General Instructions

1. Put your name in the name blank only. Test pages are numbered and can be associated together automatically.

2. Turn off (silent, not vibrate) and put away all cell phones and other communication devices.

3. This is a closed book, closed notes exam. However, one handwritten 3" × 5" notecard is allowed.

4. Show all work. Please put all of your work on the exam itself, preferably in the space provided. If you use the back of the pages, please indicate that clearly so that you will receive appropriate credit. Partial credit will not ordinarily be given for multiple-choice questions, but free-response questions may receive partial credit.

5. If you do not find the exact numerical answer, mark the answer with the closest value.

6. All multiple-choice answers must be marked clearly where indicated with at least 50% filling. Black pen or pencil is acceptable. Dark marks are more important than full boxes. If you mark one answer and need to correct it, clearly indicate the answer you intended and leave a note for the instructor below your name on the cover page.

7. Free-response answers must appear in the box provided. Answers appearing outside the box may be ignored.

8. All multiple-choice questions are 3 points. Points for free-response questions are indicated by the maximum number in the box marked "reserved for instructor."

← please encode your student number here (leave blank if you don’t know it or have your ID), and write your first and last names below.

Name: Solution
For the next four questions, use the following information. The fraction of student vehicles brought to Auburn is claimed to be 0.81. A survey of 100 students found 74 student vehicles brought to Auburn. The sample standard deviation is 0.44.

**Question 1** The alternate hypothesis is

- $\bar{X} \leq 0.81$
- $\bar{X} \neq 0.81$
- $\bar{X} = 0.81$
- $\bar{X}$ not in $[0.37, 1.25]$
- $\bar{X} = 0.74$

**Question 2** What is the estimated standard deviation of the sample mean?

- 0.0015
- 0.44
- 0.1936
- 0.37
- 0.044

\[
\frac{0.44}{\sqrt{100}} = 0.044
\]

**Question 3** Using a 91% confidence level, in what interval must the data fall to adopt the null hypothesis?

- $(-\infty, 0.885]$
- $[-1.70, 1.70]$
- $[0.751, 0.889]$
- $[0.665, 0.815]$
- $[0.735, 0.885]$

\[
\text{two-sided} \Rightarrow 0.91 = \int_{-\infty}^{\chi^2_{0.91}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx
\]

\[
\begin{align*}
0.91 & = \frac{1}{2} \left[ \int_0^{0.44} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx + \int_{0.44}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx \right] \\
& = \Phi(0.44) - \Phi(-0.44) = 1 - 2\Phi(0.44) \\
& \Rightarrow \Phi(0.44) = 0.645 \Rightarrow \bar{X} \approx 0.81
\end{align*}
\]

**Question 4** If the 100 samples follow a Gaussian distribution, what distribution can be used to calculate probabilities given that only a sample standard deviation is available?

- From CLT
- Gaussian
- Student’s t
- insufficient information

**Question 5** A stationary random signal is sampled every 0.1 s, and the following samples are obtained (in order): 1, 3, 4, 6, 4. Find the biased autocorrelation estimate for $\tau = 0.2$ s.

- 15.6
- 12.7
- 7.6
- 12.6
- 15.8

\[
\tau = 0.2 = n \Delta t = n \langle 0.1 \rangle \Rightarrow n = 2 (\text{shift})
\]

\[
\begin{array}{c}
1 \\
2 \\
4 \\
6 \\
4 \\
\hline
4+18+16 = 38
\end{array}
\]

**Question 6**

Indicate whether the function above is a valid autocorrelation function.

- valid
- invalid

*Symmetric, peak at $\tau = 0$, no flat tops, no discontinuities*
Question 7

Indicate whether the function above is a valid autocorrelation function.

- [ ] valid
- [ ] invalid

Discontinuities

Question 8

A random process has autocorrelation \( R_X(t_1, t_2) = \cos(t_2 - t_1) + 2 \) and mean \( \mu_X = -\sqrt{2} \). The most specific statement we can make about \( X(t) \) is that

- [ ] \( X \) is strict-sense stationary
- [ ] there is insufficient information to draw a conclusion
- [ ] \( X \) is wide-sense stationary
- [ ] \( X \) is not stationary in any sense

\( R_X \) only a function of \( t_2 - t_1 \) and \( \mu_X \) constant

Question 9

Which of the functions below cannot be a sample function of a random process?

- [ ] left
- [ ] right
- [ ] both can be samples of a random process

For the next four questions, we are wondering if a higher protein intake leads to weight gain. An individual varies the daily protein intake in three different months with total calorie intake and other factors held constant.

<table>
<thead>
<tr>
<th>Daily protein (p) in g</th>
<th>70</th>
<th>120</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight change (w) in lb.</td>
<td>2.0</td>
<td>4.1</td>
<td>3.4</td>
</tr>
</tbody>
</table>

Question 10

Find the unbiased sample mean of the weight change.

- [ ] 4.75
- [ ] 113.3
- [ ] 3.17
- [ ] 9.50

\[ \bar{w} = \frac{1}{3} \left( 2.0 + 4.1 + 3.4 \right) = 3.17 \]

Question 11

Find the unbiased sample standard deviation of the weight change if \( \sum(w_i - \bar{w})^2 = 2.287 \). Note: \( \bar{w} \) is the calculated sample mean.

- [ ] 0.87
- [ ] 0.76
- [ ] 2.29
- [ ] 1.14
- [ ] 1.07

\[ \sigma = \sqrt{\frac{1}{3-1} \sum (w_i - \bar{w})^2} = 1.07 \]

\[ \bar{w} \]

\[ \sigma^2 \]
Question 15  Find $f_X(x)$.

$$f_X(x) = \int f(x, y) \, dy = \int_0^1 \frac{1}{8} (xy + x + y + 2) \, dy$$

$$= \frac{1}{8} \left( \frac{1}{2} x y^2 + xy + \frac{1}{2} y^2 + 2y \right) \bigg|_0^1$$

$$= \frac{1}{8} \left( \frac{1}{2} x + x + \frac{1}{2} + 2 \right)$$

$$= \frac{1}{8} \left( \frac{3}{2} x + \frac{5}{2} \right) = \frac{1}{16} (3x + 5), \quad 0 \leq x < 2$$


Question 16  Are $X$ and $Y$ independent?

☐ yes  ☐ no  ☐ insufficient information

$$f_Y(y) = \int f(x, y) \, dx = \int_0^2 \frac{1}{8} (xy + x + y + 2) \, dx$$

$$= \frac{1}{8} \left( \frac{1}{2} x^2 y + \frac{1}{2} x^2 + xy + 2x \right) \bigg|_0^2$$

$$= \frac{1}{8} \left( 2y + 2 + 2y + 4 \right)$$

$$= \frac{1}{4} (2y + 3)$$

$$f_X(x) f_Y(y) \neq f(x, y)$$
**Corrected**

**Question 12**  Use linear regression to determine a model for weight change as a function of protein consumption. From this model, what would be the predicted weight gain in a month if the protein consumption is 50 g per day?

\[
\begin{align*}
\bar{w} &= 3.1667 \\
\bar{p} &= 0.9100 \\
\overline{w^2} &= 10.79 \\
\overline{p^2} &= 0.8281 \\
\overline{p \cdot w} &= 2.8817 \\
\end{align*}
\]

\[
\begin{align*}
m &= \frac{-\overline{p \cdot w}}{\overline{p^2} - (\overline{p})^2} = 0.02 \\
b &= \overline{w} - m \overline{p} = 0.90 \\
r &= \frac{\overline{p \cdot w} - \overline{p} \overline{w}}{\sqrt{(\overline{p^2} - (\overline{p})^2)(\overline{w^2} - (\overline{w})^2)}} = 0.76
\end{align*}
\]

**Question 13**  The correlation coefficient for this set of data tells us that

- ■ Higher protein consumption is a relatively good predictor of weight gain.
- □ Higher protein consumption has almost no relationship with weight gain.
- □ Higher protein consumption is a relatively poor predictor of weight gain.
- □ Higher protein consumption is a relatively good predictor of weight loss.

For the next four questions, consider the following joint density function.

\[
f(x, y) = \begin{cases} 
\frac{3}{8} (xy + x + y + 2) & 0 \leq x < 2, 
0 \leq y < 1 \\
0 & \text{otherwise}
\end{cases}
\]

**Question 14** \( \Pr(0 \leq X < 1, Y \geq 0) \) is

\[
= \int_0^1 \int_0^1 \frac{3}{8} (xy + x + y + 2) \, dx \, dy
= \int_0^1 \frac{1}{8} \left( \frac{1}{2} x^2 y + \frac{1}{2} x^2 + xy + 2x \right) \, dy
= \int_0^1 \frac{1}{8} \left( \frac{1}{2} y + \frac{1}{2} + y + 2 \right) \, dy
= \frac{1}{8} \left( \frac{1}{4} y^2 + \frac{1}{2} y + \frac{1}{2} y^2 + 2y \right) \bigg|_0^1 = \frac{1}{8} \left( \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 2 \right) = \frac{3.25}{8}
\]
Question 17  Find $F(x,y)$ in the interval $0 \leq x < 2$, $0 \leq y \leq 1$.

$$F(x,y) = \int_{-\infty}^{y} \int_{0}^{x} f(u,v) \, du \, dv = \int_{0}^{y} \int_{0}^{\frac{1}{8} (\frac{1}{2} u^2 v + \frac{1}{2} u^2 + uv + 2u)} \, du \, dv$$

$$= \int_{0}^{y} \frac{1}{8} \left( \frac{1}{2} u^2 v + \frac{1}{2} u^2 + uv + 2u \right) \Big|_{0}^{x} \, dv$$

$$= \int_{0}^{y} \frac{1}{8} \left( \frac{1}{4} x^2 v^2 + \frac{1}{2} x^2 v + \frac{1}{2} xy^2 + 2xy \right) \, dv$$

$$= \frac{1}{8} \left( \frac{1}{4} x^2 y^2 + \frac{1}{2} x^2 y + \frac{1}{2} xy^2 + 2xy \right)$$

Question 18  Let $Y = X + N$. For the density functions for $X$ and $N$ given below, sketch and label $f(x|y)$ for $y = 1.8$. (Height does not have to be labeled.) What is the most likely value of $X$ for $y = 1.8$?

$$f(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f_Y(y) f_X(x)}{f_Y(y)} = \frac{f_N(y-x) f_X(x)}{f_Y(y)}$$

Most likely value is $x = 1$. 

\[ [\text{Graph of } f_X(x) \text{ and } f_Y(y)] \]
For the next two questions, let \( Y(t) = 2X(t) + X(t-T) + N(t) \). \( R_X(\tau) = 2e^{-|\tau|} + 4 \). \( X(t) \) and \( N(t) \) are independent, and \( N(t) = 0 \).

**Question 19** Find \( R_Y(\tau) \).

\[
R_Y(\tau) = E[Y(t)Y(t+\tau)] = E\left[\left(2X(t) + X(t-T) + N(t)\right)\left(2X(t+\tau) + X(t+\tau-T) + N(t+\tau)\right)\right]
\]

\[
= E\left[4X(t)X(t+\tau) + 2X(t-T)X(t+\tau) + 2X(t)X(t+\tau-T) + X(t-T)X(t+\tau-T) + N(t)\left(2X(t+\tau) + X(t+\tau-T)\right) + N(t-T)N(t+\tau)\right]
\]

\[
= 4R_X(\tau) + 2R_X(\tau-T) + 2R_X(\tau+T) + R_X(\tau) + R_N(\tau)
\]

\[
= 10 + 20 + 4e^{-|\tau|} + 8 + 4e^{-|\tau-T|} + 8 + e^{-10|\tau|}
\]

**Question 20** Find the variance of \( X(t) \).

\[
\overline{X}^2 = R_X(0) = 2 + 4 = 6
\]

\[
\left(\overline{X}\right)^2 = R_X(\infty) = 4
\]

\[
\sigma^2 = \overline{X}^2 - \left(\overline{X}\right)^2 = 6 - 4 = 2
\]