MEMS Gyroscopes
- detect angular motion, but not translational motion

\[ \hat{y} \rightarrow x \]

1. Macroscale gyroscopes:
   - Newton's Second Law of Motion
     - An object in translational motion continues in that motion unless it is altered by an external force.
     - An object in rotational motion continues in that motion unless it is altered by an external torque.
     - Conservation of momentum: both linear and angular

2. Macroscale gyro's
   - Use a large spinning mass and conservation of angular momentum → measure angle of precession

3. Microscale gyro's
   - Most common approach: use a small mass and vibrate it in one direction and detect off-axis motion due to angular rotation; due to the Coriolis acceleration: $a_c$

\[ \hat{z} \rightarrow \Omega_z \]

\[ \hat{y} \rightarrow \hat{x} \rightarrow \text{object in motion in } x-y \text{ plane} \]

Coriolis accel = \( \hat{a}_c \Rightarrow a_{cx} \text{ and } a_{cy} \)

\[ a_{cx} = -2a_y \]
\[ a_{cy} = 2a_x \]
Lecture

Introduction to Coriolis Force

\[ \mathbf{r} = x \hat{x} + y \hat{y} \]

velocity vector: \[ \mathbf{\dot{r}} = V_x \hat{x} + V_y \hat{y} = \frac{d}{dt} \mathbf{r} \]

Given coordinate system rotates about z-axis at \( \mathbf{\Omega}_z = \omega \hat{z} = \dot{\omega} \hat{z} \).

Object \( T \) experiences "virtual force" in x-y plane due to \( \mathbf{\Omega}_z \) rotation.

\( \Rightarrow \) Coriolis Force

\( \Rightarrow \) results in Coriolis Acceleration in rotating x-y plane \( \mathbf{\dot{a}}_c \)

\[ \mathbf{\dot{a}}_c = 2 \mathbf{\Omega}_z \times \mathbf{\dot{r}} = 2 \Omega_z \hat{z} \times (V_x \hat{x} + V_y \hat{y}) = 2 \Omega_z V_x \hat{y} - 2 \Omega_z V_y \hat{x} \]

\( \mathbf{a}_c \) = \( 2 \Omega_z \) multiplied by \( V_x \)

Another way to look at this:

\[ s = r \theta \]

\( \dot{s} = r \dot{\theta} + \dot{r} \theta \]

\( \ddot{s} = r \ddot{\theta} + \dot{r} \dot{\theta} + \ddot{r} \theta + r \ddot{\theta} \)

\( = 2 \dot{r} \theta + \ddot{r} \theta + r \ddot{\theta} \)

\( = 2 \dot{r} \cdot \mathbf{\Omega}_z + \dot{r} \dot{\theta} + r \ddot{\theta} \)

If \( \theta = 0 \) and \( \dot{\theta} \approx 0 \) \( \Rightarrow \)

\( \ddot{s} \approx a_y = 2 V_x \Omega_z \)
If we were technicians: knowing the equation would be sufficient, but we are engineers and should know more.

Review

Unit vector → a normalized vector of length 1
Cartesian coordinates: 3 unit vectors in x, y, z directions

\[
\begin{align*}
\hat{x} &\quad \hat{y} &\quad \hat{z} \\
\hat{x} \times \hat{y} &\quad \hat{z} & \quad \hat{z} \times \hat{z} = 0
\end{align*}
\]

Cross Product of Unit Vectors • Uses the Right Hand Rule

1. \( \hat{x} \times \hat{y} = \hat{z} \)
2. \( \hat{y} \times \hat{z} = \hat{x} \)
3. \( \hat{z} \times \hat{x} = \hat{y} \)
4. \( \hat{x} \times \hat{x} = -\hat{z} \)
5. \( \hat{y} \times \hat{y} = -\hat{x} \)
6. \( \hat{z} \times \hat{z} = -\hat{y} \)
2. Consider a 2nd order model of a spring-mass-damper system in a reference frame B.

[Diagram of a spring-mass-damper system]

Assume the m can only move along x and y directions, but no rotation in B.

3. Let reference frame B be in a fixed reference frame F, where B can rotate with respect to F.

[Diagram showing rotation of B with respect to F]

\(X, Y, Z\) are in F \(\rightarrow \hat{X}, \hat{Y}, \hat{Z}\); unit vectors in F.

\(x, y, z\) are in B \(\rightarrow \hat{x}, \hat{y}, \hat{z}\); unit vectors in B.

Note: \(z\) and \(\hat{Z}\) always point in the same direction.
Relationship between \( \mathbf{R} \) and \( \mathbf{F} \):
\[
\hat{\mathbf{r}} = \hat{r}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \quad \text{and} \quad \hat{\mathbf{y}} = \hat{y}(\hat{\mathbf{x}}, \hat{\mathbf{y}})
\]
\[
\therefore \hat{\mathbf{r}} = \hat{x} \cos(\theta) + \hat{y} \sin(\theta)
\]
\[
\hat{\mathbf{y}} = -\hat{x} \sin(\theta) + \hat{y} \cos(\theta)
\]
\[\text{note: if } \theta = 0^\circ \rightarrow \hat{\mathbf{r}} = \hat{x} \]
\[\text{if } \theta = 90^\circ \rightarrow \hat{\mathbf{r}} = \hat{y} \]
\[\text{if } \theta = 0^\circ \rightarrow \hat{\mathbf{y}} = \hat{x} \]
\[\text{if } \theta = 90^\circ \rightarrow \hat{\mathbf{y}} = -\hat{x} \]

Angular Rate: \( \frac{d\theta}{dt} = \dot{\theta} = \Omega \)
Angular Acceleration: \( \frac{d\dot{\theta}}{dt} = \ddot{\theta} = \alpha \)

Derivatives of Unit Vectors

\[\frac{d}{dt} (\cos(\theta)) = -\dot{\theta} \sin(\theta) = -\Omega \sin(\theta)\]
\[\frac{d}{dt} (\sin(\theta)) = \dot{\theta} \cos(\theta) = \Omega \cos(\theta)\]
\[
\therefore \frac{d}{dt} (\hat{\mathbf{r}}) = \dot{\mathbf{r}} = \frac{d}{dt} (\hat{x} \cos(\theta) + \hat{y} \sin(\theta))
\]
\[
= -\hat{x} \sin(\theta) + \hat{y} \cos(\theta)
\]
\[
= \alpha (-\hat{x} \sin(\theta) + \hat{y} \cos(\theta))
\]

but: \( \hat{\mathbf{y}} = -\hat{x} \sin(\theta) + \hat{y} \cos(\theta) \)

\[\therefore \dot{\mathbf{r}} = \alpha \hat{\mathbf{y}} \]

\[\therefore \frac{d}{dt} (\hat{\mathbf{y}}) = \dot{\mathbf{y}} = \frac{d}{dt} (-\hat{x} \sin(\theta) + \hat{y} \cos(\theta))
\]
\[
= -\hat{x} \cos(\theta) - \hat{y} \sin(\theta)
\]
\[
= -\Omega (-\hat{x} \cos(\theta) - \hat{y} \sin(\theta))
\]
\[
= -\Omega \hat{\mathbf{r}}
\]