Two Random Variables

- Previously, we only dealt with one random variable.
- For the next few lectures, we will study two random variables.
- How they are related to each other.
- How we describe this relationship.

This is an intermediate step toward random signals.
Example

error (random variable) measured

measured voltage (random)

random voltage

random

\[ V \]

\[ + \]

\[ - \]
Joint Probability Distribution

Consider two RV's $X$ and $Y$.

Joint Probability Distribution:

Properties:

1. $0 = \mathcal{H}(\infty, \infty)$
2. $1 = \mathcal{H}(\infty, -\infty)$
3. $I = \mathcal{H}(\lambda, \infty)$

For $\lambda > 0$, $x > 0$, $0 < p(x \leq x, y \leq y) = \mathcal{H}(\lambda, X)$

where $X$ and $Y$ are RVs.
4) $F(x, y) = F(x \geq y, x \leq y) = P(x \leq y) = \lim_{y \to \infty} F(y)$

$F(x) = \begin{cases} \lambda x & x \leq \lambda \\ \infty & x > \lambda \end{cases}$

$F(y) = \begin{cases} \lambda y & y \leq \lambda \\ \infty & y > \lambda \end{cases}$

Marginal distributions

\[ F_X(x) = \begin{cases} \lambda x & x \leq \lambda \\ \infty & x > \lambda \end{cases} \quad F_Y(y) = \begin{cases} \lambda y & y \leq \lambda \\ \infty & y > \lambda \end{cases} \]

Increase is a nondecreasing function as either $x$, or $y$, or both $x$, or $y$.
Joint Density Function

\[ F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) \, du \, dv \]

\[ \lambda > \infty, \quad \infty > x > \infty, \quad 0 < (\lambda, x) \]

\[ I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda f(x, y) \, dx \, dy \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda f(x, y) \, dx \, dy = 1 \]
Joint Density Function (cont'd)

\[ p(x, y) \] is a function that describes the joint probability density of two random variables, \( X \) and \( Y \).

The marginal densities of \( X \) and \( Y \) are given by:

\[ f_X(x) = \int_{-\infty}^{\infty} p(x, y) \, dy \]
\[ f_Y(y) = \int_{-\infty}^{\infty} p(x, y) \, dx \]

The joint density function can be expressed in terms of conditional densities:

\[ p(x, y) = f_X(x) \cdot \frac{f_Y(y)}{f_X(x)} \]

The total probability is:

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \, dx \, dy = 1 \]
\[
\frac{(x - x') (x' - x)}{1} = \text{height} \\
\]

\[
\begin{cases}
0 & \text{elsewhere} \\
\frac{(x - x') (x' - x)}{1} = (x', x) f & \text{for } x \geq x' \geq x' \geq x \geq x'
\end{cases}
\]
Expected Values

Given two RVs $X$ and $Y$, the expected value of $g(X, Y)$ is called the correlation.

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \, dx \, dy$$
\[(\zeta A + \frac{1}{4} A)(\zeta X + \frac{1}{4} x)\frac{4}{I} = \begin{bmatrix} \frac{1}{4} \zeta \\
\frac{1}{4} A \end{bmatrix} \begin{bmatrix} \frac{1}{4} x \\
\frac{1}{4} A \end{bmatrix} \frac{(\frac{1}{4} x - \zeta x)(\frac{1}{4} A - \zeta A)}{I} = \]

\[xpA\rho\left(\frac{(\frac{1}{4} x - \zeta x)(\frac{1}{4} A - \zeta A)}{I}\right)\frac{1}{4} x \int \int = \{AX\} E\]

Example

\[\begin{cases} 0 & \text{for } \frac{1}{4} x - \zeta x \frac{1}{4} A - \zeta A \\
\end{cases}\]

\[(\frac{1}{4} x - \zeta x)(\frac{1}{4} A - \zeta A)\]
Example (cont'd)
Example

(a) Probability that both dimensions are larger than their mean values by 0.005 cm.

\[ P(\{X > 2.005, Y > 1.005\} | 0.020^2) = \int_{1.005}^{2.005} \int_{1.010}^{2.010} f_{XX} f_{YY} \, dX \, dY \]

Rectangular semiconductor substrate with dimensions having mean values of 1 cm and 2 cm.

Actual dimensions are independent and uniformly distributed between ±0.01 cm of their means.
Example (cont'd)

(b) Probability that the larger dimension is greater than its mean value by 0.05 cm and the smaller dimension is less than its mean value by 0.05 cm.

\[
P(\lambda > 2.005, \mu > 0.995, \sigma = 0.020) = \int_{0.995}^{2.005} \int_{0}^{0.020} \frac{0.020}{0.005} \exp\left(-\frac{y}{0.020}\right) dx\,dy = \frac{16}{1}
\]
Example (cont'd)

(c) The mean value of the area of the substrate

\[
\bar{z} = \frac{\int_{10.1}^{66.0} \int_{10.2}^{66.1} x \, dy \, dx}{\int_{10.1}^{66.0} \int_{10.2}^{66.1} 1} = \frac{\int_{10.1}^{66.0} \int_{10.2}^{66.1} (z^2 - 0.0) \, dy \, dx}{\int_{10.1}^{66.0} \int_{10.2}^{66.1} 1}
\]

\[\{XX\} E = \{V\} E = V\]
Joint Density Function (cont’d)

\[ \lambda p \left[ \begin{array}{c} 0 \\ \infty \\ x \zeta - \partial \\
\infty \\ \infty \\ 1 \\
\end{array} \right] \lambda \zeta \partial \int_{\infty}^{0} = \lambda p \int_{\infty}^{0} \lambda \zeta \partial \int_{\infty}^{0} = \lambda p \int_{\infty}^{0} \lambda \zeta \partial \int_{\infty}^{0} = 1 \\
\lambda p x p (\lambda \zeta + x \zeta) \partial A \int_{\infty}^{0} \int_{\infty}^{0} = 1 \\
\]

\[ V \] Value of

\[ \begin{align*}
0 > \lambda', 0 > x & \quad 0 \\
0 \geq \lambda', 0 \geq x & \quad (\lambda \zeta + x \zeta) \partial A
\end{align*} \]

= (\lambda', x) f \\
\[ \lambda \text{ and } X \text{ s and } Y \text{ RVs Two RVs} \]

\[ 9 = V \]

\[ \frac{\varepsilon \cdot z}{V} = \lambda p \int_{\infty}^{0} \frac{z}{V} = 1 \\
\]

Ex: Two RV’s
Joint Density Function (cont'd)

\[ \frac{9}{I} = \frac{6}{I} \times \frac{4}{I} \times 9 = \]

\[ \mathcal{L}_p(\alpha_2 + \xi) = \int_{\infty}^{0} \int_{\infty}^{0} x \cdot p(\alpha_2 - \alpha) \cdot e^{\alpha x} \cdot \int_{0}^{9} = \]

\[ \mathcal{L}_p(x+y) \int_{\{y \in \mathbb{Y} + \{x\} \in \mathbb{E} \}} = \]

\( \left( \frac{h}{I} > 8 \right) \wedge \left( \frac{z}{I} > X \right) \Rightarrow \)

\[ \mathcal{L}_p(x+y) \int_{\{y \in \mathbb{Y} + \{x\} \in \mathbb{E} \}} = \]

\[ \int_{0}^{9} \int_{0}^{9} x \cdot p(\alpha_2 - \alpha) \cdot e^{\alpha x} \cdot \int_{0}^{9} = \]

\[ \left( \frac{4}{I} > \lambda \wedge \frac{z}{I} > X \right) \]
Previously, we worked with conditional probability in terms of events. Now, we define conditional probability in terms of random variables.

\[
\frac{(B) d}{(B \cup V) d} = (B | V) d
\]

Example: What is the probability density of the actual voltage given that the measured voltage is 1.2V?

Given that the measured voltage is 1.2V, the actual voltage is within the range of the measured voltage. This is represented in the diagram by the shaded area.
Consider two RV's $X$ and $Y$ with joint distribution $F(x, y)$. First, consider the probability that is given, i.e., $\lambda = \lambda'$. Now, consider two RV's and consider the joint distribution $(\lambda', x)$.

\[
\frac{(\lambda') \lambda - (\lambda') \lambda}{(\lambda', x) \lambda - (\lambda', x) \lambda} = \frac{\lambda' \lambda > \lambda | x \leq X}{\lambda > \lambda' x \leq X} = \left[ \lambda' \lambda > \lambda | x \leq X \right] d 
\]
By using something similar to L'Hôpital's Rule, we get

\[
\frac{(x) X_f}{(\lambda', x) f} = \frac{(x | \lambda)f}{(\lambda | x) f} = \frac{(\lambda) f}{(\lambda', x) f} = (\lambda | x) f
\]
Bayes Theorem:

\[ p(x) \frac{f(x | \lambda)}{\int f(x | \lambda) f(x) \, dx} = p(\lambda | x) \frac{\int f(\lambda | x) f(x) \, dx}{\int f(\lambda | x) f(x) \, dx} \]

Total Probability:

\[ p(A | \lambda) = \int p(A | \lambda, B) p(B | \lambda) \, dB \]

\[ p(A \cap B | \lambda) = p(A | \lambda) p(B | A, \lambda) \]

Bayes Theorem (cont'd):
Conditional Probability Example

\[
\begin{align*}
&\left[\frac{\gamma + \frac{3}{2}}{2}\right] \text{ if } \left( \lambda x + y, y \right) \\
&\left[0, \frac{2}{x}\right] \text{ otherwise}
\end{align*}
\]

\[I = \gamma\]

\[= \lambda \exp((\lambda + x) \gamma) \int_{0}^{\infty} \int_{0}^{\infty} f(y) dy dx\]

Value of \(I\)

\[
I \geq \lambda, x \geq 0 \\
(\lambda + x) \gamma = (\lambda', x') f
\]

Ex.
Conditional Probability Example

(a) 
\[
\begin{align*}
\mathbb{P}(X \leq 0, Y \leq 0) &= \int_{-\infty}^{0} \int_{-\infty}^{0} f(x, y) \, dx \, dy \\
&= \int_{-\infty}^{0} \left[ \frac{1}{(4/1)X} \frac{\lambda \alpha + \frac{\lambda}{1}}{\alpha + x} \right] dx \\
&= \lambda \alpha \left[ \frac{1}{\alpha \alpha} \frac{1}{\alpha + x} \right]_{-\infty}^{0} \\
&= \lambda \alpha \left[ 1 \right] \\
&= \lambda \alpha
\end{align*}
\]

(b) 
\[
\mathbb{P}(X > 1/2, Y < 1/4) = \mathbb{P}(X > 1/2) \mathbb{P}(Y < 1/4) = \mathbb{P}(X > 1/2)
\]

\[
\begin{align*}
\mathbb{P}(X > 1/2) &= \int_{1/2}^{\infty} f(x) \, dx \\
&= \lambda \alpha \int_{1/2}^{\infty} \left( \frac{\lambda \alpha + \frac{\lambda}{1}}{\alpha + x} \right) dx \\
&= \lambda \alpha \left[ \frac{1}{\alpha \alpha} \frac{1}{\alpha + x} \right]_{1/2}^{\infty} \\
&= \lambda \alpha \left[ 0 - \frac{1}{\alpha \alpha} \frac{1}{\alpha + 1/2} \right] \\
&= \lambda \alpha \frac{1}{\alpha \alpha} \frac{1}{\alpha + 1/2} \\
&= \lambda \alpha \frac{1}{\alpha \alpha} \frac{2}{\alpha + 1/2}
\end{align*}
\]

\[
\mathbb{P}(Y < 1/4) = \int_{-\infty}^{1/4} f(y) \, dy = \frac{1}{(4/1)X} \int_{-\infty}^{1/4} \frac{\lambda \alpha + \frac{\lambda}{1}}{\alpha + y} \, dy
\]

\[
\begin{align*}
\mathbb{P}(Y < 1/4) &= \frac{1}{(4/1)X} \left[ \frac{1}{\alpha \alpha} \frac{1}{\alpha + y} \right]_{-\infty}^{1/4} \\
&= \frac{1}{(4/1)X} \left[ 1 - \frac{1}{\alpha \alpha} \frac{1}{\alpha + 1/4} \right] \\
&= \frac{1}{(4/1)X} \left[ \frac{\alpha \alpha}{\alpha \alpha} - \frac{1}{\alpha \alpha} \frac{1}{\alpha + 1/4} \right] \\
&= \frac{1}{(4/1)X} \left[ \frac{\alpha \alpha}{\alpha \alpha} - \frac{1}{\alpha \alpha} \frac{1}{\alpha + 1/4} \right]
\end{align*}
\]

\[
\begin{align*}
\mathbb{P}(X > 1/2, Y < 1/4) &= \mathbb{P}(X > 1/2) \mathbb{P}(Y < 1/4)
\end{align*}
\]
Statistical Independence

Two RV’s $X$ and $Y$ are statistically independent if and only if

$$f(x, y) = f_X(x) f_Y(y)$$

If and only if the joint density is separable.

$$\forall x \forall y \left( \int f(x) f_Y(y) = \int f_X(x) f(y) \right)$$
Statistical Independence

- If $X$ and $Y$ are independent, then

Knowledge of $y$ tells us nothing about $x$

$$f_X(x) = f(x)\cdot f_Y(y)$$

Similarly

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{f_X(x)\cdot f_Y(y)}{f_Y(y)} = f_X(x)$$
The random variables $X$ and $Y$ are uncorrelated if
\[ \langle E\{XY\} \rangle = E\{E\{X\} \} \cdot E\{E\{Y\} \} \]
Example

Answer: NO! Density function is not separable

Are they independent?

\[
\begin{align*}
&\begin{cases}
0, & \text{elsewhere} \\
\frac{e^{-x \alpha}}{\alpha + 1}, & \alpha, x \geq 0
\end{cases} \\
&= (\alpha', x) f
\end{align*}
\]

Following joint density function

Two random variables, \(X\) and \(Y\), have the
Example

Two random variables, X and Y, have the following joint density function:

\[
\begin{aligned}
&I \geq \lambda \geq 0 \quad \lambda \geq (\lambda)^{\lambda f} \\
&I \geq x \geq 0 \quad x \geq (x)^{xf}
\end{aligned}
\]

Are they independent? **YES!** Density function is separable.

Answer: YES! Density function is separable.
Statistical Independence

If \( X \) and \( Y \) are independent, then

\[
\begin{align*}
\mathbb{E}\{XY\} & = \mathbb{E}\{X\}\mathbb{E}\{Y\} \\
\mathbb{E}\{X\} \mathbb{E}\{Y\} & = \mathbb{E}\{X\}\mathbb{E}\{Y\} \\
\mathbb{E}\{X\} \mathbb{E}\{Y\} & = \mathbb{E}\{X\}\mathbb{E}\{Y\} \\
\mathbb{E}\{X\} \mathbb{E}\{Y\} & = \mathbb{E}\{X\}\mathbb{E}\{Y\}
\end{align*}
\]

\( X \) and \( Y \) are uncorrelated.
Independent vs. Uncorrelated

- Both uncorrelated and independent mean that the RV's are not related.
- They differ in how this uncorrelatedness is defined.

**Independent** - density function

\[ f_X(x) f_Y(y) = f_{X,Y}(x,y) \]

**Uncorrelated** - first and second order statistics

\[ \{X\} \mathbb{E} \{Y\} \mathbb{E} = \{XY\} \mathbb{E} \]
Example

\[
\frac{\varepsilon}{b} = 1
\]

\[
\left(\frac{\varepsilon}{\xi}\right)\left(\frac{\varepsilon}{\xi}\right) = 1
\]

\[
\left[\int_0^1 (\lambda \varepsilon + \frac{\varepsilon}{\xi})\right]\left[\int_0^1 (x + \frac{\varepsilon}{\xi})\right] = 1
\]

\[
\lambda \rho \rho (\lambda + \varepsilon)(1 + x) \int_1^0 \int_1^0 = 1
\]

(b) Find \( \lambda \)

\[
(\varepsilon + \lambda)(1 + x) = 1
\]

\[
(\varepsilon + \lambda + x \varepsilon + \lambda x) = (\lambda \varepsilon, x) f
\]

\[
\text{Answer: YES!}
\]

(a) Are \( X \) and \( Y \) independent?

\[
I \geq \lambda, x \geq 0 \quad (\varepsilon + \lambda + x \varepsilon + \lambda x) \rho = (\lambda \varepsilon, x) f
\]
\[ \frac{\mathcal{L}}{\ell} = \frac{s_1}{8} \cdot \frac{6}{\xi} = \{\Lambda\} \mathbb{E}\{X\} \mathbb{E} = \{\Lambda X\} \mathbb{E} \]

Since \( X \) and \( Y \) are independent,

\[ \frac{s_1}{8} = \quad (1 + \frac{\xi}{\ell}) \frac{s}{\xi} = \quad (\frac{\xi}{\ell} + \frac{\xi}{\ell}) \frac{s}{\xi} = \]

\[ 0 \right| [\varepsilon x + \frac{\xi}{\ell x}] \frac{s}{\xi} = \quad 0 \right| [\varepsilon x + \frac{\xi}{\ell x}] \frac{s}{\xi} = \]

\[ x \mathbb{P}(\mathcal{L} + \Lambda) \frac{s}{\xi} \Lambda \int_0^1 = \{\Lambda\} \mathbb{E} \quad x \mathbb{P}(1 + x) \frac{s}{\xi} x \int_0^1 = \{X\} \mathbb{E} \quad \{\Lambda X\} \mathbb{E} \]

Find \( \mathbb{P} \)

\[ 1 \geq \Lambda, x \geq 0 \quad (\mathcal{L} + \Lambda \mathcal{x} \Xi + \Lambda x) \gamma = (\Lambda, x) f \]

**Example**
Correlation Between RVs

- Shoe size vs. height in feet

- Data points indicate a positive correlation
Correlation Between RV’s

vision 20/x

shoe size
Correlation Between RV’s

car weight in lbs

miles per gallon
Correlation Between RV’s

We need to quantify the relationship between RV’s

Correlation $E\{XY\}$ is one way

» $X$ and $Y$ may have different mean values and scales

» Difficult to tell much about how related they are.

If we subtract the mean, we get the covariance

$$E\{(X - \bar{X})(Y - \bar{Y})\}$$

The covariance is not affected by the mean values, but we still have a problem with RV’s with different scales (i.e. $\sigma$)
Problem with Correlation
If we subtract the means and divide by the standard deviations, we get the correlation coefficient.

\[
\rho = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{\sqrt{\sum (X - \bar{X})^2 \sum (Y - \bar{Y})^2}}
\]

- If \( \rho = 1 \), the RVs are linearly related (perfectly correlated).
- If \( -1 < \rho < 0 \), the RVs are negatively correlated.
- If \( 0 < \rho < 1 \), the RVs are positively correlated.
- If \( \rho = 0 \), the RVs are not correlated (uncorrelated).

If \( m > 0 \):
- If \( \rho = 1 \), \( Y = mX + b \).
- If \( \rho = -1 \), \( Y = -mX + b \).

Correlation Coefficient
Correlation Coefficient

\[ \rho = 0.75 \]

\[ 0 = \rho \]

\[ -0.75 = \rho \]
Example

Slight Negative correlation

$$\frac{\chi_{11} / \chi_{11}}{\chi_{12} / \chi_{12}} - \frac{3}{1} = \frac{\mu \sigma^X \sigma}{\sigma X - \{X\} \sigma} = \sigma$$

$$\frac{\varepsilon}{\sigma} = \frac{\lambda x p (x + y) \lambda x}{\chi} \int_{0}^{0} \{X\} \sigma = \lambda X$$

$$\frac{\chi_{44} / \chi_{11}}{\chi_{12}} = \frac{\lambda \sigma}{\lambda X} = \chi$$

$$\varepsilon \sigma = 0$$

$$\begin{bmatrix} 0 \\ \chi + x \end{bmatrix} = (\lambda^T x) f$$
Correlation Between RV’s (cont’d)

Ex: \( Z = aX + bY \) \( a, b \) constant. What is \( \sigma_z^2 \) ?

\[
E\{Z\} = E\{aX + bY\} = a\overline{X} + b\overline{Y}
\]

\[
E\{Z^2\} = E\{(aX + bY)^2\} = a^2E\{X^2\} + 2abE\{XY\} + b^2E\{Y^2\}
\]

\[
= a^2\left[\sigma_X^2 + \overline{X}^2\right] + 2ab\left[\sigma_X\sigma_Y\rho + \overline{XY}\right] + b^2\left[\sigma_Y^2 + \overline{Y}^2\right]
\]

\[
= a^2\sigma_X^2 + 2ab\sigma_X\sigma_Y\rho + b^2\sigma_Y^2 + a^2\overline{X}^2 + 2ab\overline{XY} + b^2\overline{Y}^2
\]

\[
= a^2\sigma_X^2 + 2ab\sigma_X\sigma_Y\rho + b^2\sigma_Y^2 + \left(a\overline{X} + b\overline{Y}\right)^2
\]

\[
= a^2\sigma_X^2 + 2ab\sigma_X\sigma_Y\rho + b^2\sigma_Y^2 + \left(E\{Z\}\right)^2
\]

\[
\sigma_z^2 = \overline{Z}^2 - \overline{Z}^2
\]

\[
= a^2\sigma_X^2 + 2ab\sigma_X\sigma_Y\rho + b^2\sigma_Y^2 = \sigma_{aX + bY}^2
\]
Formulas to Remember

![Math formulas]

\[
\begin{align*}
&\zeta \varphi + d^\zeta \varphi^x \varphi + X \varphi = \zeta \varphi + X \varphi \\
&d^\zeta \varphi^x \varphi + \zeta X = \{\zeta X\} \mathcal{E} \\
&\frac{\zeta \varphi^x \varphi}{\zeta X - \{\zeta X\} \mathcal{E}} = \left\{ \left( \frac{\zeta \varphi}{\Lambda - \Lambda} \right) \left( \frac{X \varphi}{X - X} \right) \right\} \mathcal{E} = d'
\end{align*}
\]
Density Function of the Sum of Two RV's

Suppose we have two RV's $X$ and $Y$ that are statistically independent.

Let $Z = X + Y$. What is $f_Z(z)$?

First find the probability distribution function.

$$f_Z(z) = \int f_X(x) f_Y(z-x) \, dx$$

where $f_X(x)$ and $f_Y(y)$ are the probability density functions of $X$ and $Y$. Let $Z = X + Y$.

Suppose we have two RV's $X$ and $Y$ that are statistically independent.
Differentiate both sides with respect to $z$ using Leibniz's rule:

$$\lambda p(\lambda - z) X f(\lambda) \int_{-\infty}^{\infty} = (z) Z f$$

$$sp(s', t) V \left( \int p^{(i)S} \right) + (t) S \left( ((t) S', t) V - (t) Y ((t) Y', t) V \right) = sp(s', t) V \left( \int p^{(i)S} \right)$$

Density Function of the Sum of Two RV's cont'd
The probability density function of $Z = X + Y$ where $X$ and $Y$ are independent is the convolution of the density functions of $X$ and $Y$.

$$f_Z(z) = f_X(z) * f_Y(z)$$

For $Z = X - Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z) f_Y(z) \ dx$$
\[ \gamma p(\gamma - z) \hat{A}f(\gamma) X_f \int_{-\infty}^{\infty} = (z) Z_f \]

Flip and shift:

Convolution Example
Convolution Example

\( f(x) \ast g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \)

3 cases:

1. \( I < z \)
2. \( I \geq z > 0 \)
3. \( 0 \geq z \)

\( f(x) \ast g(x) = \begin{cases} 
I & \text{if } I < z \\
0 & \text{if } I \geq z > 0 \\
\lambda & \text{if } 0 \geq z
\end{cases} \)

\( (\lambda) X_f \)

\( (x) X_f \)
Convolution Example

\[ z - \alpha(1 - \alpha) = \gamma p_{(\gamma - z)} \int_{\gamma}^{\gamma} = (z)^{Zf} \quad 1 < z \]

\[ z - \alpha - 1 = \gamma p_{(\gamma - z)} \int_{z}^{0} = (z)^{Zf} \quad 1 > z > 0 \]

\[ 0 = (z)^{Zf} \quad 0 \geq z \]
Correlation

• Deterministic correlation of two functions
• Similar to convolution, but
  » No flipping – just shifting
  » When shifting, \( f(0) \) is at \( \lambda = -z \)

\[
\gamma P(\gamma + z) X f(\gamma) \int_{-\infty}^{\infty} (z) Z f = \lambda - X = Z
\]
Correlation Example

\[ f_Y(y) = \int_{-\infty}^{\infty} f_X(x) e^{-x} \, dx \]

\[ (x)_X f = \int_{-\infty}^{\infty} Z_f(z) \, dz \]

\[ \gamma p(\gamma + z) X_f(\gamma) \int_{-\infty}^{\infty} = (z) Z_f \]
Correlation Example

\[ I \rightarrow z \quad \iff \quad I < z^- \]

\[ 0 > z \geq I^- \quad \iff \quad I \geq z^- > 0 \quad \text{3 cases} \]

\[ z \geq 0 \quad \iff \quad 0 \geq z^- \]
Correlation Example

\[ 0 = (z) Z f \quad \text{I} \rightarrow z \]

\[ (1+z) - 1 = \gamma p_{(\gamma+z)-} \int_{1}^{z} = (z) Z f \quad 0 < z < 1 \]

\[ z \cdot (1 - z) = \gamma p_{(\gamma+z)-} \int_{1}^{0} = (z) Z f \quad z \geq 0 \]
Correlation Example

\[
\gamma p(\gamma + z) X_f(\gamma) \Lambda_f \int_{-\infty}^{\infty} = (z) Z_f
\]
Two Important Consequences

1) CENTRAL LIMIT THEOREM (CLT)

The PDF of sum of \( n \) independent random variables approaches a Gaussian density as \( n \) becomes large, regardless of the pdf's of the individual RV's encountered in real life (many RV's satisfy this).

The PDF of sum of \( n \) independent random variables approaches a Gaussian density as \( n \) becomes large, regardless of the pdf's of the individual RV's encountered in real life (many RV's satisfy this).
Two Important Consequences

2) If $X$ and $Y$ are Gaussian, $Z$ is also Gaussian with:

$$Z \sim \mathcal{N}(\lambda + \mu_X, \mu_Y + \mu_Z)$$