Regularized Image Restoration

$$\min_{f}\left\{\|g-Df\|^{2}+\alpha\|Lf\|^{2}\right\}$$

Minimizes the data error but biases the solution away from rough solutions

- yields a smooth "regular" restoration
- α controls the trade off between smoothing and restoring

  $\uparrow\alpha$ smoother

  $\downarrow\alpha$ rougher
- L controls the manner of the restoration

Regularized image restoration in matrix notation

$$\sum[g(m,n)-d(m,n)*f(m,n)]^{2}+\alpha\sum[l(m,n)*f(m,n)]^{2}$$

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Regularized Restoration in the Frequency Domain

$$\sum |g(m,n) - d(m,n) * f(m,n)|^2 + \alpha \sum |l(m,n) * f(m,n)|^2$$

$$\overbrace{=}^{Parseval} \frac{1}{4\pi^2} \int\int |G(\omega_m, \omega_n) - D(\omega_m, \omega_n) F(\omega_m, \omega_n)|^2 \, d\omega_m d\omega_n$$

$$+ \frac{\alpha}{4\pi^2} \int\int |L(\omega_m, \omega_n) F(\omega_m, \omega_n)|^2 \, d\omega_m d\omega_n$$

For a given frequency, note that the expressions inside the integral can be combined into the following:

$$|G - DF|^2 + \alpha \, |LF|^2$$

We use the following facts for taking derivatives with respect to complex variables:

$$\frac{df(z)}{dz} = \frac{df(z)}{dz_r} - j\frac{df(z)}{dz_i}$$

$$\frac{dAF}{dF} = 2A, \; \frac{dAF^*}{dF} = 0, \; \frac{dAFF^*}{dF} = 2AF^*$$

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The minimizer for each frequency can be found by taking a derivative and setting it to zero.

$$\frac{d}{dF}\left\{|G-DF|^2+\alpha\,|LF|^2\right\}$$

$$=\frac{d}{dF}\left\{|G|^2-G^*DF-GD^*F^*+|D|^2\,FF^*+\alpha\,|L|^2\,FF^*\right\}$$

$$=-2G^*D+2\,|D|^2\,F^*+2\alpha\,|L|^2\,F^*$$

$$-2G^*D+2\,|D|^2\,F^*+2\alpha\,|L|^2\,F^*=0$$

$$\Longrightarrow F=\frac{D^*}{|D|^2+\alpha\,|L|^2}G\quad\text{(function of }(\omega_m,\omega_n))$$

Sample to get DFT solution:

$$F(k,\ell)=\frac{D^*(k,\ell)}{|D(k,\ell)|^2+\alpha\,|L(k,\ell)|^2}G(k,\ell)$$

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Equivalence to Wiener Filter

Consider the frequency-domain filter for regularization:

$$F(\omega_m, \omega_n) = \frac{D^*(\omega_m, \omega_n)}{\left|D(\omega_m, \omega_n)\right|^2 + \alpha \left|L(\omega_m, \omega_n)\right|^2} G(\omega_m, \omega_n)$$

Setting

$$\begin{cases} \alpha = S_{nn}(\omega_m, \omega_n) \\ \left|L(\omega_m, \omega_n)\right|^2 = \frac{1}{S_{ff}(\omega_m, \omega_n)} \end{cases}$$

gives a result equivalent to a Wiener Filter.

Iterative Deblurring

In some cases, FFT-based restoration is not appropriate. For example, if a blur cannot be described with convolution then FFTs cannot be used, and iterative deblurring is necessary. Consider the regularized restoration criterion:

$$\phi(f) = \|g - Df\|^2 + \alpha \|Lf\|^2$$

- Expanding regularized image restoration yields

$$\phi(f) = (g - Df)^T (g - Df) + \alpha (Lf)^T (Lf)$$

- $\hat{f}_{k+1} = \hat{f}_k + \beta_k r_k$
- Choose r_k to reduce $\phi(\hat{f}_{(k+1)})$ the fastest

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Iterative Deblurring

Bowl interpretation: $\nabla\phi(\hat{f})$ is the steepest direction of surface f

$$\frac{d\phi}{df} = -2D^T(g - Df) + 2\alpha L^T L f = \nabla\phi(f)$$

- $\hat{f}_{k+1} = \hat{f}_k + \beta_k r_k$
- Choose r_k to reduce $\phi(\hat{f})$ the fastest
- $r_k = -\frac{1}{2}\nabla\phi(\hat{f}_k)$
- Let $r_k = -\frac{1}{2}\nabla\phi(\hat{f}) = D^T(g - D\hat{f}_k) - \alpha L^T L\hat{f}_k$
- Let $0 < \beta_k < \left\|D^T D + \alpha L^T L\right\|^{-1}$ to guarantee convergence
- Optimal $\beta_k = \frac{r_k^T r_k}{\|Dr_k\|^2 + \alpha\|Lr_k\|^2}$

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Iterative Deblurring

- Disadvantages
  - computationally demanding
- Advantages
  - parallelizable
  - handles shift variant blurs easily (FFTs alone cannot)
  - handles shift variant regularization (smoothing)

    change global α to per pixel weight for known edges
  - Handles weighted data errors (ex. dead sensor area)
  - Handles nonlinear constraints (iterative deblurring can enforce nonnegativity)
  - Handles boundary assumptions

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