1. Theoretical Analysis of Accelerometers

model: \[ \begin{array}{c}
\dot{y}(t) \\
y(t) \\
\dot{x}(t) \\
x(t)
\end{array} \]
\[ \begin{array}{c}
\ddot{y}(t) \\
k \\
C \\
m
\end{array} \]
\[ y(t) \text{ is our input} \]
\[ x(t) \text{ is our output} \]

For a constant acceleration, a steady state condition: \[ \dot{x} = \ddot{y}, \quad x = y \]

\[ F_{\text{inertial}} = F_{\text{spring}} \]
\[ m\ddot{x} = k\dot{d}, \quad d = \text{spring displacement} \]
\[ ma = kd \]
\[ d = a \frac{m}{k} = \frac{a}{w_n^2} \quad \text{since} \quad w_n = \sqrt{\frac{k}{m}} \]

Define: 
Sensitivity \( S = \frac{m}{k} = \frac{1}{w_n^2} \rightarrow d = aS \)

Trade Off: 
wide bandwidth sensor: large \( w_n \rightarrow \) low sensitivity
high sensitivity sensor: large \( S \rightarrow \) low bandwidth

a) Damping Ratio, \( \xi \) for an accelerometer

another model: \[ \begin{array}{c}
\dot{y}(t) \\
y(t) \\
\dot{x}(t) \\
x(t) \\
f(t)
\end{array} \]
\[ \begin{array}{c}
k \\
C \\
m \\
f(t)
\end{array} \]

\( f(t) = \text{inertial force} = ma(t) \)

System dynamics: \[ m\ddot{x} + c\dot{x} + k\dot{x} = f(t) = ma(t) \]
\[ X(s) s^2 + X(s) s \frac{c}{m} + X(s) \frac{k}{m} = A(s) \]
\[ \frac{X(s)}{A(s)} = \frac{1}{s^2 + \frac{c}{m} s + \frac{\omega_n^2}{m}} = \frac{1}{s^2 + 2\xi \omega_n s + \omega_n^2} \]

Plot of \( \frac{\left| X(s) \right|}{Y(s)} v.s. \omega : \)

\[ \frac{1}{\omega_n^2} \quad \frac{\xi}{\omega_n} \]

For accelerometers, \( \xi = 1 \) usually \( \rightarrow \) critically damped

\( \rightarrow \) fast response time

\( \rightarrow \) no overshoot in the time response

b. Frequency Domain Analysis of an accelerometer with \( \xi = 1 \)

For the system spring model: \( F_s = k_{sys} d \)

where \( d(t) = y(t) - x(t) \) \( \rightarrow \) earlier: \( d = aS = a \frac{m}{m} = \frac{a}{\omega_n} \), \( a = \text{constant} \)

or \( D(s) = Y(s) - X(s) \)

From Transmissibility (earlier in the semester):

\[ T(s) = \frac{X(s)}{Y(s)} = \frac{2\xi \omega_n s + \omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \]

\( \therefore X(s) = Y(s) T(s) \)

and \( D(s) = Y(s) - X(s) \)

\[ = Y(s) - Y(s) T(s) \]

\[ = Y(s)(1 - T(s)) \]
\[ 1 - T(s) = \frac{s^2 + 2\beta \omega_n s + \omega_n^2}{s^2 + 2\beta \omega_n s + \omega_n^2} - \frac{2\beta \omega_n s + \omega_n^2}{s^2 + 2\beta \omega_n s + \omega_n^2} \]

\[ = \frac{s^2}{s^2 + 2\beta \omega_n s + \omega_n^2} \]

\[ \therefore D(s) = Y(s) \left[ \frac{s^2}{s^2 + 2\beta \omega_n s + \omega_n^2} \right] \]

\[ s = 1 \]

From last time:

- For a constant acceleration \( \dddot{y} = \alpha \rightarrow y(t) = \frac{1}{2}at^2 \)
  - and \( y(0) = \frac{a}{s^2} \)

Note:

\[ s^2 + 2\omega_n s + \omega_n^2 = (s + \omega_n)^2 \]

\[ \therefore D(s) = \frac{a}{s(s + \omega_n)^2} \]

We want to find \( \dddot{y}(t) = \mathcal{F}^{-1} \{ D(s) \} \)

- Use Partial Fraction Expansion

\[ \frac{a}{s(s + \omega_n)^2} = \frac{k_1}{s} + \frac{k_2}{s + \omega_n} + \frac{k_3}{(s + \omega_n)^2} \]

\[ k_1 = \frac{a}{(s + \omega_n)^2} \bigg|_{s=0} = \frac{a}{\omega_n^2} \]

\[ k_3 = \frac{a}{s} \bigg|_{s=\omega_n} = -\frac{a}{\omega_n} \]
For \( K_2 \):

\[
\frac{a}{s(1+sw_0)^2} = \frac{K_1}{s} + \frac{K_2}{s+sw_0} + \frac{K_3}{s(1+sw_0)^2}
\]

or

\[
a = K_1(s+sw_0)^2 + K_2(s+sw_0) + K_3 s
\]

\[= K_1(s^2+2sw_0s+w_0^2) + K_2(s^2+w_0s) + K_3 s \quad \text{Eq A}
\]

\[
\frac{dA}{ds} = 0 = K_1(2s+2w_0) + K_2(2s+w_0) + K_3 \quad \text{Eq B}
\]

\[
\frac{dB}{ds} = 0 = 2K_1 + 2K_2 + 0
\]

\[
; K_2 = -K_1 = -\frac{a}{w_0^2}
\]

\[
; B(s) = \frac{a}{w_0^2 s} - \frac{a}{w_0^2(s+sw_0)} - \frac{a}{w_0(s+sw_0)^2}
\]

Use Laplace Transform Tables:

\[
\frac{K}{s} \rightarrow K
\]

\[
\frac{1}{s-K} \rightarrow e^{kt}
\]

\[
\frac{1}{(s-K)^2} \rightarrow te^{kt}
\]

\[
\therefore \quad d(t) = \mathcal{L}^{-1}[B(s)] = \frac{a}{w_0^2} - \frac{a}{w_0^2} e^{-w_0t} - \frac{a}{w_0} + e^{-w_0t}
\]

\[\text{steady state term} \quad \text{transient term : time constant } t_c = \frac{1}{w_0} \]

\[
; d(t) \bigg|_{s.s.} = \frac{a}{w_0^2} = \frac{am}{K} = aS \rightarrow \text{same as before}
\]

\[
; \text{if you know } S \rightarrow \text{measure spring deflection to determine } a
\]

\[
\Rightarrow \text{plot on next page: } d(t) \text{ vs } t \text{ for } a=1m/s^2 \text{ and } w_0=1rad/s \text{ with } S=1
\]
\[ d(t) \text{ Vs Time for } a=1\text{m/s}^2, \, \omega_n=1\text{rad/s} \]
1) Accelerometer Structures

\[ \text{Steady State: } F_r = F_w \]
\[ Kd = ma \]
\[ d = a \frac{m \omega_n}{K} = a S \]
\[ S = \text{Sensitivity, } S = \frac{m \omega_n}{K} = \frac{1}{\omega_n^2} \]
\[ [S] = S^2 \]

Often, the proof mass is made as large as possible. Often, the springs are made to be stiff (K large, \(\omega_n\) small)

\[ K \propto \frac{E_n t^3}{L^2} \]

result: for high \(S\) want small \(K\), big \(m\)

2) Bulk Micromachined Accelerometer Designs

- similar architecture to pressure sensors

a) single cantilever spring design

micromachined Si part:

Cross-section of whole device

- Motion of \(\mu\)
- Glass, anodically bonded to Si
- Hermetically sealed chamber
- Some gas at \(P_0\) (at \(T_0\)) \(\Rightarrow \beta T_0\)

piece resistor to measure spring strain
\[ R = f(\mu \text{ deflection}) \]
b) Double-clamped beam-spring design

4 piezoresistors: 2 in tension, 2 in compression; same as bossed pressure sensor

\[ R + \Delta R \quad \text{and} \quad R - \Delta R \quad \text{if PM moving up} \]
\[ R - \Delta R \quad \text{and} \quad R + \Delta R \quad \text{if PM moving down} \]

→ use Wheatstone bridge

c) Capacitive Sensing → also similar to Pressure Sensors

\[ \frac{A}{d} \quad C = \frac{\varepsilon_0 \varepsilon_r A}{d_0 + dd} \rightarrow \text{Single capacitance} \]

i) Differential Capacitance

\[ 2 \text{electrodes: } C_{\text{top}} = \frac{\varepsilon_0 \varepsilon_r A}{d_0 + dd} \]
\[ 2 \text{electrodes: } C_{\text{bot}} = \frac{\varepsilon_0 \varepsilon_r A}{d_0 + dd} \]

\[ \text{Diagram: } V_{\text{in}} \rightarrow C_{\text{top}} \rightarrow V_{\text{out}} \rightarrow C_{\text{bot}} \]
1) Other spring designs are also used

\[
\text{Frame} \xrightarrow{4 \text{ springs}} \text{Frame} \xrightarrow{8 \text{ springs}} \text{a lot of wasted space}
\]

2) Consider this design:

\[
\text{proof mass} \quad \text{note: be cautious of bending modes}
\]

3) This design yields a larger mass and longer springs (smaller \( k \))

\[
S^2 = \frac{m}{K} \Rightarrow m \uparrow \quad \therefore S^\uparrow
\]

3. Surface Micromachined Accelerometer Designs

- can build out of polysi deposited on SiO\(_2\) [sacrificial layer]

Start with:

\[
\text{Si wafers} \quad \text{thin polysi layer} \quad \text{thin SiO}_2 \quad \text{layer}
\]

Similar to Si wafer, but SOI device layer may be much thicker than polysi layer
Example of a lateral motion accelerometer using interdigitated teeth to measure capacitance:

\[ A_0 \quad 1 \quad C = \frac{\varepsilon_0 \varepsilon_r + f(t)}{d_0} \]

Overlap area = \( f \times \lambda(t) \)

- Holes allow SiO\(_2\) to be etched from under PM

Differential capacitance sensing

Could also be built using an SOI wafer

4) All these designs are "open loop" accelerometers

\[ \text{Output signal} = f(\text{PM displacement}) \]