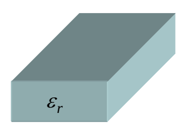
Waveguides
Part 2

- Rectangular Waveguides
  - Dielectric Waveguide
  - Optical Fiber

Dielectric Waveguide

Let us consider the simpler case of a rectangular slab of waveguide.



Snell's Law of Reflection

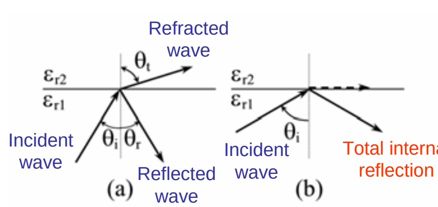
$$\theta_i = \theta_r$$

Snell's Law of Refraction

$$\frac{\beta_1}{\beta_2} = \frac{\sin\theta_t}{\sin\theta_i}$$

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1} \text{ and } \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

Case(1): $\theta_i < (\theta_i)_{critical}$ Case(ii): $\theta_i > (\theta_i)_{critical}$



Critical Angle:

$$(\theta_i)_{critical} = \sin^{-1}\left(\frac{\sqrt{\varepsilon_{r2}}}{\sqrt{\varepsilon_{r1}}}\right)$$

When the incident angle is greater than the critical angle, the wave is totally reflected back and this phenomenon is known as *Total internal reflection*.

Dielectric Waveguide

Index of refraction:

The *index of refraction*, *n*, is the ratio of the speed of light in a vacuum to the speed of light in the unbounded medium, or

$$n = \frac{\text{Velocity of light in Free Space}}{\text{Velocity of light in the medium}} = \frac{c}{u_u} = \sqrt{\mu_r \varepsilon_r}$$

Where $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$

$$u_u = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_o \mu_r \varepsilon_o \varepsilon_r}} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \frac{1}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

In nonmagnetic material $\mu_r = 1$

$$n = \sqrt{\varepsilon_r}$$

Snell's Law of Refraction can be expressed in terms of refractive index:

Critical Angle: $(\theta_i)_{critical} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

Snell's Law of Refraction: $\frac{n_1}{n_2} = \frac{\sin\theta_t}{\sin\theta_i}$

Dielectric Waveguide

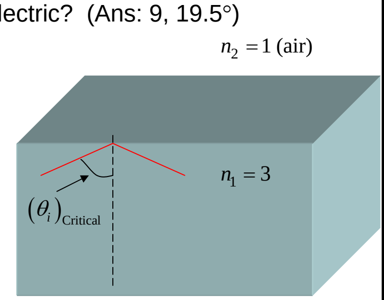
Example

D7.5: A slab of dielectric with index of refraction 3.00 sits in air. What is the relative permittivity of the dielectric? At what angle from a normal to the boundary will light be totally reflected within the dielectric? (Ans: 9, 19.5°)

What is the relative permittivity of the dielectric?

$$n_1 = \sqrt{\varepsilon_{r1}} \Rightarrow \varepsilon_{r1} = n_1^2$$

$$\varepsilon_{r1} = 3^2 = 9$$



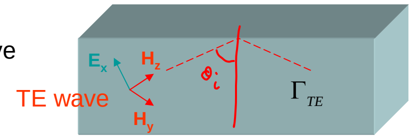
At what angle from a normal to the boundary will light be totally reflected within the dielectric?

$$(\theta_i)_{critical} = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}\left(\frac{1}{3}\right) = 19.5^\circ$$

Dielectric Waveguide

TE wave

The reflection coefficient of a TE plane wave (See Chapter 5) is given by



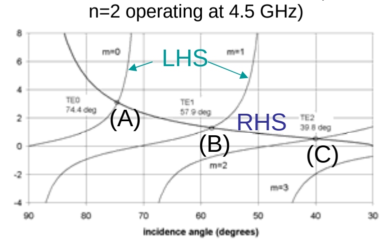
$$\Gamma = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

Using Snell's Law of refraction

$$\Gamma_{TE} = \frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - (n_2/n_1)^2}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - (n_2/n_1)^2}}$$

$$\phi_{TE} = 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - (n_2/n_1)^2}}{\cos\theta_i}\right)$$

TE modes (50 mm thick dielectric of ε_r = 4 or n=2 operating at 4.5 GHz)



Possible modes can be obtained by evaluating the phase expression for various values of m.

LHS RHS

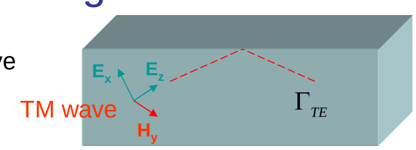
$$\tan\left(\frac{a\beta_1\cos\theta_i}{2} - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - (n_2/n_1)^2}}{\cos\theta_i}$$

For this example only three TE modes are possible;
A) TE_0 at θ_i = 74.4°,
B) TE_1 at θ_i = 57.9°, and
C) TE_2 at θ_i = 39.8°.

Dielectric Waveguide

TM wave

The reflection coefficient of a TM plane wave (See Chapter 5) is given by

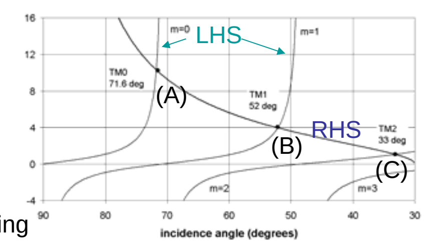


$$\Gamma_{TM} = \frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i}$$

Using Snell's Law of refraction

$$\Gamma_{TE} = \frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - (n_2/n_1)^2}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - (n_2/n_1)^2}}$$

TM modes (50 mm thick dielectric of ε_r = 4 or n=2 operating at 4.5 GHz)



Possible modes can be obtained by evaluating the phase expression for various values of m.

LHS RHS

$$\tan\left(\frac{a\beta_1\cos\theta_i}{2} - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - (n_2/n_1)^2}}{(n_2/n_1)^2\cos\theta_i}$$

For this example only three TM modes are possible;
A) TM_0 at θ_i = 71.6°,
B) TM_1 at θ_i = 52°, and
C) TM_2 at θ_i = 33°.

Dielectric Waveguide

LHS | RHS

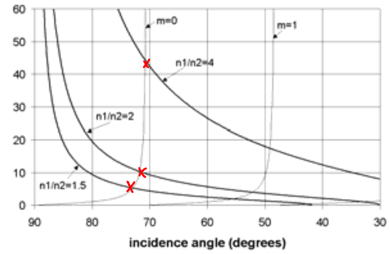
$$\tan\left(\frac{a\beta_1\cos\theta_i}{2}-\frac{m\pi}{2}\right)=\frac{\sqrt{\sin^2\theta_i-\left(n_2/n_1\right)^2}}{\left(n_2/n_1\right)^2\cos\theta_i}$$

A larger ratio of n_1/n_2 results in

a) a lower critical angle and therefore

b) more propagating modes.

RHS for various m



For single mode operation:

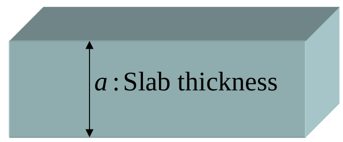
$$\frac{a}{\lambda_o}<\frac{1}{2}\frac{1}{\sqrt{n_1^2-n_2^2}} \Rightarrow a<\frac{1}{2}\frac{\lambda_o}{\sqrt{n_1^2-n_2^2}}$$

(or)

Using

$$\lambda_o=\frac{c}{f}$$

$$f<\frac{1}{2}\frac{c}{a\sqrt{n_1^2-n_2^2}}$$



Dielectric Waveguide

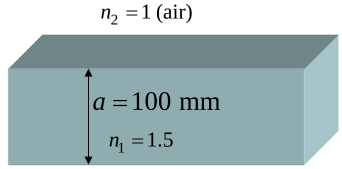
Example

D7.6: Suppose a polyethylene dielectric slab of thickness 100 mm exists in air. What is the maximum frequency at which this slab will support only one mode?

From Table E.2, for polyethylene

$$\varepsilon_{r1}=2.26 \Rightarrow n_1=\sqrt{2.26}=1.5$$

$n_2=1$ (air)



$n_2=1$ (air)

The maximum frequency at which this slab will support only one mode is

$$f<\frac{1}{2}\frac{c}{a\sqrt{n_1^2-n_2^2}}$$

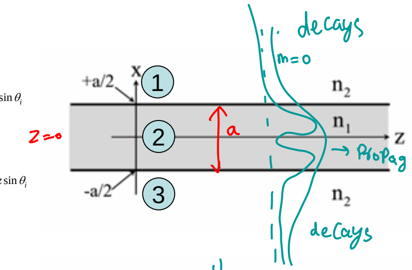
$$f_{\max}=\frac{1}{2}\frac{c}{a\sqrt{n_1^2-n_2^2}}=\frac{1}{2}\frac{\left(3\times10^8\right)}{\left(100\times10^{-3}\right)\sqrt{(1.5)^2-(1.0)^2}}=1.2\text{ GHz}$$

Dielectric Waveguide

Field Equations: The field equations can be obtained by solving Maxwell's equations with the appropriate boundary conditions.

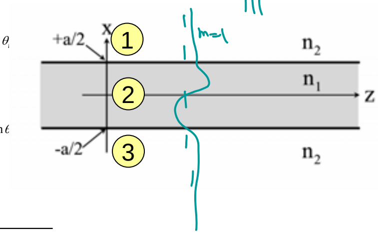
Even Modes:

1. For $x > a/2: E_y = E_o \cos\left(\beta_1 (a/2)\cos\theta_i\right) e^{-\alpha_2(x-a/2)} e^{-j\beta_1 z \sin\theta_i}$
2. $E_y = E_o \cos\left(\beta_1 x \cos\theta_i\right) e^{-j\beta_1 z \sin\theta_i}$ (m = 0, 2, 4 ...)
3. For $x < -a/2: E_y = E_o \cos\left(\beta_1 (a/2)\cos\theta_i\right) e^{+\alpha_2(x+a/2)} e^{-j\beta_1 z \sin\theta_i}$



Odd Modes:

1. For $x > a/2: E_y = E_o \sin\left(\beta_1 (a/2)\cos\theta_i\right) e^{-\alpha_2(x-a/2)} e^{-j\beta_1 z \sin\theta_i}$
2. $E_y = E_o \sin\left(\beta_1 x \cos\theta_i\right) e^{-j\beta_1 z \sin\theta_i}$ (m = 1, 3, 5 ...)
3. For $x < -a/2: E_y = -E_o \sin\left(\beta_1 (a/2)\cos\theta_i\right) e^{\alpha_2(x+a/2)} e^{-j\beta_1 z \sin\theta_i}$



The phase constant in medium 1 is $\beta_e = \beta_1 \sin\theta_i$

The attenuation in medium 2 is $\alpha_2 = \beta_1 \sqrt{\sin^2\theta_i - (n_2/n_1)^2}$

The effective wavelength in the guide is $\lambda_e = \dfrac{\lambda_u}{\sin\theta_i} = \dfrac{\lambda_o}{n_1 \sin\theta_i}$

The propagation velocity is $u_p = \dfrac{\omega}{\beta_e} = \dfrac{c}{n_1 \sin\theta_i}$

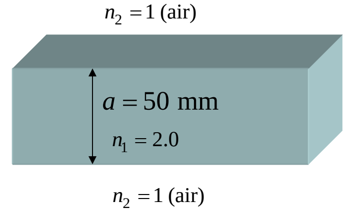
Dielectric Waveguide

Example

D7.6: Find λ_e and u_p at 4.5 GHz for the TE_0 mode in a 50 mm thick $n_1 = 2.0$ dielectric in air. (Ans: 35 mm and 1.6 x 10^8 m/s)

From Fig. 7.16, the critical incident angle for the TE_0 mode

TE_0 at $\theta_i = 74.4°$



The effective wavelength in the guide is

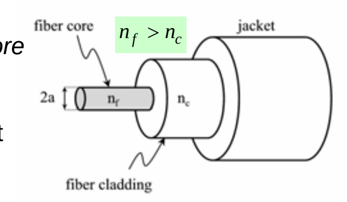
$$\lambda_e = \frac{\lambda_u}{\sin\theta_i} = \frac{\lambda_o}{n_1 \sin\theta_i} = \frac{c}{f n_1 \sin\theta_i} = \frac{\left(3\times10^8\right)}{\left(4.5\times10^9\right)(2)\sin\left(74.4^\circ\right)} = 35 \text{ mm}$$

The propagation velocity is

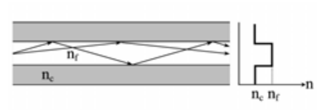
$$u_p = \frac{\omega}{\beta_e} = \frac{c}{n_1 \sin\theta_i} = \frac{\left(3\times10^8\right)}{(2)\sin\left(74.4^\circ\right)} = 1.6\times10^8 \text{ m/s}$$

Optical Fiber

A typical optical fiber is shown in Figure. The fiber *core* is completely encased in a fiber *cladding* that has a slightly lesser value of refractive index. Signals propagate along the core by total internal reflection at the core-cladding boundary.



A cross section of the fiber with rays traced for two different incident angles is shown. If the phase matching condition is met, these rays each represent propagating modes.
The abrupt change in *n* is characteristic of a step-index fiber. Optical fiber designed to support only one propagating mode is termed *single-mode fiber*. More than one mode propagates in *multi-mode fiber*.



In step-index optical fiber, a single mode will propagate so long as the wavelength is big enough such that

$$\lambda > \frac{2\pi a\sqrt{n_f^2 - n_c^2}}{k_{01}}$$

where k_{01} is the first root of the zeroth order Bessel function, equal to 2.405

Optical Fiber

For step-index multi-mode fiber, the total number of propagating modes is approximately

$$N = 2\left(\frac{\pi a}{\lambda}\right)^2 \left(n_f^2 - n_c^2\right)$$

Example 7.3: Suppose we have an optical fiber core of index 1.465 sheathed in cladding of index 1.450.

What is the maximum core radius allowed if only one mode is to be supported at a wavelength of 1550 nm?

$$\lambda > \frac{2\pi a\sqrt{n_f^2 - n_c^2}}{k_{01}} \Rightarrow a < \frac{k_{01}\lambda}{2\pi\sqrt{n_f^2 - n_c^2}} \Rightarrow a < \frac{(2.405)(1550x10^{-9}m)}{2\pi\sqrt{(1.465)^2 - (1.450)^2}} \text{ or } a < 2.84\mu m$$

How many modes are supported at this maximum radius for a source wavelength of 850 nm?

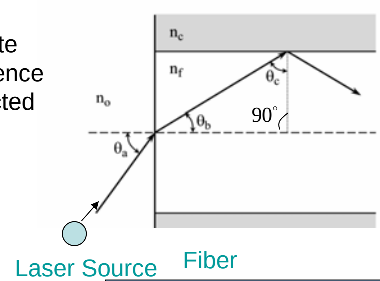
$$N = 2\left(\frac{\pi(2.84x10^{-6}m)}{850x10^{-9}m}\right)^2 \left((1.465)^2 - (1.450)^2\right) = 9.6$$

The fiber supports 9 modes!

Optical Fiber

Numerical Aperture

Light must be fed into the end of the fiber to initiate mode propagation. As Figure shows, upon incidence from air (n_o) to the fiber core (n_f) the light is refracted by Snell's Law:



$$n_o \sin\theta_a = n_f \sin\theta_b$$

$$\sin^2\theta_b + \cos^2\theta_b = 1$$

$$n_o \sin\theta_a = n_f \sqrt{1-\cos^2\theta_b}$$

$$\cos\theta_b = \cos\left(90^\circ - \theta_c\right) = \sin\theta_c$$

$$n_o \sin\theta_a = n_f \sqrt{1-\sin^2\theta_c}$$

The sum of the internal angles in a triangle is 180 deg.

$$\theta_c + \theta_b + 90^\circ = 180^\circ$$

$$\theta_b = 90^\circ - \theta_c$$

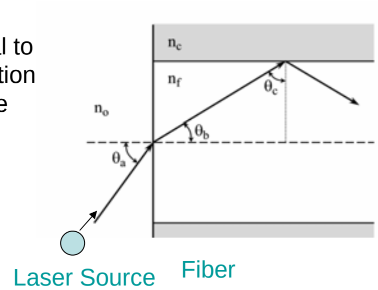
The *numerical aperture*, NA, is defined as

$$NA = \sin\theta_a = \frac{n_f\sqrt{1-\sin^2\theta_c}}{n_o}$$

Optical Fiber

Numerical Aperture

The incident light make an angle θ_c with a normal to the core–cladding boundary. A necessary condition for propagation is that θ_c exceed the critical angle $(\theta_i)_{critical}$, where



$$\sin\left(\theta_i\right)_{crit} = \frac{n_c}{n_f}$$

Therefore, the *numerical aperture*, NA, can be written as

$$NA = \frac{n_f\sqrt{1-\sin^2\theta_c}}{n_o} \xrightarrow{\sin\left(\theta_i\right)_{crit} = \frac{n_c}{n_f}} NA = \frac{\sqrt{n_f^2 - n_c^2}}{n_o}$$

Optical Fiber

Numerical Aperture

Example 7.4: Let's find the critical angle within the fiber described in Example 7.3. Then we'll find the acceptance angle and the numerical aperture.

The critical angle is

$$\theta_c = \sin^{-1}\left(\frac{n_c}{n_f}\right) = \sin\left(\frac{1.450}{1.465}\right) = 81.8^\circ.$$

The acceptance angle

$$\theta_a = \sin^{-1}\left(\frac{\sqrt{(1.465)^2 - (1.450)^2}}{1}\right) = 12.1^\circ.$$

Finally, the numerical aperture is

$$NA = \sin\theta_a = 0.209.$$

Optical Fiber

Signal Degradation

Intermodal Dispersion: Let us consider the case when a single-frequency source (called a *monochromatic* source) is used to excite different modes in a multi-mode fiber. Each mode will travel at a different angle and therefore each mode will travel at a different propagation velocity. The pulse will be spread out at the receiving end and this effect is termed as the intermodal dispersion.

Waveguide Dispersion: The propagation velocity is a function of frequency. The spreading out of a finite bandwidth pulse due to the frequency dependence of the velocity is termed as the waveguide dispersion.

Material Dispersion: The index of refraction for optical materials is generally a function of frequency. The spreading out of a pulse due to the frequency dependence of the refractive index is termed as the material dispersion.

Attenuation

Electronic Absorption: The photonic energy at short wavelengths may have the right amount of energy to excite crystal electrons to higher energy states. These electrons subsequently release energy by phonon emission (i.e., heating of the crystal lattice due to vibration).

Vibrational Absorption: If the photonic energy matches the vibration energy (at longer wavelengths), energy is lost to vibrational absorption.

Optical Fiber

Graded-Index Fiber

One approach to minimize dispersion in a multimode fiber is to use a graded index fiber (or GRIN, for short).

The index of refraction in the core has an engineered profile like the one shown in Figure. Here, higher order modes have a longer path to travel, but spend most of their time in lower index of refraction material that has a faster propagation velocity.

Lower order modes have a shorter path, but travel mostly in the slower index material near the center of the fiber.

The result is the different modes all propagate along the fiber at close to the same speed. The GRIN therefore has less of a dispersion problem than a multimode step index fiber.