A methodology of synthesis of lossy ladder filters

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***Abstract*—In the paper, issues concerning the synthesis of ladder reactance analog filter have been raised. In all existing algorithms of synthesis of such filters it is assumed that the reactance elements are ideal ones. Since actual reactance elements are lossy, which means the Q-factor has a finite value, the frequency response of real filters differs widely from the desired frequency response, assumed before designing the filter. In this article, a new approach to the synthesis of reactance ladder filters has been introduced. In this approach, the lossiness of reactance elements is taken into consideration during the synthesis process. This method allows to design a ladder filter with desired frequency response with the use of lossy capacitors and what is more important, lossy inductors.**

I. INTRODUCTION

A ladder passive reactance filter is made up of inductors and capacitors arranged series and shunt alternately, or of series or parallel tuned LC circuits. Many varieties of such filters are used: doubly terminated symmetric or asymmetric filters, or singly terminated with resistive load or without resistive load. Predominantly, filters with Butterworth, Chebychev, inverse Chebychev and Cauer frequency responses are used. Each of these responses has different, specific properties.

The classical synthesis process of ladder reactance filter begins with designing a lowpass prototype. The reactance elements in the prototype are then transformed, if necessary, to obtain a highpass, bandpass or bandstop filter. Next, the values of all filter elements are scaled. There are two main purposes of scaling: to produce more practical or simply feasible values of filter elements, e.g. load resistance (magnitude scaling) and/or to move the filter passband towards the required range of frequencies (frequency scaling). Almost always, transformation and scaling take place at the same time, with the use of appropriately modified formulae.

For the most popular sorts of prototype lowpass filters, the values of elements have been derived and put into tables [6]. Thus, when designing a filter, there is no need to start from the desired frequency response and to recalculate them anew. Therefore, synthesis of a ladder reactance filter is very simple and requires relatively less computation, particularly when using tables of element values to obtain the prototype filter.

There are several ways of implementation of such ladder filters. They are built as reactance ladders according to the obtained circuit scheme, or as ladders with simulated inductors. The latter are commonly used in integrated circuit technologies. The subject of analog filter design was recently undertaken by [1], but that work focuses rather on noise and non-linearity of integrated filters with simulated inductors. Fig. 1 shows a lowpass, seventh-order Butterworth filter. Fig. 2 shows an example of implementation of a third-order Butterworth filter, which has one series inductor in its prototype. This inductance has been realized with the use of two gyrators, each of which has been built with two operational transconductance amplifiers OTA-C. Assuming the gyrators are ideal, the filter shown in Fig. 2 has exactly the same frequency response as its passive LC prototype. Another way of realizing ladder filters using differential voltage current conveyor (DVCC) is described in [2]. For yet other examples of realizations of ladder filters see [3][4].

Unfortunately, for every possible realization of a ladder reactance filter, its magnitude Bode plot has a different shape than it was intended to have. The reason of this fact is that the real reactance elements used to build the filter are lossy. To determine how big the reactance element lossiness is, the so called quality factor (Q-factor) is used. This dimensionless parameter is particularly important for inductors. Real inductors have low values of the quality factor and consequently high values of lossiness. It concerns real inductors made simply of wounded wire as well as conductors simulated with real gyrators loaded with capacitors, or with frequency-dependent negative resistors (FDNR) or generalized impedance converter (GIC) etc. For example, inductors simulated with a real gyrator loaded with a capacitor, have lossiness because of non-zero values of the gyrator's parasitic admittances. Additional issues concerning simulated inductances with gyrators are described in [5].

In this article, we will use the term lossiness rather than

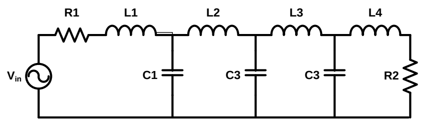

Figure 1. A seventh-order lowpass Butterworth ladder filter.

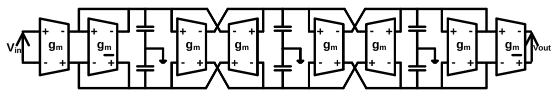

Figure 2. An example implementation of the third-order lowpass Butterworth ladder filter.

the quality factor. The use of the term lossiness means series resistance, measured in ohms, for inductors and leakage conductance, measured in siemens, for capacitors.

Further in the article, units will be omitted for simplicity of notification. Models of real inductors and real capacitors are shown in Fig. 3.

The greater the lossiness of reactance elements used to build a filter, the greater the difference between the frequency response of the filter and the frequency response that the filter was intended to have. Fig. 4 shows the comparison of magnitude Bode plots and poles location on the complex plane for a seventh-order Chebychev lowpass prototype filter and its realization with real, lossy elements.

In classical algorithms of ladder filter synthesis it is assumed that the elements used for the filter implementation will be lossless, which is obviously impossible to accomplish. As we will see later in this article, it is possible to replenish the synthesis algorithm so that the lossiness of the reactance elements will be taken into consideration already when the filter is being designed. A filter obtained in this way can have exactly the same frequency response as the non-feasible filter designed using the classical algorithm and built with ideal reactance elements.

As it is shown in Fig. 4, the lossiness of the reactance elements changes the location of poles on the complex plane, and this implies the deformation of the magnitude response of the filter. With some additional assumptions it is however possible to modify the values of capacitance and inductance of the reactance elements so as to move the poles back into place. This is the key concept of the synthesis algorithm introduced in this article.

II. SINGLY TERMINATED LADDER FILTERS.

There are two configurations of singly terminated ladder filters: with resistive load and without resistive load, as shown in Fig. 5.

A. *Ladder filters with resistive load.*

Let us take into consideration a singly terminated fourth-order Chebychev lowpass prototype with a transfer function obtained using approach from [7].

$$Tp = \frac{1}{1 + 2.44475\,s + 3.17051\,s^2 + 2.17713\,s^3 + 1.20699\,s^4} \tag{1}$$

Having this filter designed using the classical synthesis

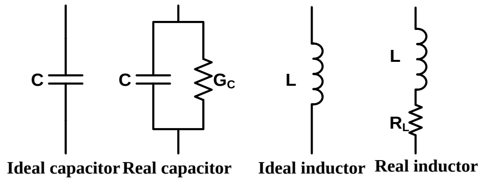

Figure 3. Models of real inductor and real capacitor.

algorithm, that is without considering the reactance elements' lossiness, a schematics as in Fig. 6 can be drawn.

In order to include the lossiness, all of the inductors and capacitors in the schematic have to be replaced by their real models from Fig. 3. In this way we get a circuit shown in Fig. 7.

Let us now set the example value of the lossiness $RL1 = RL2 = GC1 = GC2 = 0.1$, although quite obviously this value can be different for each element. After that, the transfer function of the filter shown in Fig. 7 is as follows

$$\begin{aligned}
Ts = {} & R2\,/ \\
& 0.201 + 1.0301R2 + (0.01C1 + 1.01L1 + 1.01L2 + 0.101C1R2 + \\
& +0.201C2R2 + 0.201L1R2 + 0.101L2R2)s + (0.1C1L1 + 0.1C1L2 + \\
& +0.1L1L2 + 0.01C1C2R2 + 1.01C1L1R2 + 1.01C2L1R2 + \\
& +0.01C1L2R2 + 1.01C2L2R2 + 0.01L1L2R2)s^2 + (C1L1L2 + \\
& +0.1C1C2L1R2 + 0.1C1C2L2R2 + 0.1C1L1L2R2 + 0.1C2L1L2R2)s^3 + \\
& +C1C2L1L2R2s^4
\end{aligned} \tag{2}$$

Comparing the coefficients of denominators of transfer functions (1) and (2), we get a system of five non-linear equations with the unknowns $R2, L1, L2, C1, C2$

$$\begin{aligned}
Ts = {} & R2\,/ \\
& 0.201 + 1.0301R2 + \\
& +(0.01C1 + 1.01L1 + 1.01L2 + 0.101C1R2 + 0.201C2R2 + 0.201L1R2 + 0.101L2R2)s + \\
& +(0.1C1L1 + 0.1C1L2 + 0.1L1L2 + 0.01C1C2R2 + 1.01C1L1R2 + 1.01C2L1R2 + \\
& +0.01C1L2R2 + 1.01C2L2R2 + 0.01L1L2R2)s^2 + \\
& +(C1L1L2 + 0.1C1C2L1R2 + 0.1C1C2L2R2 + 0.1C1L1L2R2 + 0.1C2L1L2R2)s^3 + \\
& +C1C2L1L2R2s^4
\end{aligned} \tag{3}$$

The system (3) can be solved analytically. There are twenty four different solutions, two of which are real and can be actually used to build a filter.

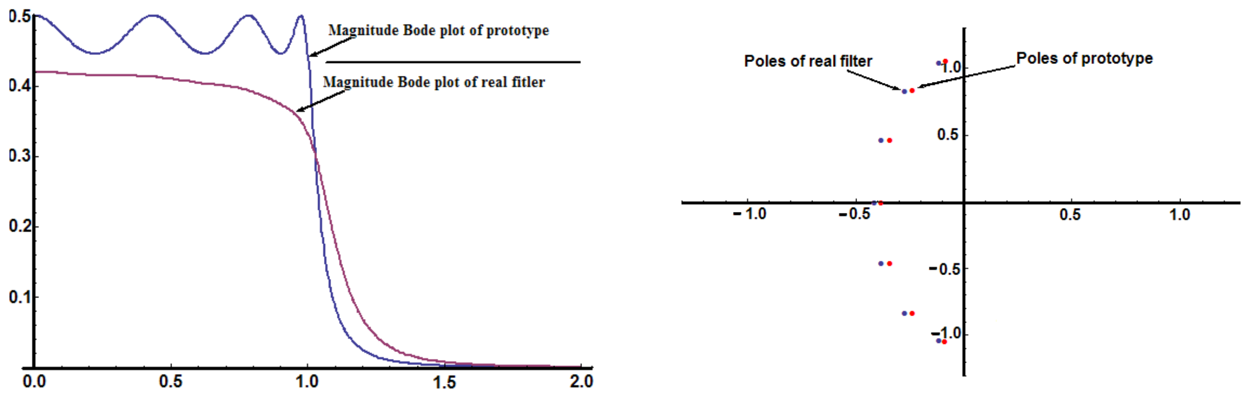

Figure 4. Magnitude Bode plots of the seventh-order lowpass Chebychev ladder filter and the pole location of its transfer function.

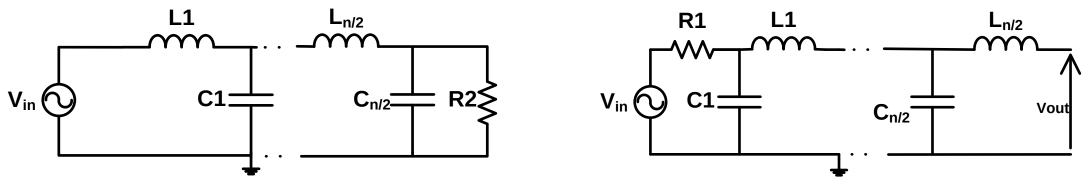

Figure 5. Singly terminated filters with resistive load a) and without resistive load b).

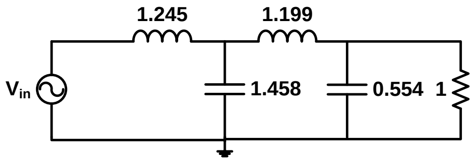

Figure 6. A fourth-order lowpass Chebychev ladder filter with resistive load.

$$\begin{cases} C1 = 4.759993153,\ C2 = 1.180227393,\ R2 = 0.7756528492, \\ \quad L1 = 0.1856518514,\ L2 = 1.491986786 \\ C1 = 1.924134578,\ C2 = 0.9020147578,\ R2 = 0.7756528492, \\ \quad L1 = 0.8885120233,\ L2 = 1.009076015 \end{cases} \tag{4}$$

After substituting the coefficients $R2, L1, L2, C1, C2$ in (2) with either of the solutions (4) we get the same transfer function as the transfer function of the prototype filter (1), except for the coefficient in the numerator.

$$Tp = \frac{0.775653}{1 + 2.44475\ s + 3.17051\ s^2 + 2.17713\ s^3 + 1.20699\ s^4} \tag{5}$$

Fig. 8 shows the schematic of the ladder filter built from real, lossy reactance elements, the values of which are determined by one of the solutions (4).

Since the numerator of the transfer function (5) has the value of less than 1, the filter designed with the lossy reactance elements has a bigger attenuation than the prototype filter – but this fact is of no great importance. The magnitude plot of the obtained filter has exactly the same shape as it was intended before the synthesis process. The magnitude Bode plot of this filter is shown in Fig. 9 with the line ③.

The ladder filter, built according to the circuit shown in Fig. 6, but with real reactance elements, will have a different frequency response than the desired one. The magnitude Bode plot of this filter is shown in Fig. 9, with the line ②. Apart from the bigger attenuation, which is of lesser importance, the shape of the plot is different and what is most important, the cut-off frequency is shifted.

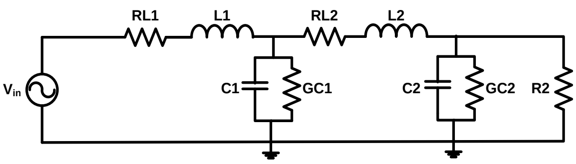

Figure 7. A fourth-order lowpass Chebychev ladder filter with resistive load, built with real reactance elements.

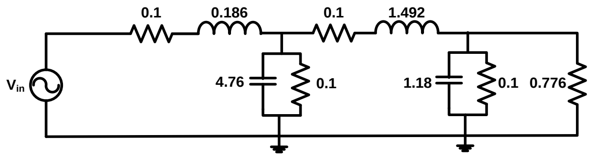

Figure 8. The filter from Fig. 9, built with real reactance elements according to (4).

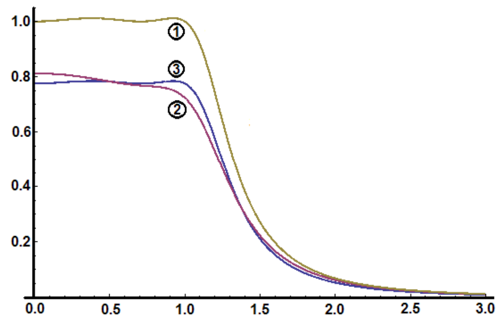

Figure 9. The magnitude Bode plots of the prototype filter ①, the real filter ② and the filter shown in Fig. 8 ③.

B. Ladder filters without resistive load.

The filter with the transfer function (1) can also be realized as a circuit without resistive load. The schematic of such filter is shown in Fig. 10.

After replacing the reactance elements with their real models according to Fig. 3, we get a circuit shown in Fig. 11.

The transfer function of this filter, made of lossy elements, with the value of lossiness $RL1 = RL2 = GC1 = GC2 = 0.1$ is as shown below

$$\begin{aligned} Ts = 1/ \\ & 1.0301 + 0.201R1 + (0.101C1 + 0.201C2 + 0.201L1 + 0.101L2 + \\ & +1.01C1R1 + 1.01C2R1 + 0.01L2R1)s + (0.01C1C2 + 1.01C1L1 + \\ & +1.01C2L1 + 0.01C1L2 + 1.01C2L2 + 0.01L1L2 + 0.1C1L2R1 + \\ & 0.1C2L2R1)s^2 + (0.1C1C2L1 + 0.1C1C2L2 + 0.1C1L1L2 + \\ & +0.1C2L1L2 + C1C2L2R1)s^3 + C1C2L1L2s^4 \end{aligned} \tag{6}$$

Comparing the coefficients of denominators of the transfer functions (6) and (1), we get a system of five non-

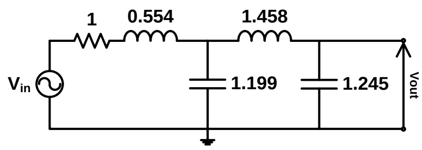

Figure 10. A fourth-order lowpass Chebychev ladder filter without resistive load.

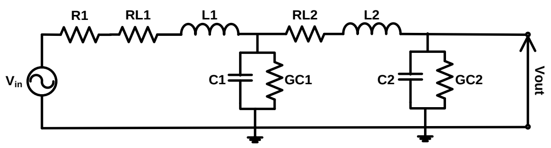

Figure 11. The filter without resistive load, built with real reactance elements.

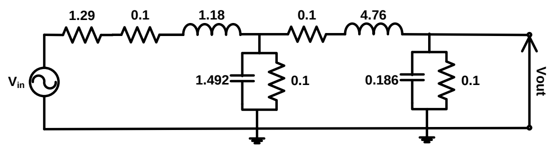

Figure 12. The filter from Fig. 11, built with real reactance elements according to (9).

linear equations as previously, with the unknowns $R1, L1, L2, C1, C2$

$$\begin{cases} 1.0301 + 0.201R1 = 1 \\ 0.101C1 + 0.201C2 + 0.201L1 + 0.101L2 + 1.01C1R1 + \\ +1.01C2R1 + 0.01L2R1 = 2.44475 \\ 0.01C1C2 + 1.01C1L1 + 1.01C2L1 + 0.01C1L2 + \\ +1.01C2L2 + 0.01L1L2 + 0.1C1C2R1 + \\ +0.1C1L2R1 + 0.1C2L2R1 = 3.17051 \\ 0.1C1C2L1 + 0.1C1C2L2 + 0.1C1L1L2 + 0.1C2L1L2 + \\ +C1C2L2R1 = 2.17713 \\ C1C2L1L2 = 1.20699 \end{cases} \quad (7)$$

Again there are two real solutions

$$\begin{cases} \mathrm{C1} = 9.855369094,\ \mathrm{C2} = 58.79309583,\ \mathrm{R1} = -0.1497512438, \\ \qquad \mathrm{L1} = 0.05587480537,\ \mathrm{L2} = 0.03728107491 \\ \mathrm{C1} = 0.2489640550,\ \mathrm{C2} = 0.07104167957,\ \mathrm{R1} = -0.1497512438, \\ \qquad \mathrm{L1} = 4.239975232,\ \mathrm{L2} = 16.09498154 \end{cases} \quad (8)$$

This time, the obtained solutions are worthless because of the sign of $R1$. It was not necessary to solve the system (8), because by looking at the first equation, we can easily notice that in any existing solution of the system (8) will appear $\mathrm{R1} = -0.1497512438$. The numerator of the transfer function (1) has the value of 1, which means that by comparing the coefficients of denominators we are looking for a filter with the same shape of the magnitude Bode plot and with the same attenuation. Since the attenuation of the filter is not of the greatest importance, we can allow the numerator to have a different value. Thus dividing the numerator and the denominator of the transfer function (6) by $R1$, and then comparing coefficients of the denominator to (1), we get a system of equations which has two real positive solutions.

$$\begin{cases} \mathrm{C1} = 1.491987044,\ \mathrm{C2} = 0.1856499926,\ \mathrm{R1} = 1.289236546, \\ \qquad \mathrm{L1} = 1.180230733,\ \mathrm{L2} = 4.760010868 \\ \mathrm{C1} = 1.009071373,\ \mathrm{C2} = 0.8885171440,\ \mathrm{R1} = 1.289236546, \\ \qquad \mathrm{L1} = 0.9020135856,\ \mathrm{L2} = 1.924128515 \end{cases} \quad (9)$$

The ladder filter built with the lossy elements according to one of the solutions (9) and the circuit Fig. 11 is shown in Fig. 12.

The magnitude Bode plots of the prototype filter ①, the real filter ② and the filter made of lossy elements ③ according to Fig. 12, are shown in Fig. 13.

As we can note by comparing Fig. 9 and Fig. 13, the magnitude Bode plots obtained in this way for the filter without resistive load are exactly the same as for the filter with resistive load.

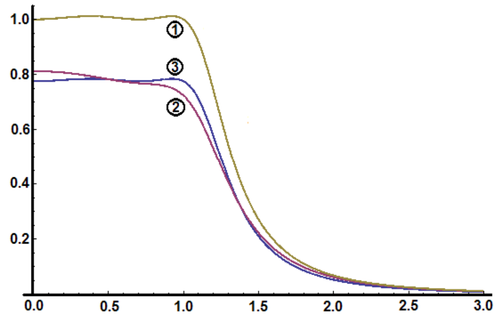

Figure 13. The magnitude Bode plots of the prototype filter ①, the real filter ② and the filter shown in Fig. 12 ③.

III. Lossiness limitations for singly terminated ladder filters.

In both considered cases we set the same (0.1) value of lossiness for each reactance element. Both designed filters have exactly the same magnitude Bode plots with attenuation value of 0.775653 at the frequency $\omega = 0$. The questions appear now, whether we can introduce any value for lossiness, no matter how high, and if not, what the upper limit is for it, and how the introduced lossiness affects the attenuation of the filter.

In both filter configurations considered in 1a) and 1b) the upper limit of lossiness is 0.189. For a greater value of lossiness the system of equations does not have any real positive solutions. The relationship between the attenuation of the filter and the lossiness of reactance elements for both above-mentioned filters has been shown in Fig. 14 with line ①.

The synthesis method presented in 1a), even though it let us easily design a singly terminated ladder filter with resistive load, had to be modified slightly for the filter without resistive load. Without this modification, the above-mentioned system of equations did not have real positive solutions. The mentioned modification was simple: it was enough to reduce the transfer function by the resistance R1 and only after that compare the denominators' coefficients. It appears that if we choose a different expression to reduce the transfer function by, we will be able to introduce even greater values of lossiness, and the system of equations will still have real positive solutions. If we reduce transfer function (6) so that the constant has the value of 1, the lossiness limit will increase from 0.189 to 0.254 for both considered filters. Moreover, the filter's attenuation will be lower for the particular values of lossiness. The relationship between the attenuation and the lossiness of reactance elements has been shown in Fig. 14 with the line ②.

IV. Doubly terminated ladder filters.

The synthesis method for doubly terminated ladder filters will be discussed, like for singly terminated filters,

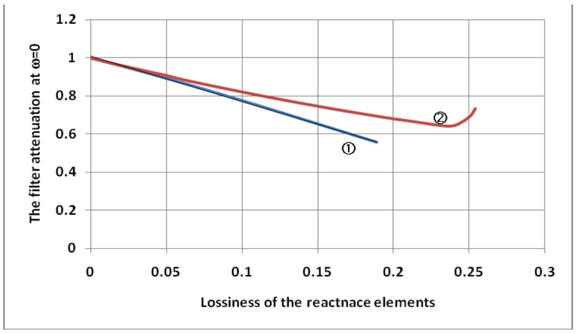

Figure 14. Relationship between attenuation and lossiness of reactance elements without reducing the transfer function ① and after reducing the transfer function ②.

using the example of a fourth-order Chebychev lowpass prototype. The transfer function of such filter is as follows:

$$Tp = \frac{0.849126}{2 + 4.9s + 6.34s^2 + 4.35s^3 + 2.41s^4} \tag{10}$$

and its schematic has been show in Fig. 15.

Just as it was done in the previous examples, we replace the reactance elements with their real, lossy models according to Fig. 3. Thus, we get the circuit shown in Fig. 16.

We set the value of the lossiness $RL1 = RL2 = GC1 = GC2 = 0.1$, as before. Having determined the transfer function of this filter we compare the coefficients of its denominator to the denominator of the transfer function (10) in a similar way as it was done before. In this way we get the following system of equations.

$$\begin{cases} 0.201 + 1.01\,R1 + 1.0301\,R2 + 0.201\,R1\,R2 = 2 \\ 0.01\,C1 + 1.01\,L1 + 1.01\,L2 + 0.1\,C1\,R1 + 0.1\,L2\,R1 + \\ +0.101\,C1\,R2 + 0.201\,C2\,R2 + 0.201\,L1\,R2 + 0.101\,L2\,R2 + \\ +1.01\,C1\,R1\,R2 + 1.01\,C2\,R1\,R2 + 0.01\,L2\,R1\,R2 = 5 \\ 0.1\,C1\,L1 + 0.1\,C1\,L2 + 0.1\,L1\,L2 + C1\,L2\,R1 + 0.01\,C1\,C2\,R2 + \\ + 1.01\,C1\,L1\,R2 + 1.01\,C2\,L1\,R2 + 0.01\,C1\,L2\,R2 + 1.01\,C2\,L2\,R2 + \\ + 0.01\,L1\,L2\,R2 + 0.1\,C1\,C2\,R1\,R2 + 0.1\,C1\,L2\,R1\,R2 + \\ +0.1\,C2\,L2\,R1\,R2 = 6.116 \\ C1\,L1\,L2 + 0.1\,C1\,C2\,L1\,R2 + 0.1\,C1\,C2\,L2\,R2 + \\ +0.1\,C1\,L1\,L2\,R2 + 0.1\,C2\,L1\,L2\,R2 + C1\,C2\,L2\,R1\,R2 = 4.456 \\ C1\,C2\,L1\,L2\,R2 = 2.097 \end{cases} \tag{11}$$

This time the obtained system of equations has six unknowns and only five equations, therefore we have to establish one of the unknowns. The numerator of the transfer function of the filter shown in Fig. 16 has only the value of R2, thus it is this unknown that we establish, thereby having the influence on the filter's attenuation. Setting $R2 = 0.3$ we get, apart from complex solutions,

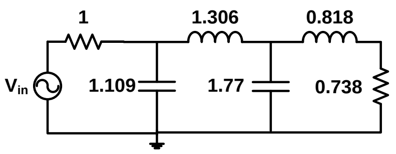

Figure 15. A doubly terminated fourth-order lowpass Chebychev ladder filter.

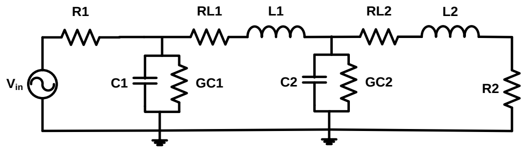

Figure 16. A doubly terminated fourth-order lowpass Chebychev ladder filter built with real reactance elements.

two solutions which are real and positive.

$$\begin{cases} C1 = 0.7953046216,\ C2 = 1.566204861,\ R1 = 1.120192968, \\ \qquad L1 = 1.569217023,\ L2 = 0.9579526492 \\ C1 = 0.8331791988,\ C2 = 1.623699805,\ R1 = 1.120192968, \\ \qquad L1 = 1.655198890,\ L2 = 0.8362088968 \end{cases} \tag{12}$$

Magnitude Bode plots of the doubly terminated prototype filter ①, the real filter ② and the filter made of lossy elements ③ are shown in Fig. 17.

The maximum value of lossiness which can be introduced to the reactance elements depends on the previously set value of $R2$. The relationship between these two quantities is shown in Fig. 18.

Every ladder filter can be realized with an inductor as the first element in the ladder. Fig. 19 a) shows the schematic of a fourth-order Chebychev lowpass prototype, in which the first element of the ladder is an inductor. Schematics of this filter after replacing the reactance elements with their lossy models is shown in Fig. 19 b).

The transfer function of the prototype filter is as follows

$$Tp = \frac{1.15087}{2 + 4.9s + 6.34s^2 + 4.35s^3 + 2.41s^4} \tag{12}$$

Similarly as in the previous examples, we replace the

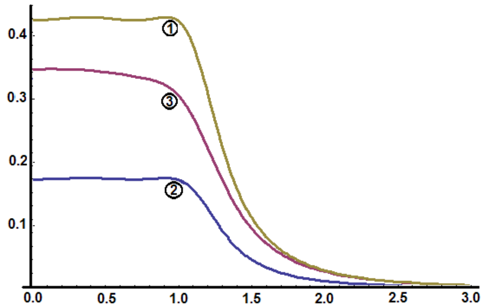

Figure 17. The magnitude Bode plots of the prototype filter ①, the real filter ③ and the filter shown in Fig. 16 ②.

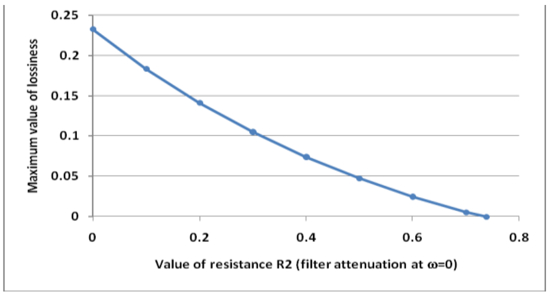

Figure 18. The relationship between the maximum value of lossiness and the value of resistance R2.

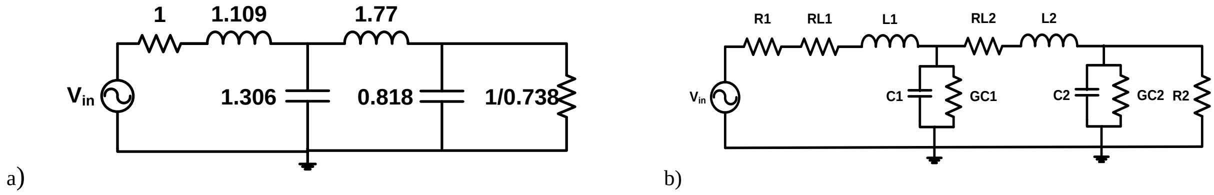

Figure 19. A doubly terminated fourth-order lowpass Chebychev ladder filter, before a) and after b) replacing reactance elements with their real models.

reactance elements with the real models, we calculate the transfer function and we create a system of equations by comparing the denominators' coefficients. This system, like the previous one, has five equations and six unknowns. However in this case, the real and positive solutions will occur after setting R2 values from the range $1.6 - 2.09$. A value of R2 set from this range causes the filter to have a lower attenuation than the prototype filter. This is an obvious advantage of the filter shown in Fig. 19 over the filter shown in Fig. 15 if we design these filters with the presented method.

V. CONCLUSIONS

The synthesis algorithm introduced in the article makes it possible to design a passive ladder, Butterworth or Chebychev filter with the use of lossy reactance elements. The filter obtained in this way can have exactly the same frequency response as the non-feasible prototype filter. The described method, although conceptually very simple, is not suitable for practical applications without some modification. Analytical solving of the obtained systems of non-linear equations is complicated as well as pointless. When designing a filter we do not need to know all existing solutions; it is enough to know only one of them. Moreover, for filters of higher order, it is not possible to solve such a system analytically. Thus we have to make use of numerical methods, for example one of the varieties of the Newton method. Here, some additional problems occur, e.g. singularity of the Jacobian, choice of the initial point or lack of convergence. These issues are the subject of further research. Apart from that, additional research concerns adapting this promising method for filters the transfer function of which has zeros on the complex plane.

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