Understanding Minimization . . .

Logic function: \[ F = \overline{a} \overline{b} + \overline{a}b + bd + cd \]
Reducing products:

\[ F = a\overline{b} + \overline{a}b + bd + cd \]
\[ = \overline{b}(a + \overline{a}) + bd + cd \]
\[ = \overline{b}1 + bd + cd \]
\[ = \overline{b}(c + \overline{c}) + bd + cd \]
\[ = bd + \overline{b}c + cd + \overline{b}\overline{c} \]
\[ = bd + \overline{b}c + \overline{b}\overline{c} \]
\[ = bd + \overline{b}(c + \overline{c}) \]
\[ = bd + \overline{b} \]

Distributivity
Complementation
Identity
Complementation
Distributivity
Consensus theorem
Distributivity
Complement, identity
... Understanding Minimization

Reduced SOP: \[ F = bd + \overline{b} \]

Diagram:
- b inputs to NOT gate
- d inputs to AND gate
- AND gate outputs to OR gate
- OR gate outputs to F
. . . Understanding Minimization

Reducing literals:

\[ F = bd + \overline{b} \]
\[ = d + \overline{b} \]

Absorption theorem

Exercise: This circuit uses 8 transistors in CMOS technology. Can you redesign it with 6 transistors? *(Hint: Use de Morgan’s Theorem.)*
Or, Could Use Karnaugh Map

http://www.ee.calpoly.edu/media/uploads/resources/KarnaughExplorer_1.html
Logic Minimization

- Generally means
  - In SOP form:
    - Minimize number of products (reduce gates) and
    - Minimize literals (reduce gate inputs)
  - In POS form:
    - Minimize number of sums (reduce gates) and
    - Minimize literals (reduce gate inputs)
Product or Implicant or Cube

Any set of literals ANDed together.

Minterm is a special case where all variables are present. It is the largest product.

A minterm is also called a 0-implicant of 0-cube.

A 1-implicant or 1-cube is a product with one variable eliminated:

¬D = ABC(D + ¬D) = ABC

Obtained by combining two adjacent 0-cubes
Cubes (Implicants) of 4 Variables

What is this?

Minterm or 0-implicant or 0-cube
A \ \bar{B} \ \bar{C} \ \bar{D}

1-implicant or 1-cube
ABD
Growing Cubes, Reducing Products

1-implicant or 1-cube
$\overline{A}B D$

2-implicant or 2-cube
$BD$

What is this?

1-implicant or 1-cube
$ABD$
Largest Cubes or Smallest Products

What is this?

3-implicant or 3-cube

A
B
C
D
Implication and Covering

A larger cube covers a smaller cube if all minterms of the smaller cube are included in the larger cube.

A smaller cube implies (or subsumes) a larger cube if all minterms of the smaller cube are included in the larger cube.
Implicants of a Function

Minterms, products, cubes that imply the function.

\[ F = \overline{AB} + BD + \overline{ACD} + AB\overline{CD} \]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
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<tr>
<td>6</td>
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</tr>
</tbody>
</table>

\( A \) \( B \) \( C \) \( D \)

\( \overline{A} \) \( \overline{B} \) \( \overline{C} \) \( \overline{D} \)
Prime Implicant (PI)

A cube or implicant of a function that cannot grow larger by expanding into other cubes.
Essential Prime Implicant (EPI)

If among the minterms subsuming a prime implicant (PI), there is at least one minterm that is covered by this and only this PI, then the PI is called an essential prime implicant (EPI).

Also called essential prime cube (EPC).

Why not this?
Redundant Prime Implicant (RPI)

If each minterm subsuming a prime implicant (PI) is also covered by other essential prime implicants, then that PI is called a redundant prime implicant (RPI).

Also called redundant prime cube (RPC).
Selective Prime Implicant (SPI)

- A prime implicant (PI) that is neither EPI nor RPI is called a selective prime implicant (SPI).
- Also called selective prime cube (SPC).
- SPIs occur in pairs.
Minimum Sum of Products (MSOP)

- Identify all prime implicants (PI) by letting minterms and implicants grow.

- Construct MSOP with PI only:
  - Cover all minterms
  - Use only essential prime implicants (EPI)
  - Use no redundant prime implicant (RPI)
  - Use cheaper selective prime implicants (SPI)
  - A good heuristic – Choose EPI in ascending order, starting from 0-implicant, then 1-implicant, 2-implicant, . . .
Example: $F = \sum m(1,3,4,5,8,9,13,15)$

MSOP:

$$F = \overline{A} \overline{B} D + \overline{A} B \overline{C} + A B D + A \overline{B} \overline{C}$$
Example: $F=\Sigma m(1,3,5,7,8,10,12,13,14)$

MSOP:
$$F = \bar{A}D + A\bar{D} + AB\bar{C}$$
Functions with Don’t Care Minterms

F(A,B,C) = \sum m(0,3,7) + d(4,5)

Include don’t care minterms when beneficial.

F = B \overline{C} + \overline{A} \overline{B} \overline{C}

F = B \overline{C} + \overline{B} \overline{C}
Five-Variable Function

\[ F(A,B,C,D,E) = \sum m(0,1,4,5,6,13,14,15,22,24,25,28,29,30,31) \]

A = 0

\[
\begin{array}{cccc}
0 & 1 & 4 & 1 \\
1 & 1 & 5 & 1 \\
3 & 7 & 15 & 1 \\
2 & 6 & 14 & 1 \\
\end{array}
\]

B

A = 1

\[
\begin{array}{cccc}
16 & 20 & 28 & 24 \\
17 & 21 & 29 & 25 \\
19 & 23 & 31 & 27 \\
18 & 22 & 30 & 26 \\
\end{array}
\]

B

D

E

C

F = \overline{A} \overline{B} \overline{D} + AB \overline{D} + BCE + CDE
## Multiple-Output Minimization

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Individual Output Minimization

Need five products.

F1

A

F2

A

C

D

B

C

D

B

Fall 2015, Oct 23... ELEC2200-002 Lecture 5
Global Minimization

Need four products.

F1

F2
Minimized SOP and POS

\[ F(A,B,C,D) = \sum m(1,3,4,7,11) + d(5,12,13,14,15) \]

\[ = \Pi M(0,2,6,8,9,10) D(5,12,13,14,15) \]

\[ F = B \overline{C} + \overline{A}D + CD \]

\[ \overline{F} = \overline{B} \overline{D} + C \overline{D} + A \overline{C} \]

\[ F = (B + D)(\overline{C} + D)(\overline{A} + C) \]
SOP and POS Circuits

\[ F(A,B,C,D) = \sum m(1,3,4,7,11) + d(5,12,13,14,15) = \prod M(0,2,6,8,9,10) D(5,12,13,14,15) \]

Are two circuits functionally identical?
How Don’t Cares Occur

Consider two crossroads:
- A highway with traffic signals, red (R), yellow (Y) and green (G), and
- A rural road with red (r) and green (g) signals.

Here R, Y, G, r and g are Boolean variables; a 1 implies light is on, 0 means light is off.

Highway signals R, Y and G are controlled by a computer.

Rural traffic can cross highway (g = 1) only when R = 1, Y = G = 0, and r = g.
Completely Specified Function

<table>
<thead>
<tr>
<th>minterm</th>
<th>R</th>
<th>Y</th>
<th>G</th>
<th>g</th>
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<tr>
<td>0</td>
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<td>1</td>
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<tr>
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</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ g = R \overline{Y} \overline{G} \]
Incompletely Specified Function

Additional condition: Exactly one highway light can be on.

Φ or X denotes “don’t care”
Absorption Theorem

For two Boolean variables: A, B
A + A B = A

Proof:
Consensus Theorem

For three Boolean variables: $A$, $B$, $C$

$A \land B + \bar{A} \land C + B \land C = A \land B + \bar{A} \land C$

Proof:
Growing Implicants to PI

\[ F = AB + \overline{CD} + \overline{ABCD} \]

- **initial implicants**

\[ = AB + \overline{ABCD} + BCD + \overline{CD} \]

- **consensus th.**

\[ = AB + BCD + \overline{CD} \]

- **absorption th.**

\[ = AB + BCD + \overline{CD} + BD \]

- **consensus th.**

\[ = AB + \overline{CD} + BD \]

- **absorption th.**
Identifying EPI

- Find all prime implicants.
- From prime implicant SOP, remove a PI.
- Apply consensus theorem to the remaining SOP.
- If the removed PI is generated, then it is either an RPI or an SPI.
- If the removed PI is not generated, then it is an EPI.
Example

PI SOP: \( F = A \overline{D} + \overline{A} \overline{C} + \overline{C}D \)

Is AD an EPI?

\[ F - \{AD\} = \overline{A} \overline{C} + \overline{C}D, \text{ no new PI can be generated} \]

Hence, AD is an EPI. Similarly, A \( \overline{C} \) is an EPI.
Example (Cont.)

PI SOP: \( F = A \overline{D} + \overline{A} \overline{C} + \overline{CD} \)

Is \( \overline{CD} \) an EPI?

\[
F - \{ \overline{CD} \} = A \overline{D} + \overline{A} \overline{C} = A \overline{D} + \overline{A} \overline{C} + \overline{C} D
\]

(Consensus theorem)

Hence \( \overline{C} D \) is not an EPI
(it is an RPI)

Minimum SOP:
\( F = A \overline{D} + \overline{A} \overline{C} \)
Finding MSOP

1. Start with minterm or cube SOP representation of Boolean function.
2. Find all prime implicants (PI).
3. Include all EPI’s in MSOP.
4. Find the set of uncovered minterms, \{UC\}.
5. MSOP is minimum if \{UC\} is empty. \textbf{DONE.}
6. For a minterm in \{UC\}, include the largest PI from remaining PI’s (non-EPI’s) in MSOP.
7. Go to step 4.
Finding Uncovered Minterms, \{UC\}

\[ \{UC\} = (\{PI\} \# \{PMSOP\}) \# \{DC\} \]

Where:

- \{PI\} is set of all prime implicants of the function.
- \{PMSOP\} is any partial SOP.
- \{DC\} is set of don’t care minterms.
- Sharp (\#) operation between Boolean expressions \( X \) and \( Y \), \( X \# Y \), is the set of minterms covered by \( X \) that are not covered by \( Y \).
Example: # (Sharp) Operation

\[ AD \# \overline{CD} = \{A \ B \ C \ D, \ A \overline{B} \ C \ D\} \]

\[ \overline{CD} \# AD = \{ \overline{A} \ B \ C \ D, \ \overline{A} \overline{B} \overline{C} \ D\} \]
Minterms Covered by a Product

A product from which k variables have been eliminated, covers $2^k$ minterms.

Example: For four variables, A, B, C, D

Product AC covers $2^2 = 4$ minterms:

1) $A \overline{B} C \overline{D}$
2) $A \overline{B} C D$
3) $A B C \overline{D}$
4) $A B C D$

Obtained by inserting the eliminated variables in all possible ways.
Quine-McCluskey

Willard V. O. Quine  
1908 – 2000

Edward J. McCluskey  
b. 1929
Quine-McCluskey Tabular Minimization Method


Textbook, Section 3.9, pp. 211-225.
Q-M Tabular Minimization

Minimizes functions with many variables.

Begin with minterms:

- Step 1: Tabulate minterms in groups of increasing number of true variables.
- Step 2: Conduct linear searches to identify all prime implicants (PI).
- Step 3: Tabulate PI’s vs. minterms to identify EPI’s.
- Step 4: Tabulate non-essential PI’s vs. minterms not covered by EPI’s. Select minimum number of PI’s to cover all minterms.

MSOP contains all EPI’s and selected non-EPI’s.
F(A,B,C,D) = \sum m(2,4,6,8,9,10,12,13,15)

Q-M Step 1: Group minterms with 1 true variable, 2 true variables, etc.

<table>
<thead>
<tr>
<th>Minterm</th>
<th>ABCD</th>
<th>Groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0010</td>
<td>1: single 1</td>
</tr>
<tr>
<td>4</td>
<td>0100</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0110</td>
<td>2: two 1's</td>
</tr>
<tr>
<td>9</td>
<td>1001</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1100</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1101</td>
<td>3: three 1's</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
<td>4: four 1's</td>
</tr>
</tbody>
</table>
Q-M Step 2

Find all implicants by combining minterms, and then combining products that differ in a single variable: For example,

2 and 6, or \( \overline{A} \overline{B} C \overline{D} \) and \( \overline{A} B C \overline{D} \rightarrow \overline{A} C \overline{D} \), written as 0 – 1 0.

Try combining a minterm (or product) with all minterms (or products) listed below in the table.

Include resulting products in the next list.

If minterm (or product) does not combine with any other, mark it as PI.

Check the minterm (or product) and repeat for all other minterms (or products).
Step 2 Executed on Example

<table>
<thead>
<tr>
<th>List 1</th>
<th>List 2</th>
<th>List 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minterm</td>
<td>ABCD</td>
<td>PI?</td>
</tr>
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<td>X</td>
</tr>
<tr>
<td>4</td>
<td>0100</td>
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<td>X</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
<td>X</td>
</tr>
</tbody>
</table>
## Step 3: Identify EPI’s

<table>
<thead>
<tr>
<th>Covered by EPI →</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>12</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
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<tr>
<td>Minterms →</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
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<td></td>
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<td>4</td>
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<td>8</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>PI_1 is EPI</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>PI_2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
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<tr>
<td>PI_3</td>
<td></td>
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<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
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</tr>
<tr>
<td>PI_4</td>
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<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>PI_5</td>
<td></td>
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<td></td>
<td></td>
<td>x</td>
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<td>PI_6</td>
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<td>x</td>
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<tr>
<td>PI_7 is EPI</td>
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<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>
Step 4: Cover Remaining Minterms

<table>
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<tr>
<th>Remaining minterms →</th>
<th>2</th>
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<th>10</th>
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<tbody>
<tr>
<td>PI_2</td>
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<td>x</td>
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</tr>
<tr>
<td>PI_3</td>
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<td>PI_4</td>
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<td></td>
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<td>x</td>
</tr>
<tr>
<td>PI_5</td>
<td></td>
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<td></td>
<td>x</td>
</tr>
<tr>
<td>PI_6</td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>

Integer linear program (ILP), available from Matlab and other sources: Define integer \{0,1\} variables, \(x_k = 1\), select PI_k; \(x_k = 0\), do not select PI_k.

Minimize \(\sum_k x_k\), subject to constraints:
\[
\begin{align*}
  & x_2 + x_3 \geq 1 \\
  & x_4 + x_5 \geq 1 \\
  & x_2 + x_4 \geq 1 \\
  & x_3 + x_6 \geq 1
\end{align*}
\]

A solution is \(x_3 = x_4 = 1\), \(x_2 = x_5 = x_6 = 0\), or select PI_3, PI_4.
Linear Programming (LP)

A mathematical optimization method for problems where some “cost” depends on a large number of variables.

An easy to understand introduction is:


Very useful tool for a variety of engineering design problems.

Available in software packages like Matlab.

Courses on linear programming are available in Math, Business and Engineering departments.
Q-M MSOP Solution and Verification

- \( F(A,B,C,D) = PI_1 + PI_3 + PI_4 + PI_7 \)
  \[= 1-0- + -010 + 01-0 + 11-1 \]
  \[= A \bar{C} + \bar{B} C \bar{D} + \bar{A} B \bar{D} + A B D \]

- See Karnaugh map.

EPI’s in MSOP

Non-EPI’s in MSOP

Non-EPI’s not in MSOP
QM Minimizer on the Web

http://quinemccluskey.com/
Function with Don’t Cares
\[ F(A,B,C,D) = \sum m(4,6,8,9,10,12,13) + \sum d(2, 15) \]

Q-M Step 1: Group “all” minterms with 1 true variable, 2 true variables, etc.

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<td>4</td>
<td>0100</td>
<td>2: two 1’s</td>
</tr>
<tr>
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<tr>
<td>6</td>
<td>0110</td>
<td>3: three 1’s</td>
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<tr>
<td>9</td>
<td>1001</td>
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<tr>
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</tr>
<tr>
<td>12</td>
<td>1100</td>
<td></td>
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<tr>
<td>13</td>
<td>1101</td>
<td>4: four 1’s</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
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</table>
Step 2: Same As Before on “All” Minterms

<table>
<thead>
<tr>
<th>List 1</th>
<th>List 2</th>
<th>List 3</th>
</tr>
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<tbody>
<tr>
<td><strong>Minterm</strong></td>
<td><strong>ABCD</strong></td>
<td><strong>PI?</strong></td>
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<tr>
<td>2</td>
<td>0010</td>
<td>X</td>
</tr>
<tr>
<td>4</td>
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<tr>
<td>8</td>
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<td>X</td>
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<tr>
<td>6</td>
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<td>9</td>
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<td>X</td>
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<tr>
<td>10</td>
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<tr>
<td>12</td>
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<tr>
<td>13</td>
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<td>X</td>
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Step 3: Identify EPI’s Ignoring Don’t Cares

<table>
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<td>4</td>
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<td>8</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>13</td>
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<tr>
<td>PI1 is EPI</td>
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<tr>
<td>PI2</td>
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<td>PI4</td>
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<tr>
<td>PI6</td>
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<tr>
<td>PI7</td>
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Table: Covered by EPI and Minterms
Step 4: Cover Remaining Minterms

<table>
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</thead>
<tbody>
<tr>
<td>PI_2</td>
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<tr>
<td>PI_3</td>
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<tr>
<td>PI_4</td>
<td>x</td>
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</tr>
<tr>
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<tr>
<td>PI_6</td>
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<td>x</td>
</tr>
</tbody>
</table>

Integer linear program (ILP), available from Matlab and other sources: Define integer \(\{0,1\}\) variables, \(x_k = 1\), select PI_k; \(x_k = 0\), do not select PI_k.
Minimize \(\sum_k x_k\), subject to constraints:
\[
\begin{align*}
  x_4 + x_5 & \geq 1 \\
  x_2 + x_4 & \geq 1 \\
  x_3 + x_6 & \geq 1
\end{align*}
\]
A solution is \(x_3 = x_4 = 1\), \(x_2 = x_5 = x_6 = 0\), or select PI_3, PI_4.
Q-M MSOP Solution and Verification

- \( F(A,B,C,D) = PI_1 + PI_3 + PI_4 \)
  \[= 1-0- + -010 + 01-0\]
  \[= A \overline{C} + \overline{B} C \overline{D} + \overline{A} B \overline{D} \]

- See Karnaugh map.
Minimized Circuit

A B C D
\[ \overline{A} \overline{B} \overline{C} \overline{D} \]

F

A B C D
\[ \begin{align*}
\text{PI1} \\
\text{PI3} \\
\text{PI4}
\end{align*} \]
Further Reading

- Incompletely specified functions: See Example 3.25, pages 218-220.
- Multiple output functions: See Example 3.26, pages 220-222.