Contents

4 PLANAR KINEMATICS OF RIGID BODY

A rigid body is an idealized model of an object that does not deform, or change shape. A rigid body is by definition an object with the property that the distance between every pair of points of a rigid body is constant. Although any object does deform as it moves, if its deformation is small one may approximate its motion by modeling it as a rigid body.

4.1 Types of motion

The rigid body motion is described with respect to a reference frame (coordinate system) relative to which the motions of the points of the rigid body and its angular motion are measured. In many situations it is convenient to use a reference frame that is fixed with respect to the earth.

Rotation about a fixed axis. Each point of the rigid body on the axis is stationary, and each point not on the axis moves in a circular path about the axis as the rigid body rotates, Fig. 4.1(a).

Translation. Each point of the rigid body describes parallel paths, Fig. 4.1(b). Every point of a rigid body in translation has the same velocity and acceleration. The motion of the rigid body may be described the motion of a single point.

Planar motion. Consider a rigid body intersected by a plane fixed relative to a given reference frame, Fig. 4.1(c). The points of the rigid body intersected by the plane remain in the plane for two-dimensional, or planar, motion. The fixed plane is the plane of the motion. Planar motion or complex motion exhibits a simultaneous combination of rotation and translation. Points on the rigid body will travel non-parallel paths, and there will be, at every instant, a center of rotation, which will continuously change location.

The rotation of a rigid body about a fixed axis is a special case of planar motion.

4.2 Rotation about a fixed axis

Figure 4.2 shows a rigid body rotating about a fixed axis a. The reference line b is fixed and it is perpendicular to the fixed axis a, $b \perp a$. The body-fixed line c rotates with the rigid body and it is perpendicular to the fixed axis a, $c \perp a$. The angle θ between the reference line and the body-fixed line describes the position, or orientation, of the rigid body about the fixed axis. The angular velocity (rate of rotation) of the rigid body is

$$\omega = \frac{d\theta}{dt} = \dot{\theta}, \tag{4.1}$$

and the angular acceleration of the rigid body is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}. \tag{4.2}$$

The velocity of a point P, of the rigid body, at a distance r from the fixed axis is tangent to its circular path (Fig. 4.2) and is given by

$$v = r\omega. \tag{4.3}$$

The normal and tangential acceleration of P are

$$a_t = r\alpha, \quad a_n = \frac{v^2}{r} = r\omega^2. \tag{4.4}$$

4.3 Relative velocity of two points of the rigid body

Figure 4.3 shows a a rigid body in planar translation and rotation. The position vector of the point A of the rigid body is $\mathbf{r}_A = \mathbf{OA}$, and the position vector of the point B of the rigid body is $\mathbf{r}_B = \mathbf{OB}$. The point O is the origin of a given reference frame. The position of point A relative to point B is the vector $\mathbf{BA}$. The position vector of point A relative to point B is related to the positions of A and B relative to O by

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{BA}. \tag{4.5}$$

The derivative of the Eq. (4.5) with respect to time gives

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}. \tag{4.6}$$

where $\mathbf{v}_A$ and $\mathbf{v}_B$ are the velocities of A and B relative to the reference frame. The velocity of point A relative to point B is

$$\mathbf{v}_{AB} = \frac{d\mathbf{BA}}{dt}.$$

Since A and B are points of the rigid body, the distance between them, $BA = |\mathbf{BA}|$, is constant. That means that relative to B, A moves in a circular path as the rigid body rotates. The velocity of A relative to B is therefore tangent to the circular path and equal to the product of the angular velocity ω of the rigid body and BA

$$v_{AB} = |\mathbf{v}_{AB}| = \omega\, BA \tag{4.7}$$

The velocity $\mathbf{v}_{AB}$ is perpendicular to the position vector $\mathbf{BA}$, $\mathbf{v}_{AB} \perp \mathbf{BA}$. The sense of $\mathbf{v}_{AB}$ is the sense of ω, Fig. 4.3. The velocity of A is the sum of the velocity of B and the velocity of A relative to B.

4.4 Angular velocity vector of a rigid body

Euler's theorem: a rigid body constrained to rotate about a fixed point can move between any two positions by a single rotation about some axis through the fixed point.

With Euler's theorem the change in position of a rigid body relative to a fixed point A during an interval of time from t to $t + dt$ may be expressed as a single rotation through an angle $d\theta$ about some axis. At the time t the rate of rotation of the rigid body about the axis is its angular velocity $\omega = d\theta/dt$, and the axis about which it rotates is called the *instantaneous axis of rotation.*

The angular velocity vector of the rigid body, denoted by $\boldsymbol{\omega}$, specifies both the direction of the instantaneous axis of rotation and the angular velocity. The vector $\boldsymbol{\omega}$ is defined to be parallel to the instantaneous axis of rotation (Fig. 4.4), and its magnitude is the rate of rotation, the absolute value of ω. The direction of $\boldsymbol{\omega}$ is related to the direction of the rotation of the rigid body through a right-hand rule: you point the thumb of your right hand in the direction of $\boldsymbol{\omega}$, the fingers curl around $\boldsymbol{\omega}$ in the direction of the rotation.

Figure 4.5 shows two points A and B of a rigid body. The rigid body has the angular velocity $\boldsymbol{\omega}$. The velocity of A relative to B is given by the equation

$$\mathbf{v}_{AB} = \frac{d\mathbf{BA}}{dt} = \boldsymbol{\omega} \times \mathbf{BA}. \tag{4.8}$$

Proof. The point A is moving at the present instant in a circular path relative to the point B. The radius of the path is $|\mathbf{BA}| \sin\beta$, where β is the angle between the vectors $\mathbf{BA}$ and $\boldsymbol{\omega}$. The magnitude of the velocity of A relative to B is equal to the product of the radius of the circular path and the angular velocity of the rigid body, $|\mathbf{v}_{AB}| = (|\mathbf{BA}| \sin\beta)|\omega|$, which is the magnitude of the cross product of $\mathbf{BA}$ and $\boldsymbol{\omega}$ or

$$\mathbf{v}_{AB} = \boldsymbol{\omega} \times \mathbf{BA}.$$

The relative velocity $\mathbf{v}_{AB}$ is perpendicular to $\boldsymbol{\omega}$ and perpendicular to $\mathbf{BA}$.

Substituting Eq. (4.8) into Eq. (4.6), for the relation between the velocities of two points of a rigid body in terms of its angular velocity is obtained

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB} = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{BA}. \tag{4.9}$$

4.5 Instantaneous center

The *instantaneous center* of a rigid body is a point whose velocity is zero at the instant under consideration. Every point of the rigid body rotates about the instantaneous center at the instant under consideration.

The instantaneous center may be or may not be a point of the rigid body. When the instantaneous center is not a point of the rigid body the rigid body is rotating about an external point at that instant.

Figure 4.6 shows two points A and B of a rigid body and their directions of the motion Δ_A and Δ_B

$$\mathbf{v}_A || \Delta_A \quad \text{and} \quad \mathbf{v}_B || \Delta_B,$$

where $\mathbf{v}_A$ is the velocity of point A, and $\mathbf{v}_B$ is the velocity of point B.

Through the points A and B perpendicular lines are drawn to their directions of motion

$$d_A \perp \Delta_A \quad \text{and} \quad d_B \perp \Delta_B.$$

The perpendicular lines intersect at the point C

$$d_A \in d_B = C.$$

The velocity of point C in terms of the velocity of point A is

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{AC},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body. Since the vector $\boldsymbol{\omega} \times \mathbf{AC}$ is perpendicular to $\mathbf{AC}$

$$(\boldsymbol{\omega} \times \mathbf{AC}) \perp \mathbf{AC}$$

this equation states that the direction of motion of C is parallel to the direction of motion of A

$$\mathbf{v}_C || \mathbf{v}_A. \tag{4.10}$$

The velocity of point C in terms of the velocity of point B is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{BC}.$$

The vector $\boldsymbol{\omega} \times \mathbf{BC}$ is perpendicular to $\mathbf{BC}$

$$(\boldsymbol{\omega} \times \mathbf{BC}) \perp \mathbf{BC}$$

so this equation states that the direction of motion of C is parallel to the direction of motion of B

$$\mathbf{v}_C || \mathbf{v}_B. \tag{4.11}$$

But C cannot be moving parallel to A and parallel to B, so Eqs. (4.10) and (4.11) are contradictory unless $\mathbf{v}_C = \mathbf{0}$. So the point C, where the perpendicular lines through A and B to their directions of motion intersect, is the instantaneous center. This is a simple method to locate the instantaneous center of a rigid body in planar motion.

If the rigid body is in translation (the angular velocity of the rigid body is zero) the instantaneous center of the rigid body C moves to infinity.

4.6 Relative acceleration of two points of the rigid body

The velocities of two points A and B of a rigid body in planar motion relative to a given reference frame with the origin at point O are related by, Fig. 4.7

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}.$$

Taking the time derivative of this equation, one may obtain

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{AB}.$$

where $\mathbf{a}_A$ and $\mathbf{a}_B$ are the accelerations of A and B relative to the origin O of the reference frame and $\mathbf{a}_{AB}$ is the acceleration of point A relative to point B. Because the point A moves in a circular path relative to the point B as the rigid body rotates, $\mathbf{a}_{AB}$ has a normal component and a tangential component, Fig. 4.7

$$\mathbf{a}_{AB} = \mathbf{a}^n_{AB} + \mathbf{a}^t_{AB}$$

The normal component points toward the center of the circular path (point B) and its magnitude is

$$|\mathbf{a}^n_{AB}| = |\mathbf{v}_{AB}|^2/|\mathbf{BA}| = \omega^2 BA.$$

The tangential component equals the product of the distance $BA = |\mathbf{BA}|$ and the angular acceleration α of the rigid body

$$|\mathbf{a}^t_{AB}| = \alpha BA.$$

The velocity of the point A relative to the point B in terms of the angular velocity vector, $\boldsymbol{\omega}$, of the rigid body is given by Eq. (4.8)

$$\mathbf{v}_{AB} = \boldsymbol{\omega} \times \mathbf{BA}.$$

Taking the time derivative of this equation, one may obtain

$$\begin{aligned} \mathbf{a}_{AB} &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{BA} + \boldsymbol{\omega} \times \mathbf{v}_{AB} \\ &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA}). \end{aligned}$$

Defining the angular acceleration vector $\boldsymbol{\alpha}$ to be the rate of change of the angular velocity vector,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}, \tag{4.12}$$

the acceleration of A relative to B is

$$\mathbf{a}_{AB} = \boldsymbol{\alpha} \times \mathbf{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA}).$$

The velocities and accelerations of two points of a rigid body in terms of its angular velocity and angular acceleration are

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{BA}, \tag{4.13}$$

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA}). \tag{4.14}$$

In the case of planar motion, the term $\boldsymbol{\alpha} \times \mathbf{BA}$ in Eq. (4.14) is the tangential component of the acceleration of A relative to B and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA})$ is the normal component (Fig. 4.7). Equation (4.14) may be written for planar motion in the form

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{BA} - \omega^2 \mathbf{BA}. \tag{4.15}$$

4.7 Motion of a point that moves relative to a rigid body

A reference frame that moves with the rigid body is a *body fixed* reference frame. Figure 4.8 shows a rigid body RB, in motion relative to a primary reference frame with its origin at point O_0, XO_0YZ. The primary reference frame is a fixed reference frame or an earth fixed reference frame. The unit vectors $\boldsymbol{\imath}_0, \boldsymbol{\jmath}_0$, and $\mathbf{k}_0$ of the primary reference reference frame are constant.

The body fixed reference frame, $xOyz$, has its origin at a point O of the rigid body ($O \in RB$), and is a moving reference frame relative to the primary reference. The unit vectors $\boldsymbol{\imath}, \boldsymbol{\jmath}$, and $\mathbf{k}$ of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame.

The position vector of a point P of the rigid body ($P \in RB$) relative to the origin, O, of the body fixed reference frame is the vector **OP**. The velocity of P relative to O is

$$\frac{d\mathbf{OP}}{dt} = \mathbf{v}_{PO} = \boldsymbol{\omega} \times \mathbf{OP},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body. The unit vector $\boldsymbol{\imath}$ may be regarded as the position vector of a point P of the rigid body (Fig. 4.8), and its time derivative may be written as $\frac{d\boldsymbol{\imath}}{dt} = \dot{\mathbf{i}} = \boldsymbol{\omega} \times \boldsymbol{\imath}$. In a similar way the time derivative of the unit vectors $\boldsymbol{\jmath}$ and $\mathbf{k}$ may be obtained. The expressions

$$\frac{d\boldsymbol{\imath}}{dt} = \dot{\mathbf{i}} = \boldsymbol{\omega} \times \boldsymbol{\imath},$$
$$\frac{d\boldsymbol{\jmath}}{dt} = \dot{\mathbf{j}} = \boldsymbol{\omega} \times \boldsymbol{\jmath},$$
$$\frac{d\mathbf{k}}{dt} = \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k}. \tag{4.16}$$

are known as Poisson's relations.

The position vector of a point A (the point A is not assumed to be a point of the rigid body), relative to the origin O_0 of the primary reference frame is, Fig. 4.9

$$\mathbf{r}_A = \mathbf{r}_O + \mathbf{r},$$

where

$$\mathbf{r} = \mathbf{OA} = x\boldsymbol{\imath} + y\boldsymbol{\jmath} + z\mathbf{k}$$

is the position vector of A relative to the origin O, of the body fixed reference frame, and x, y, and z are the coordinates of A in terms of the body fixed reference frame. The velocity of the point A is the time derivative of the position vector $\mathbf{r}_A$

$$\mathbf{v}_A = \frac{d\mathbf{r}_O}{dt} + \frac{d\mathbf{r}}{dt} = \mathbf{v}_O + \mathbf{v}_{AO} =$$
$$\mathbf{v}_O + \frac{dx}{dt}\boldsymbol{\imath} + x\frac{d\boldsymbol{\imath}}{dt} + \frac{dy}{dt}\boldsymbol{\jmath} + y\frac{d\boldsymbol{\jmath}}{dt} + \frac{dz}{dt}\mathbf{k} + z\frac{d\mathbf{k}}{dt}.$$

Using Eqs. (4.16), the total derivative of the the position vector $\mathbf{r}$ is

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k} + \boldsymbol{\omega} \times \mathbf{r}.$$

The velocity of A relative to the body fixed reference frame is a local derivative

$$\mathbf{v}_{Arel} = \frac{\partial \mathbf{r}}{\partial t} = \frac{dx}{dt}\boldsymbol{\imath} + \frac{dy}{dt}\boldsymbol{\jmath} + \frac{dz}{dt}\mathbf{k} = \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k}, \tag{4.17}$$

A general formula for the total derivative of a moving vector $\mathbf{r}$ may be written as

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial t} + \boldsymbol{\omega} \times \mathbf{r}. \tag{4.18}$$

This relation is known as the *transport theorem*. In operator notation the transport theorem is written as

$$\frac{d}{dt}() = \frac{\partial}{\partial t}() + \boldsymbol{\omega} \times (). \tag{4.19}$$

The velocity of the point A relative to the primary reference frame is

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}, \tag{4.20}$$

Equation (4.20) expresses the velocity of a point A as the sum of three terms:

- the velocity of a point O of the rigid body,
- the velocity $\mathbf{v}_{Arel}$ of A relative to the rigid body, and
- the velocity $\boldsymbol{\omega} \times \mathbf{r}$ of A relative to O due to the rotation of the rigid body.

The acceleration of the point A relative to the primary reference frame is obtained by taking the time derivative of Eq. (4.20)

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO}, \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \end{aligned} \tag{4.21}$$

where

$$\mathbf{a}_{Arel} = \frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{d^2 x}{dt^2}\mathbf{\imath} + \frac{d^2 y}{dt^2}\mathbf{\jmath} + \frac{d^2 z}{dt^2}\mathbf{k}, \tag{4.22}$$

is the acceleration of A relative to the body fixed reference frame or relative to the rigid body. The term

$$\mathbf{a}_{Cor} = 2\boldsymbol{\omega} \times \mathbf{v}_{Arel},$$

is called the Coriolis acceleration force.

In the case of planar motion, Eq. (4.21) becomes

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO}, \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} - \boldsymbol{\omega}^2\mathbf{r}, \end{aligned} \tag{4.23}$$

The motion of the rigid body (RB) is described relative to the primary reference frame. The velocity $\mathbf{v}_A$ and the acceleration $\mathbf{a}_A$ of point a point A are relative to the primary reference frame. The terms $\mathbf{v}_{Arel}$ and $\mathbf{a}_{Arel}$ are the velocity and acceleration of point A relative to the body fixed reference frame i.e., they are the velocity and acceleration measured by an observer moving with the rigid body, Fig. 4.10.

If A is a point of the rigid body, $A \in RB$, $\mathbf{v}_{Arel} = \mathbf{0}$ and $\mathbf{a}_{Arel} = \mathbf{0}$.

Motion of a point relative to a moving reference frame

The velocity and acceleration of an arbitrary point A relative to a point

O of a rigid body, in terms of the body fixed reference frame, are given by Eqs. (4.20) and (4.21)

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{OA} \tag{4.24}$$

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{OA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OA}). \tag{4.25}$$

These results apply to any reference frame having a moving origin O and rotating with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ relative to a primary reference frame (Fig. 4.11). The terms $\mathbf{v}_A$ and $\mathbf{a}_A$ are the velocity and acceleration of an arbitrary point A relative to the primary reference frame. The terms $\mathbf{v}_{Arel}$ and $\mathbf{a}_{Arel}$ are the velocity and acceleration of A relative to the secondary moving reference frame i.e., they are the velocity and acceleration measured by an observer moving with the secondary reference frame.

Inertial reference frames

A reference frame is inertial if one may use it to apply Newton's second law in the form $\sum \mathbf{F} = m\mathbf{a}$.

Figure 4.12 shows a nonaccelerating, nonrotating reference frame with the origin at O_0, and a secondary nonrotating, earth centered reference frame with the origin at O. The nonaccelerating, nonrotating reference frame with the origin at O_0 is assumed to be an inertial reference. The acceleration of the earth, due to the gravitational attractions of the sun, moon, etc., is $\mathbf{g}_O$. The earth centered reference frame has the acceleration $\mathbf{g}_O$, too.

Newton's second law for an object A of mass m, using the hypothetical nonaccelerating, nonrotating reference frame with the origin at O_0, may be written as

$$m\mathbf{a}_A = m\mathbf{g}_A + \sum \mathbf{F}, \tag{4.26}$$

where $\mathbf{a}_A$ is the acceleration of A relative to O_0 $\mathbf{g}_A$ is the resulting gravitational acceleration, and $\sum \mathbf{F}$ is the sum of all other external forces acting on A.

Using Eq. (4.25) the acceleration of A relative to O_0 is

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel},$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the earth centered reference frame and the acceleration of the origin O is equal to the gravitational acceleration of the earth $\mathbf{a}_O = \mathbf{g}_O$. The earth-centered reference frame does not rotate ($\boldsymbol{\omega} = \mathbf{0}$).

If the object A is on or near the earth, its gravitational acceleration $\mathbf{g}_A$ due to the attraction of the sun, etc., is nearly equal to the gravitational acceleration of the earth $\mathbf{g}_O$, and Eq. (4.26) becomes

$$\sum \mathbf{F} = m\mathbf{a}_{Arel}. \tag{4.27}$$

One may apply Newton's second law using a nonrotating, earth centered reference frame if the object is near the earth.

In most applications, Newton's second law may be applied using an earth fixed reference frame. Figure 4.13 shows a nonrotating reference frame with its origin at the center of the earth O and a secondary earth fixed reference frame with its origin at a point B. The earth fixed reference frame with the origin at B may be assumed to be an inertial reference and

$$\sum \mathbf{F} = m\mathbf{a}_{Arel}, \tag{4.28}$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the earth fixed reference frame.

The motion of an object A may be analyzed using a primary inertial reference frame with its origin at the point O, Fig. 4.14. A secondary reference frame with its origin at B undergoes an arbitrary motion with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. The Newton's second law for the object A of mass m is

$$\sum \mathbf{F} = m\mathbf{a}_A, \tag{4.29}$$

where $\mathbf{a}_A$ is the acceleration of A acceleration relative to O. Equation (4.29) may be written in the form

$$\sum \mathbf{F} - m[\mathbf{a}_B + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{BA})] = m\mathbf{a}_{Arel}, \quad (4.30)$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the secondary reference frame. The term $\mathbf{a}_B$ is the acceleration of the origin B of the secondary reference frame relative to the primary inertial reference. The term $2\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is the Coriolis acceleration, and the term $-2m\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is called the Coriolis force.

This is Newton's second law expressed in terms of a secondary reference frame undergoing an arbitrary motion relative to an inertial primary reference frame.

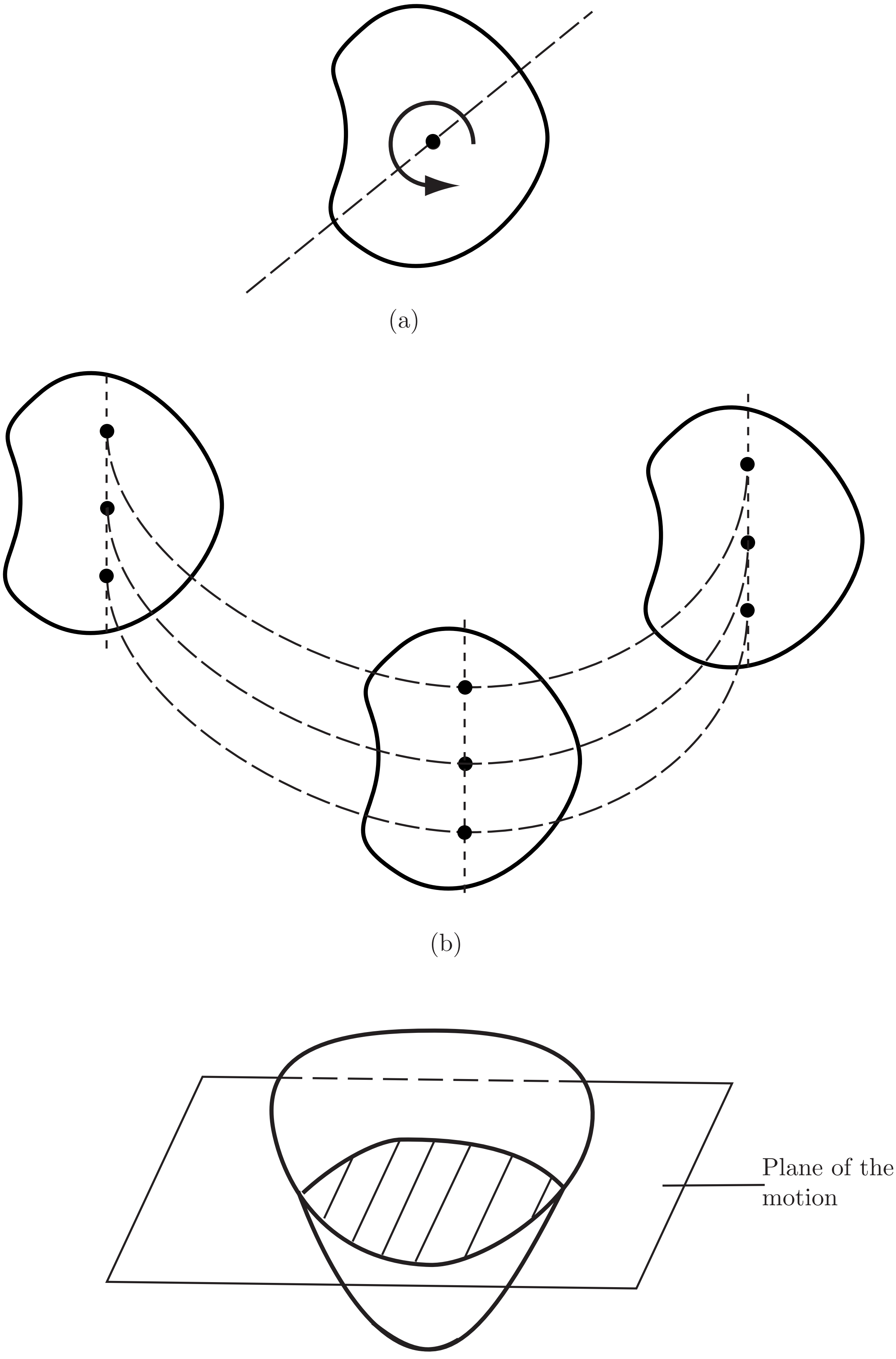


Figure 4.1

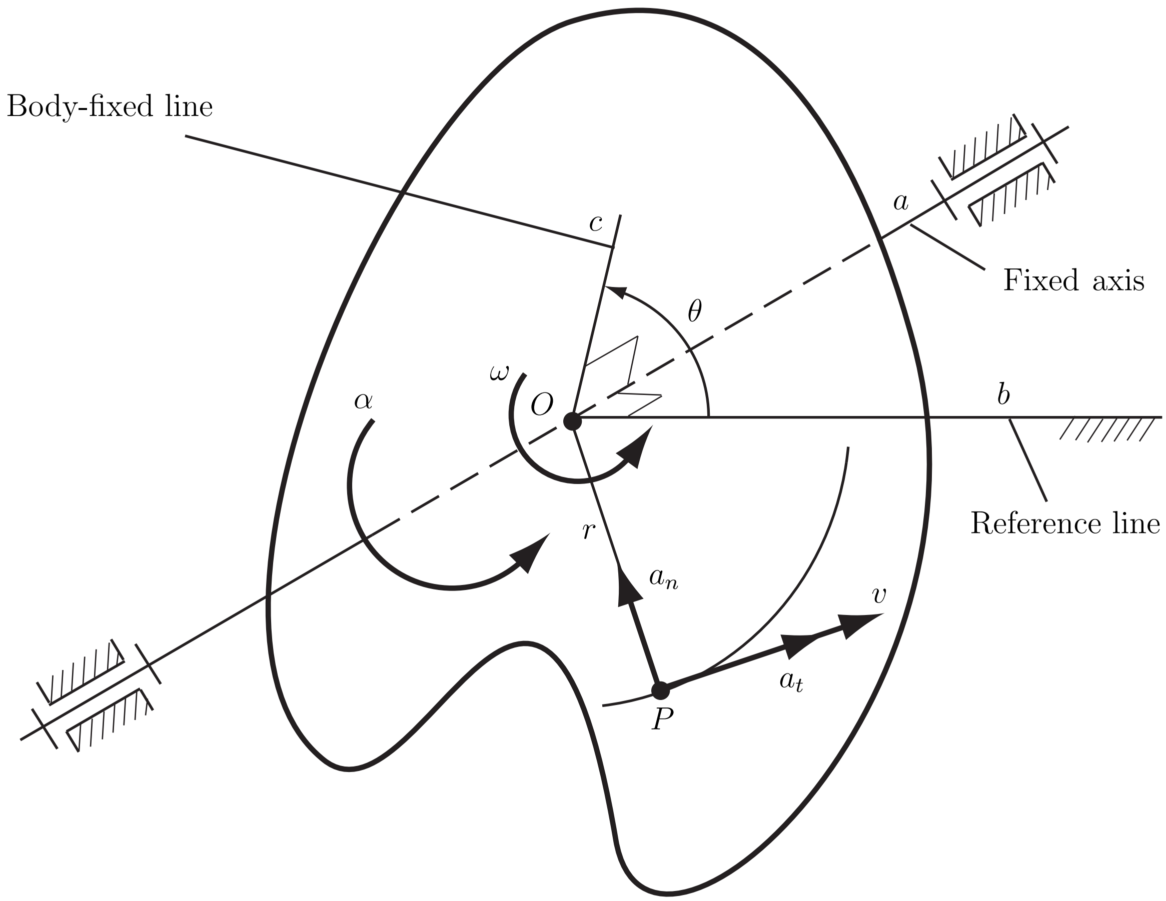
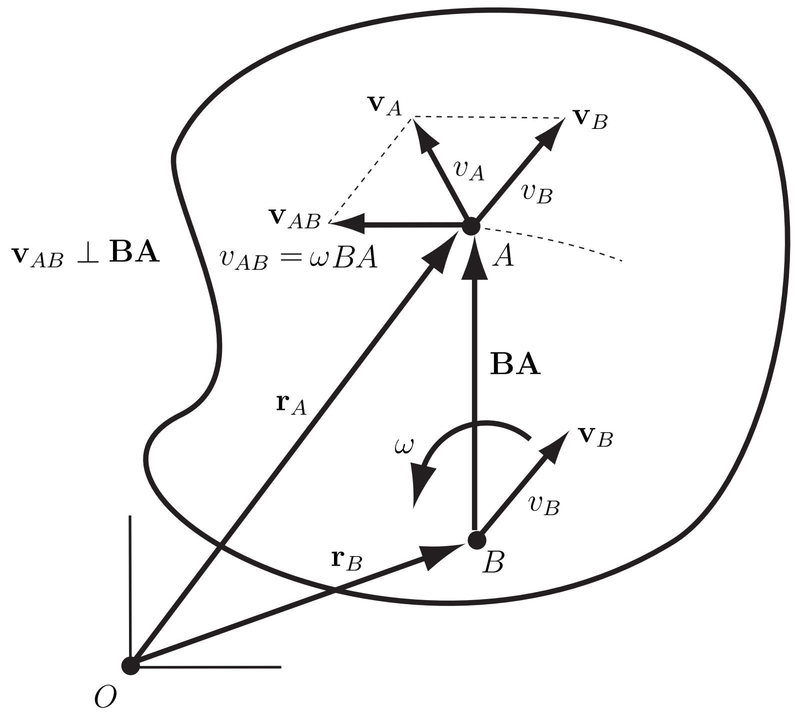


Figure 4.2



$\mathbf{v}_A$
$\mathbf{v}_B$
v_A
v_B
$\mathbf{v}_{AB}$
$\mathbf{v}_{AB} \perp \mathbf{BA}$
$v_{AB} = \omega BA$
A
$\mathbf{BA}$
$\mathbf{r}_A$
ω
$\mathbf{v}_B$
v_B
$\mathbf{r}_B$
B
O

Figure 4.3

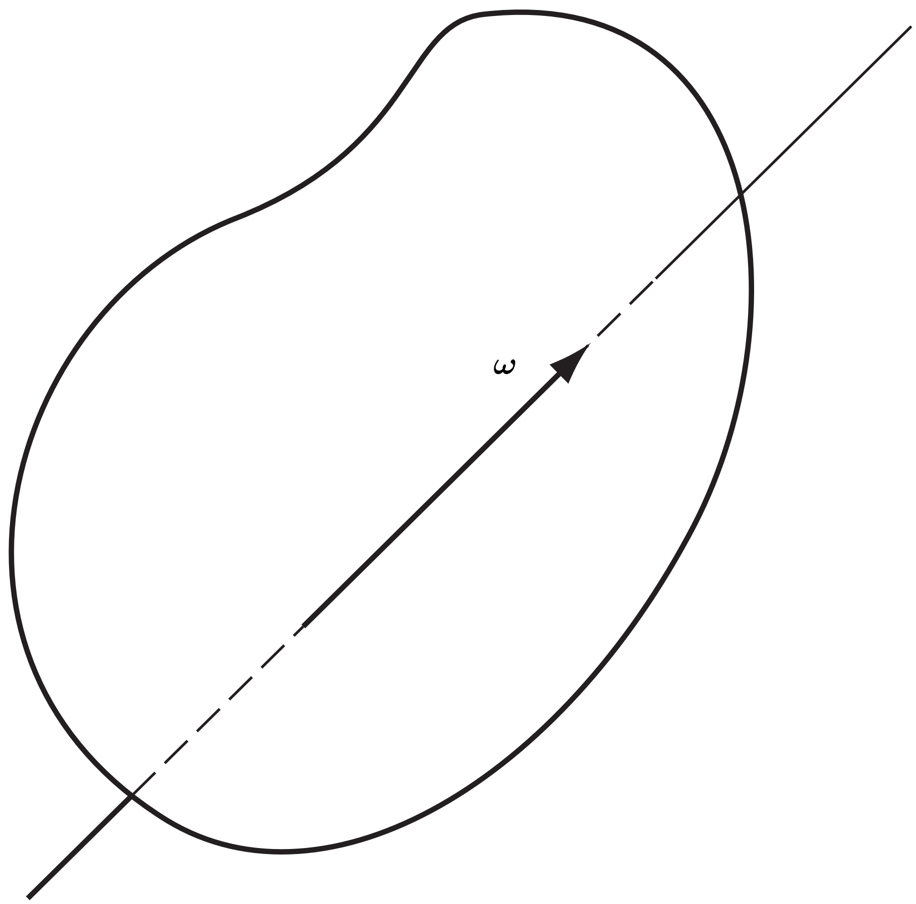
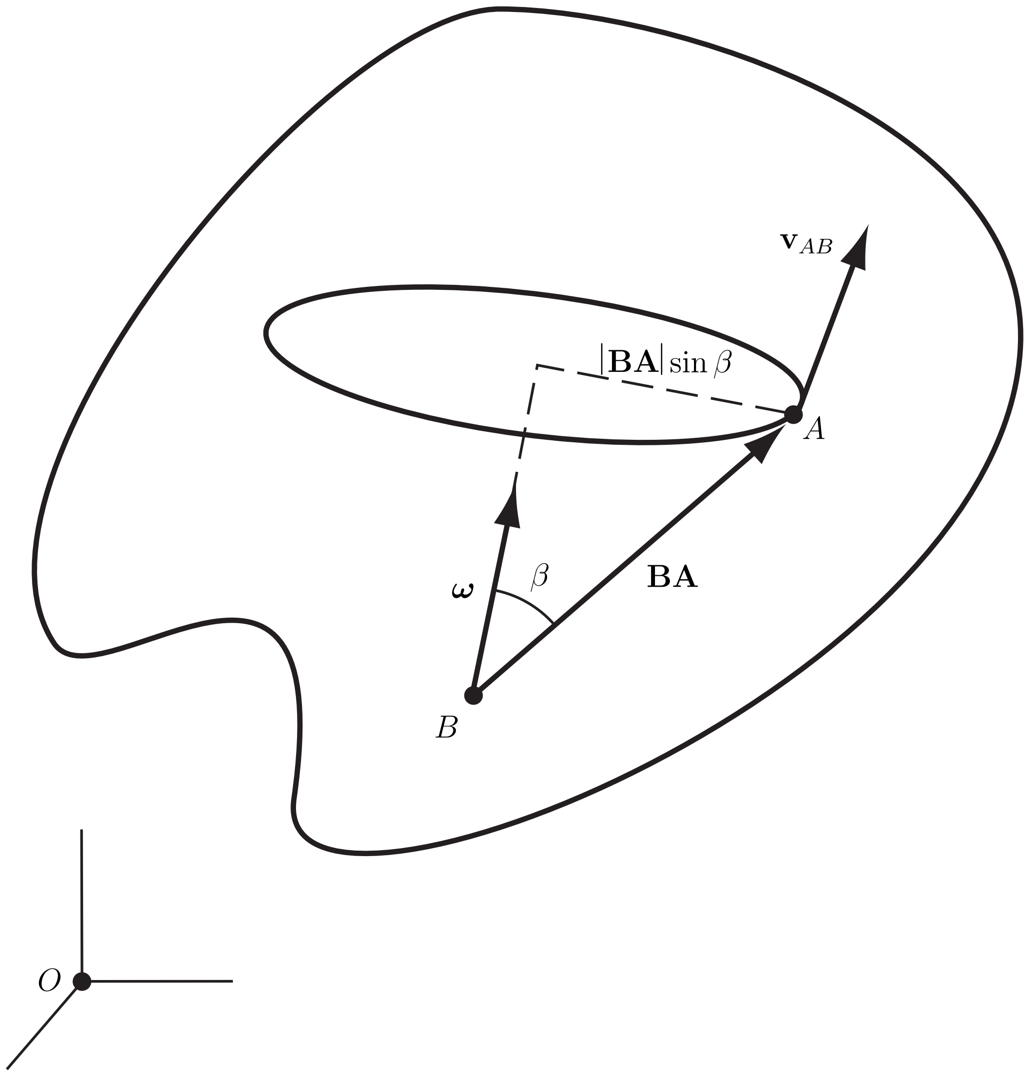
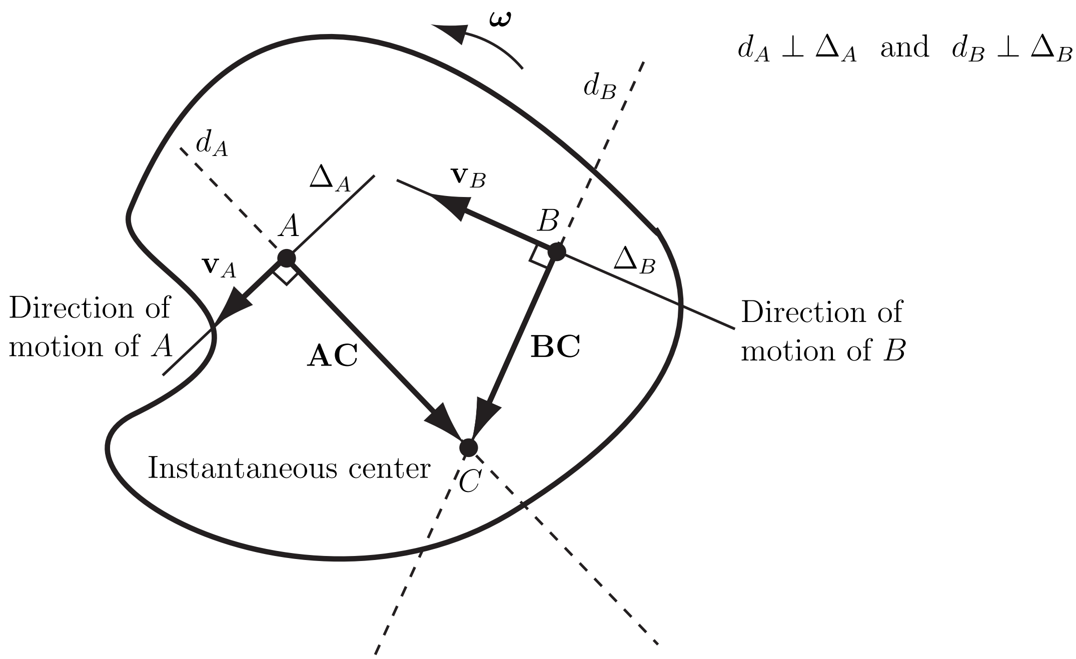


Figure 4.4



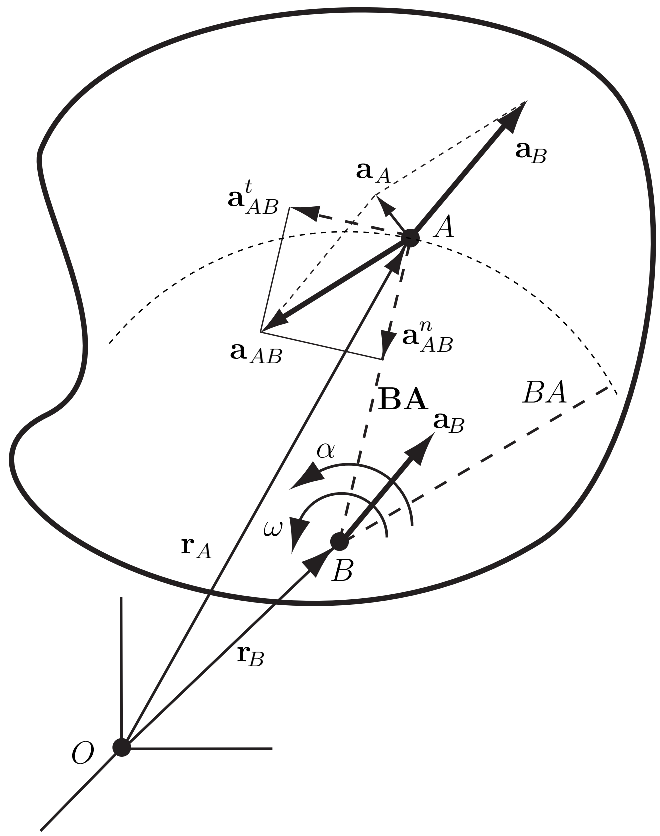
$\mathbf{v}_{AB}$
$|\mathbf{BA}|\sin\beta$
A
$\boldsymbol{\omega}$
β
$\mathbf{BA}$
B
O

Figure 4.5



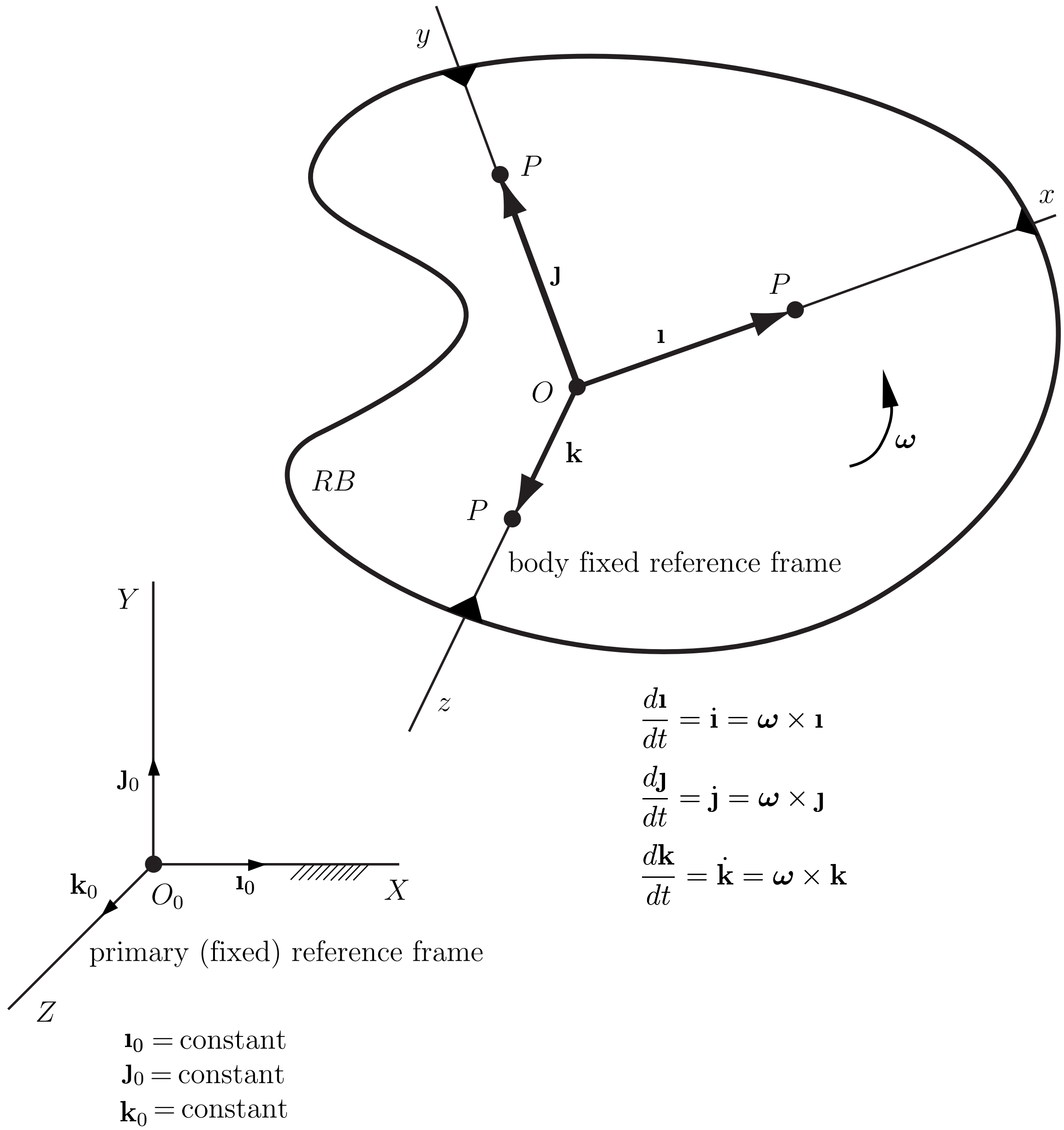
ω
$d_A \perp \Delta_A$ and $d_B \perp \Delta_B$
d_B
d_A
Δ_A
$\mathbf{v}_B$
A
B
Δ_B
$\mathbf{v}_A$
Direction of motion of A
Direction of motion of B
$\mathbf{AC}$
$\mathbf{BC}$
Instantaneous center
C

Figure 4.6



$\mathbf{a}_B$
$\mathbf{a}_A$
$\mathbf{a}^t_{AB}$
A
$\mathbf{a}^n_{AB}$
$\mathbf{a}_{AB}$
BA
$\mathbf{BA}$
$\mathbf{a}_B$
α
ω
$\mathbf{r}_A$
B
$\mathbf{r}_B$
O

Figure 4.7



y
P
x
P
ȷ
ı
O
k
ω
RB
P
body fixed reference frame
z
$\frac{d\boldsymbol{\imath}}{dt} = \dot{\mathbf{i}} = \boldsymbol{\omega} \times \boldsymbol{\imath}$
$\frac{d\boldsymbol{\jmath}}{dt} = \dot{\mathbf{j}} = \boldsymbol{\omega} \times \boldsymbol{\jmath}$
$\frac{d\mathbf{k}}{dt} = \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k}$
Y
$\boldsymbol{\jmath}_0$
$\mathbf{k}_0$
O_0
$\boldsymbol{\imath}_0$
X
primary (fixed) reference frame
Z
$\boldsymbol{\imath}_0 =$ constant
$\boldsymbol{\jmath}_0 =$ constant
$\mathbf{k}_0 =$ constant

Figure 4.8

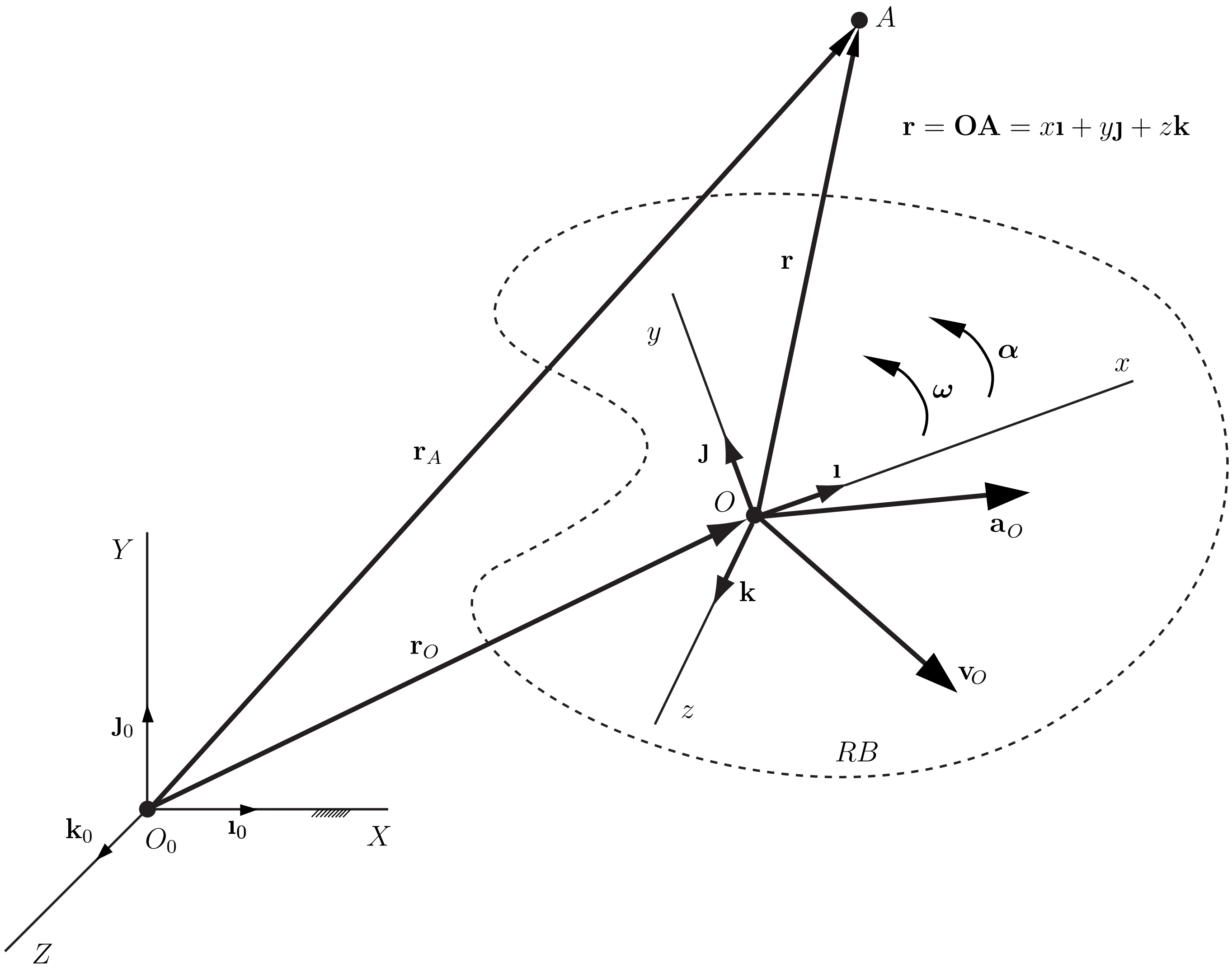
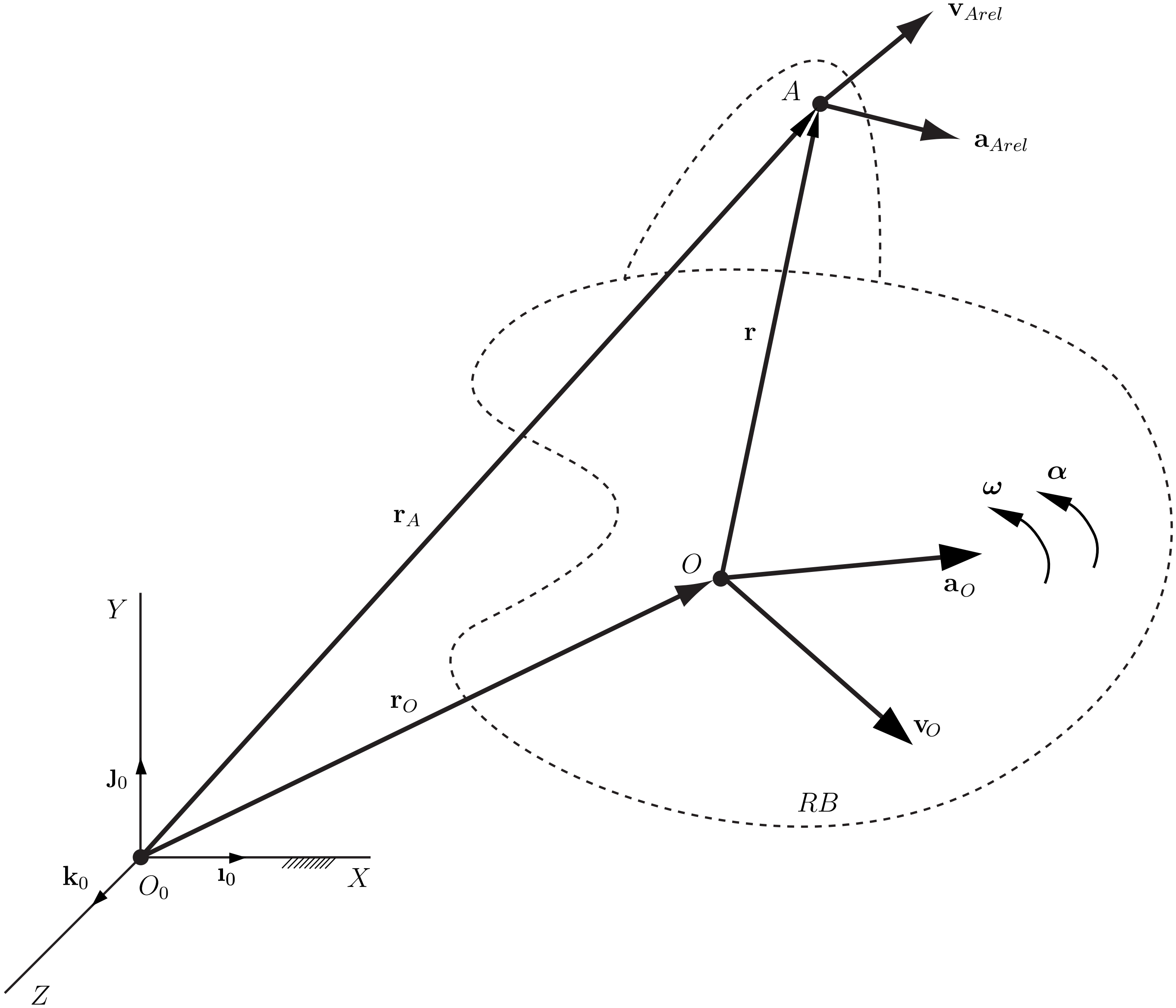


Figure 4.9



$\mathbf{v}_{Arel}$
A
$\mathbf{a}_{Arel}$
$\mathbf{r}$
$\boldsymbol{\omega}$
$\boldsymbol{\alpha}$
$\mathbf{r}_A$
O
$\mathbf{a}_O$
Y
$\mathbf{r}_O$
$\mathbf{v}_O$
$\mathbf{j}_0$
RB
$\mathbf{k}_0$
O_0
$\mathbf{i}_0$
X
Z

Figure 4.10

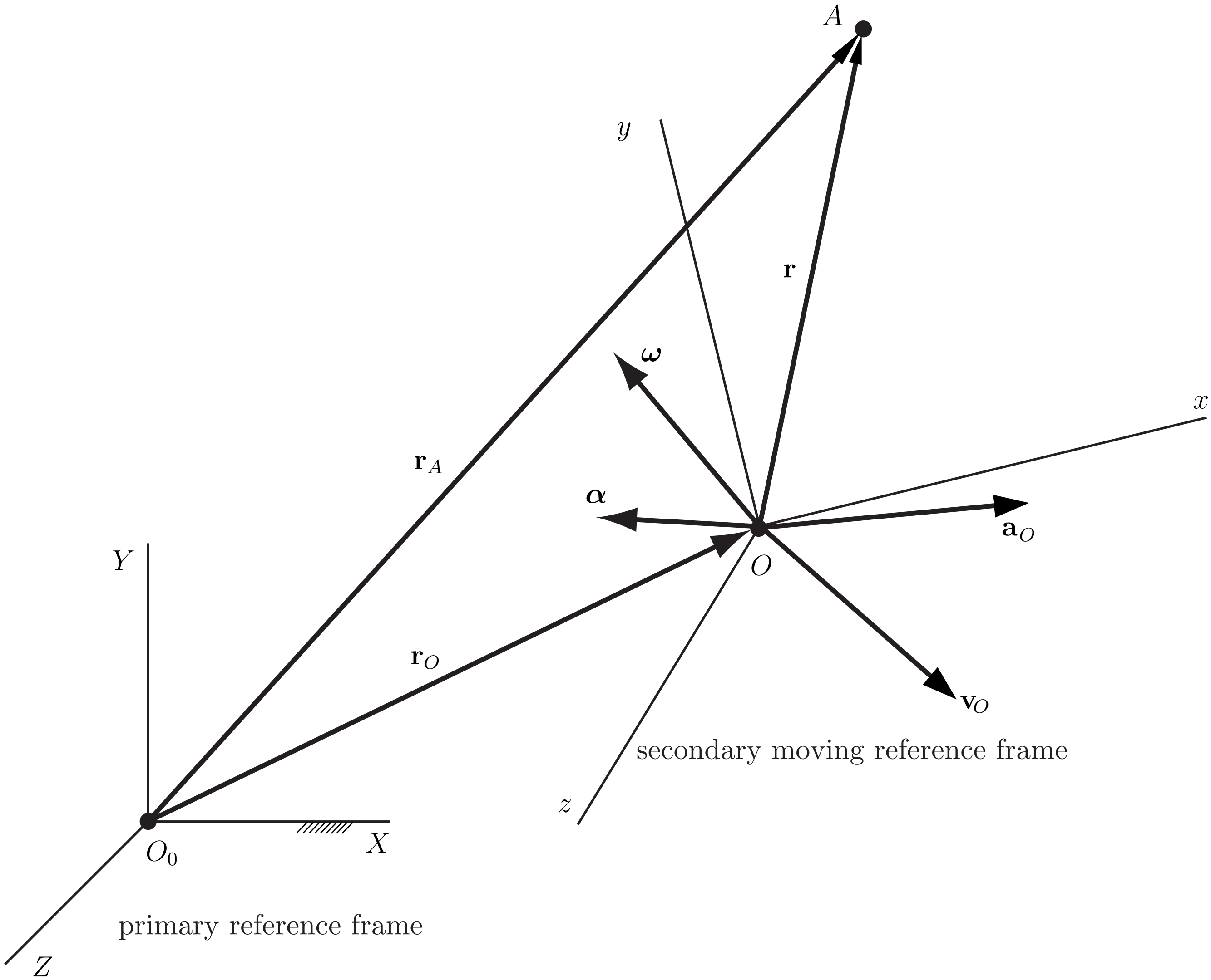

A
y
r
ω
x
r_A
α
a_O
Y
O
r_O
v_O
secondary moving reference frame
z
O_0
X
primary reference frame
Z

Figure 4.11

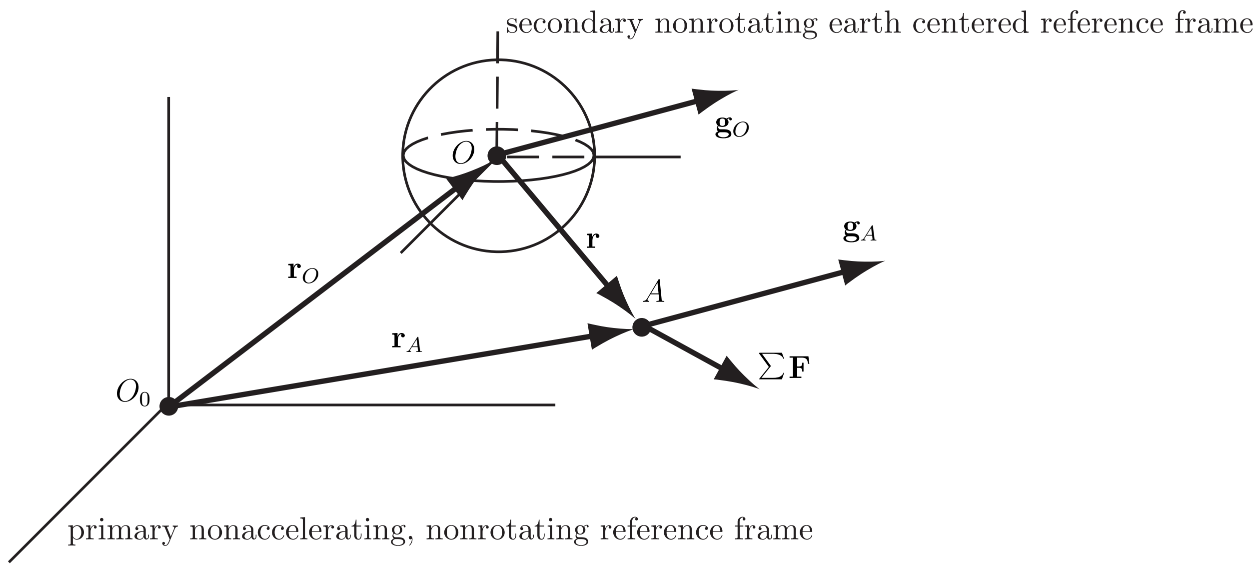

secondary nonrotating earth centered reference frame
$\mathbf{g}_O$
O
$\mathbf{r}$
$\mathbf{g}_A$
$\mathbf{r}_O$
A
$\mathbf{r}_A$
$\sum \mathbf{F}$
O_0
primary nonaccelerating, nonrotating reference frame

Figure 4.12

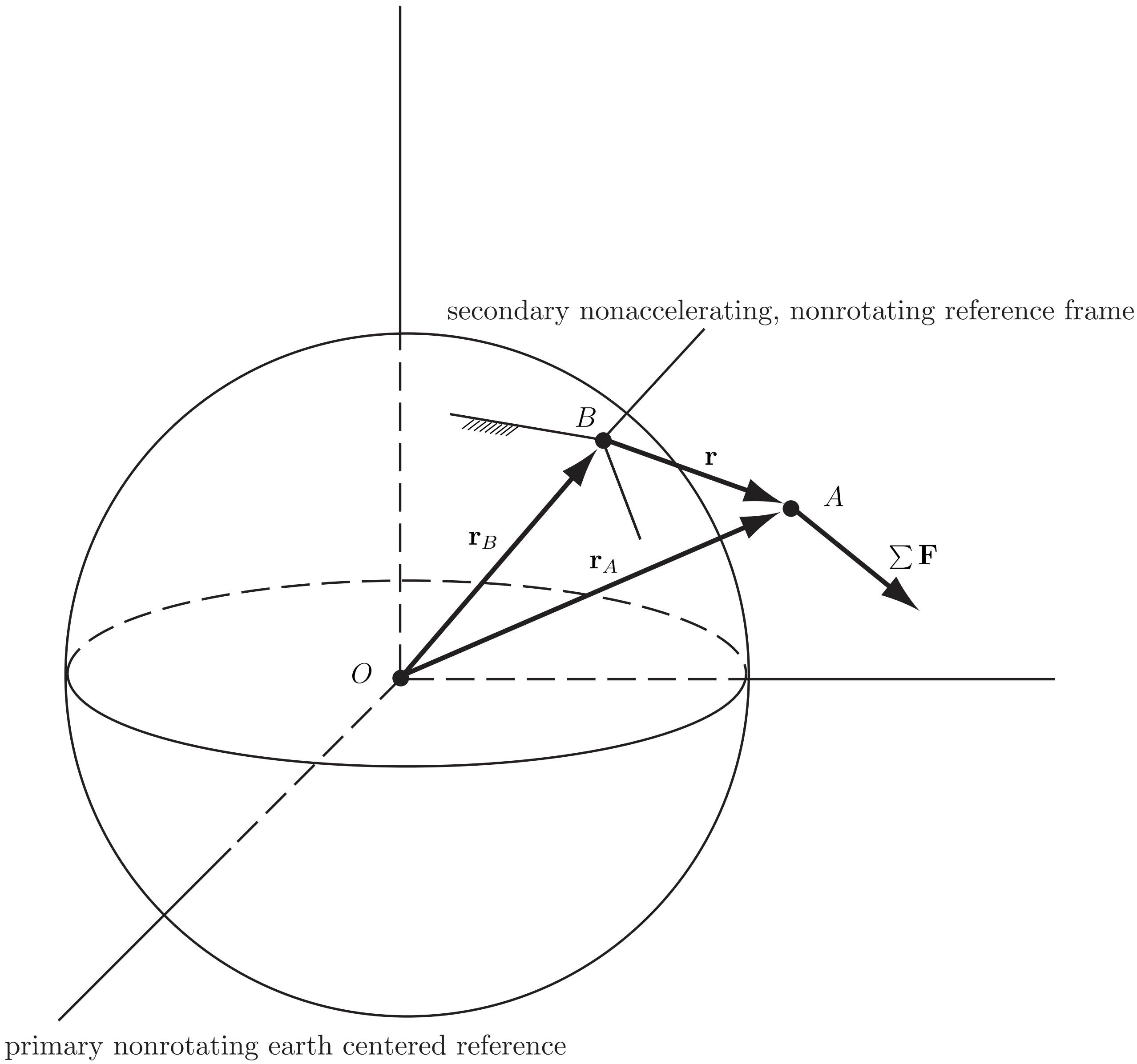

secondary nonaccelerating, nonrotating reference frame
B
r
A
$\mathbf{r}_B$
$\mathbf{r}_A$
$\sum \mathbf{F}$
O
primary nonrotating earth centered reference

Figure 4.13

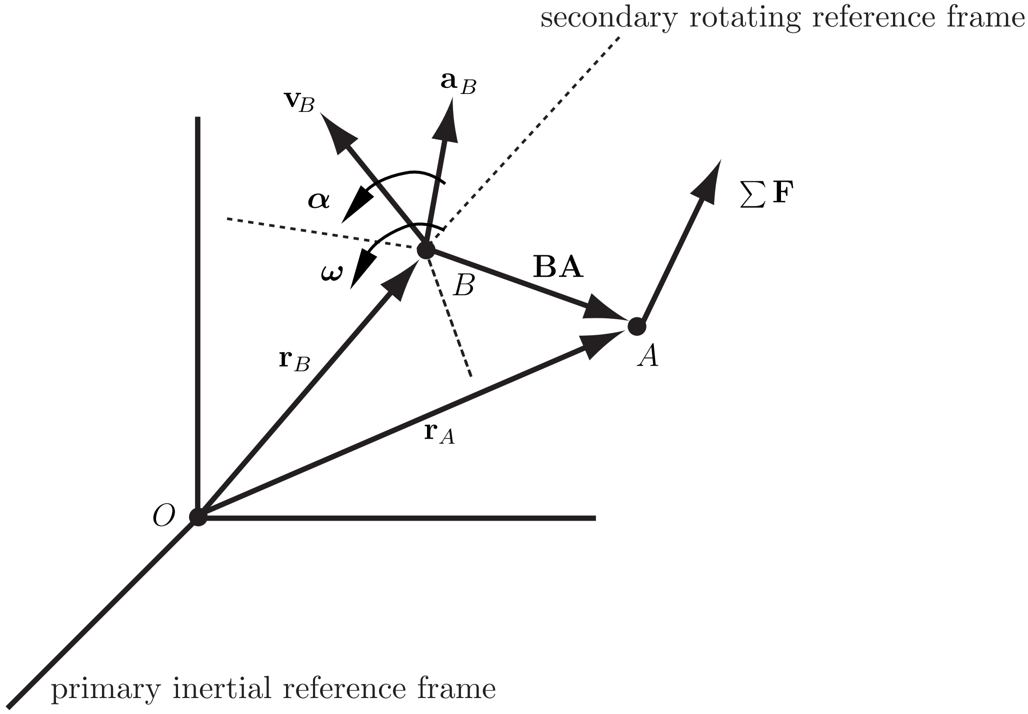

secondary rotating reference frame
$\mathbf{v}_B$
$\mathbf{a}_B$
$\boldsymbol{\alpha}$
$\sum \mathbf{F}$
$\boldsymbol{\omega}$
B
$\mathbf{BA}$
A
$\mathbf{r}_B$
$\mathbf{r}_A$
O
primary inertial reference frame

Figure 4.14