use Eqs. (27–29) to solve for the values of the functions at step \( \eta + \Delta\eta \). This would be known as an explicit integration method, and it is not advised as in can run into stability and accuracy problems, yet it does illustrate the fundamental numerical procedure: the integration is carried out step–by–step.
A problem, of course, is that the value of $f_2 = f''$ is not specified at the surface. This quantity relates to the velocity gradient at the wall, since

$$\frac{\partial u}{\partial y} \bigg|_{y=0} = u_\infty \left. \frac{d \eta}{d \eta} \frac{df'}{d \eta} \right|_{\eta=0} = u_\infty \left( \frac{u_\infty}{\nu x} \right)^{1/2} f''(0) \tag{30}$$

What we do know is the free–stream condition of Eq. (26). Such information is impossible to build into the integration scheme; rather, it would be one of the outcomes of the integration for a sufficiently large value of $\eta$. An iteration strategy needs to be adopted to match the free–stream condition. The procedure would be to start with a trial value of $f''(0)$, integrate the equations to some limit $\eta_{\text{max}}$, and identify the value of $f(\eta_{\text{max}})$ corresponding to the trial value. A correct value of $f''(0)$ will produce $f'(\eta_{\text{max}}) = 1$. If this condition is not met, an intelligent new estimate of $f''(0)$ is set and the procedure is repeated. You should see that this is nothing more than finding the root to a nonlinear equation; i.e.,

$$F[f''(0)] = 0, \quad \text{with} \quad F = f'(\eta_{\text{max}}) - 1 \tag{31}$$

and the standard numerical methods for solving such equations can be used here. This iteration strategy of using integration from an initial point to solve a two–point boundary value problem is often referred to as a shooting method; i.e., the integration shoots from the starting point and attempts to hit the correct end point.

An implementation of this procedure in mathematical software packages is not difficult; Mathematica, Matlab and Maple have built–in black–box routines to perform numerical integration and to find roots of nonlinear equations. An example of a code for Mathematica is shown below.

The first module provides a solution to the DE via the NDSolve command. The two variables are $f''(0)$ and $\eta_{\text{max}}$. The solution is represented by an interpolation function – which represents an interpolating polynomial for $f(\eta)$ valid for $0 \leq \eta \leq \eta_{\text{max}}$.

```mathematica
bde[fp20_, etamax_] := Module[{bc1, bc2, bc3, de, soln},
  bc1 = f[0] == 0;
  bc2 = f'[0] == 0;
  bc3 = f''[0] == fp20;
  de = f[eta] f''[eta] + 2 f''''[eta] == 0;
  soln = NDSolve[{de, bc1, bc2, bc3}, f[eta], {eta, 0, etamax}];
  soln[[1,1]]
]
```

The next module uses the previous module to calculate the error $f'(\eta_{\text{max}}) - 1$ as a function of $f''(0)$;

```mathematica
minfunc[fp20_, etamax_] := Module[{soln, uinf},
  soln = bde[fp20, etamax];
  uinf = Evaluate[D[f[eta] /. soln, {eta, 1}]] /. eta -> etamax;
  uinf - 1
]
```

The rootfinding step to obtain the correct $f''(0)$ value can be written in one simple line. A value of $\eta_{\text{max}} = 10$ is used.

```mathematica
fp2 = fp20 /. FindRoot[minfunc[fp20, 10] == 0, {fp20, {0.5, 0.6}}][[1]]
0.332058
```

The code returns a value for $f''(0) = 0.332$. Shown in the figure are results for $f'$ and $f''$ as a function of $\eta$. Note again that these functions represent dimensionless velocity and shear stress profiles in the boundary layer.
The results show that the edge of the boundary layer corresponds to \( \eta \approx 6 \).

From Eqs. (18) and (30), the shear stress at the wall will be given by

\[
\tau_s = \rho \nu u_\infty \left( \frac{u_\infty}{\nu x} \right)^{1/2} f''(0) = 0.332 \rho \nu u_\infty \left( \frac{u_\infty}{\nu x} \right)^{1/2}
\]  

and the dimensionless coefficient of friction \( C_f \) is

\[
C_f = \frac{2 \tau_s}{\rho u_\infty^2} = 0.664 Re_x^{-1/2}
\]  

5.3 Homework Exercises

1. Complete the steps between Eqs. (22) and (23).

2. Consider the case of a laminar boundary layer on a flat plate with a constant and uniform heat flux.

   (a) The energy equation for the boundary layer will appear as

   \[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}
   \]

   Reformulate this DE using the dimensionless dependent variables

   \[
   \mathcal{T} = \frac{T - T_s}{T_\infty - T_s}, \quad \mathcal{u} = \frac{u}{u_\infty}, \quad \mathcal{v} = \frac{v}{u_\infty}
   \]

   In making the substitutions, recognize that \( T_s \) will be, in general, a function of \( x \).

   (b) Now use the similarity assumption, so that

   \[
   \mathcal{u} = f'(\eta), \quad \mathcal{v} = \delta'(x) \left( \eta f'(\eta) - f(\eta) \right), \quad \mathcal{T} = \mathcal{T}(\eta)
   \]

   with

   \[
   \eta = \frac{y}{\delta(x)}, \quad \delta(x) = \left( \frac{x \nu}{u_\infty} \right)^{1/2}
   \]

   Show that a similarity solution is possible only if

   \[
   \Delta T = T_s - T_\infty = K x^{\gamma}
   \]

   for constant \( K \) and any power \( \gamma \) except -1.

   (c) For the constant heat flux case, show that \( \gamma = 1/2 \).
(d) The ODE for $T(\eta)$ will be linear and inhomogeneous, and will satisfy the two boundary conditions

$$T(0) = 0, \quad T(\eta \to \infty) \to 1$$

In general, an iterative method would be required to solve the ODE, in which 1) a guess for $T'(0)$ is made; 2) the ODE is integrated out from $\eta = 0$ to a sufficiently large value of $\eta_{\text{max}}$; 3) an error is computed using $|1 - T(\eta_{\text{max}})|$; and 4) the process continued until the outer BC is met. Since the problem for $T$ is linear (although not homogeneous), it turns out that a solution to the problem can actually be obtained from two separate numerical integrations of one–point boundary value problems. That is, a solution to this problem can be obtained by superimposing (or adding) two separate solutions, via

$$T = 1 + T_1 + CT_2$$

where $C$ is a constant (to be determined), and $T_1$ and $T_2$ each satisfy the homogeneous form of the ODE for $T$ along with the surface conditions,

$$T_1(0) = -1, \quad T_1'(0) = 0, \quad T_2(0) = 0, \quad T_2'(0) = 1$$

Both $T_1$ and $T_2$ will become constant for sufficient large $\eta_{\text{max}}$, and the values of these constants would be identified by numerical integration of the ODEs for $T_1$ and $T_2$. Show how you would use the numerical solution for $T_1$ and $T_2$ to obtain $C$.

(e) Using Mathematica or MatLab, numerically solve the ODE for $T$ to get the temperature profile in the BL as a function of the similarity variable $\eta$. I will provide you with a code to calculate the dimensionless stream function $f$; you will need to use this in your solution. Use $Pr = 1$ in your calculations.

(f) From the numerical solution, show that, for $Pr = 1$,

$$Nu_x = 0.458 \ Re_x^{1/2}$$

(g) Revise the Mathematica stream function–vorticity code for the case of uniform surface heat flux, and plot the steady–state dimensionless surface temperature distribution as a function of $x$, for $Re_L = 10^5$ and $Pr = 1$. Compare the numerical results with those predicted from laminar boundary layer theory.