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8 Disk Friction and Flexible Belts

8.1 Disk Friction

The sliding surfaces are present in most machine components (bearings, gears, cams, etc.) and it is desirable to minimize the friction in order to reduce energy loss and wear. In contrast, clutches and brakes depend on friction in order to function. The function of a clutch is to permit smooth, gradual connection and disconnection of two elements having a common axis of rotation. A brake acts similarly except that one of the elements is fixed.

In pivot bearings, clutch plates, and disk brakes there is friction between circular surfaces under distributed normal pressure. Two flat circular disks are considered in Fig. 8.1. The figure shows a simple disk clutch with one driving and one driven surface. Driving friction between the two develops when they are forced together. The disks can be brought into contact under an axial force P. The maximum moment that this clutch can transmit is equal to the moment M required to slip one disk against the other. The elemental frictional force acting on an elemental area is

$$dF_f = \mu\, p\, dA,$$

where p is the normal pressure at any location between the plates, μ is the coefficient of friction, and $dA = r\, dr\, d\theta$ is the area of the element.

The moment of this elemental friction force about the shaft axis is

$$dM = \mu\, p\, r\, dA,$$

and the total moment is

$$M = \int\int \mu\, p\, r\, dA,$$

where the integral is evaluated over the area of the disk.

The coefficient of friction, μ, is assumed to be constant. If the disk surfaces are new, flat, and well supported it is assumed that the pressure p is uniform over the entire surface so that

$$P = \pi\, R^2\, p.$$

The total frictional moment becomes

$$M = \int\int \mu\, \frac{\mu\, P}{\pi\, R^2}\, r\, dA = \frac{\mu\, P}{\pi\, R^2} \int_0^{2\pi} \int_0^{R} r^2\, dr\, d\theta = \frac{2}{3} \mu\, P\, R. \qquad (8.1)$$

The total moment is equal to a friction force $\mu\, P$ acting at a distance $2R/3$ from the shaft center. If the friction disks are rings, as shown in Fig. 8.2, the frictional moment is

$$M = \frac{\mu\, P}{\pi\, R^2} \int_0^{2\pi} \int_{R_i}^{R_o} r^2\, dr\, d\theta = \frac{2}{3} \mu\, P\, \frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}, \tag{8.2}$$

where R_o and R_i are the inside and outside radii.

It is reasonable to assume that after the initial wearing-in period is over, the surfaces retain their new relative shape and further wear is therefore constant over the surface. This wear depends on both the pressure p and the circumferential distance traveled. The distance traveled is proportional to r. Therefore the following expression may be written:

$$r\, p = K,$$

where K is a constant that is determined from the equilibrium condition for the axial forces

$$P = \int p\, dA = K \int_0^{2\pi} \int_0^R dr\, d\theta = 2\, \pi\, K\, R.$$

The constant K is

$$K = \frac{P}{2\, \pi\, R}.$$

With $p\, r = P/(2\, \pi\, R)$, the frictional moment is

$$M = \int \int \mu\, p\, r\, dA = \frac{\mu\, P}{2\, \pi\, R} \int_0^{2\pi} \int_0^R r\, dr\, d\theta = \frac{1}{2}\, \mu\, P\, R. \tag{8.3}$$

The frictional moment for worn-in plates is, therefore, only (1/2)/(2/3)=3/4, as much as for new surfaces. If the friction disks are rings of inside radius R_i and outside radius R_o, the frictional moment for worn-in surfaces is

$$M = \frac{1}{2}\, \mu\, P\, (R_o + R_i). \tag{8.4}$$

Actual clutches employ N friction interfaces transmitting torque in parallel. The number of friction interfaces N is an even number. For a clutch with N friction interfaces, Eq. (8.4) is modified to give

$$M = \frac{1}{2}\, \mu\, P\, (R_o + R_i)\, N. \tag{8.5}$$

The ratio of inside to outside radius is a parameter in the design of clutches. The maximum moment for a given outside radius is obtained when [7]

$$R_i = R_0 \sqrt{\frac{1}{3}} = 0.58\, R_0, \tag{8.6}$$

and the proportions commonly used range from $R_i = 0.45\, R_0$ to $R_i = 0.80\, R_0$.

Disk clutches can be designed to operate either "dry" or "wet" with oil. Most multiple-disk clutches, including those used in automotive automatic transmissions, operate wet.

8.2 Flexible Belts

In the design of belt drives and band brakes the impending slippage of flexible cables, belts, and ropes over sheaves and drums is important. Figure 8.3(a) shows a drum subjected to the two belt tensions T_1 and T_2, the moment M necessary to prevent rotation, and a bearing reaction R.

Figure 8.3(b) shows the free-body diagram of an element of the belt of length $r\,d\theta$. The forces acting on the differential element are calculated using the equilibrium of the element. The tension increases from T at the angle θ to $T + dT$ at the angle $\theta + d\theta$. The normal force which acts on the differential element of area is a differential dN. The friction force, $\mu\, dN$ is impending motion and acts on the belt in a direction to oppose slipping.

The equation for the equilibrium of forces in the t-direction gives

$$T \cos\frac{d\theta}{2} + \mu\, dN = (T + dT) \cos\frac{d\theta}{2} \quad \text{or} \quad \mu\, dN = dT, \tag{8.7}$$

where the cosine of the differential quantity is unity in the limit ($\cos d\theta/2 \approx 1$).

Equilibrium of forces in the n-direction gives

$$dN = (T + dT) \sin\frac{d\theta}{2} + T \sin\frac{d\theta}{2} \quad \text{or} \quad dN = T\, d\theta, \tag{8.8}$$

where the sine of the differential angle is the angle in the limit ($\sin d\theta/2 \approx d\theta/2$) and the product of two differentials is neglected in the limit compared with the first-order differentials ($dT\, d\theta \approx 0$).

The two equilibrium relations Eqs. (8.7), (8.8) give

$$\frac{dT}{T} = \mu\, d\theta,$$

and integrating between corresponding limits T_1 and T_2 [with M in the direction shown in Fig. 8.3(a) $\Longrightarrow T_2 > T_1$]:

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^{\phi} \mu\, d\theta \quad \text{or} \quad \ln \frac{T_2}{T_1} = \mu\, \phi,$$

where ϕ is the total angle of belt contact expressed in radians.

The tension T_2 is

$$T_2 = T_1\, e^{\mu\, \phi}. \tag{8.9}$$

If a rope were wrapped around a drum n times, the total angle of belt contact is

$$\phi = 2\, \pi\, n.$$

Equation (8.9) also applies to belt drives where both the belt and the pulley are rotating at constant speed and describes the ratio of belt tensions for impending slippage (or slippage).

The centrifugal force acting on a flat belt creates a tension of [7]

$$T_c = m'\, V^2 = m'\, \omega^2\, r^2, \tag{8.10}$$

where m' is the mass per unit length of belt, V is the belt speed, and r is the pulley radius. Equation (8.9) becomes

$$\frac{T_2 - T_c}{T_1 - T_c} = e^{\mu\, \phi}. \tag{8.11}$$

The centrifugal force tends to reduce the angles of wrap ϕ.

For a V-belt of angle β [see Fig. 8.3(c)], Eq. (8.11) becomes

$$\frac{T_2 - T_c}{T_1 - T_c} = e^{\mu\, \phi / \sin\beta}. \tag{8.12}$$

8.3 Examples

Example 8.1. The automobile disk brake, shown in Fig. 8.4, consists of a flat-faced rotor and caliper which contains a disk pad on each side of the rotor. The inside radius is R_i and the outside radius is R_o. The forces behind the two pads are eqaul to P and μ is the coefficient of friction. The normal pressure p is uniform distributed over the pad. Show that the moment applied to the hub is independent of the angular span α of the pads.

Solution.

The force acting on the pads is

$$P = p\,A = p\int_0^\alpha \int_{R_i}^{R_o} r\,dr\,d\theta = \frac{p}{2}\int_0^\alpha (R_o^2 - R_i^2)\,d\theta = \frac{p}{2}\,(R_o^2 - R_i^2)\,\alpha.$$

The moment applied to the hub is

$$\begin{aligned} M &= 2\int \mu\,p\,r\,dA = 2\mu\,p\int_0^\alpha \int_{R_i}^{R_o} r^2\,dr\,d\theta = \frac{2\mu\,p}{3}\,(R_o^3 - R_i^3)\,\alpha \\ &= \frac{2\mu}{3}\,\frac{2\,P}{(R_o^2 - R_i^2)\,\alpha}\,(R_o^3 - R_i^3)\,\alpha = \frac{4\mu\,P}{3}\,\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}. \end{aligned}$$

The expression of the moment M shows no dependence with the angular span α of the pads. The pressure variation with the angle θ would not change the moment M.

Example 8.2. The basic disk clutch, shown in Fig. 8.2, has the outside disk diameter of 6 in. The kinetic coefficient of friction is 0.3 and the maximum disk allowable pressure is 100 psi. The disk clutch is designed to transmit a moment of 400 lb·in. Determine the appropriate value of the inside diameter and the clamping force.

Solution.

The maximum moment for a given outside radius is obtained from Eq. (8.6)

$$R_i = 0.58\,R_0 = 0.58\,(3) = 1.74 \text{ in.}$$

The greatest pressure occurs at the inside radius. The design of a clutch of inside radius R_i and allowable pressure p_{max} is based on

$$p\,r = K = p_{max}\,R_i. \tag{8.13}$$

The total moment that can be developed over the entire interface is

$$\begin{aligned} M &= \int \mu\, p\, r\, dA = \int_{R_i}^{R_o} \mu\,(p\, r)\,(2\pi\, r\, dr) = \int_{R_i}^{R_o} 2\pi\,\mu\, p_{max}\, R_i\, r\, dr \\ &= \pi\,\mu\, p_{max}\, R_i\,(R_o^2 - R_i^2), \end{aligned} \tag{8.14}$$

or

$$M = \pi\,(0.3)\,(100)\,(1.74)\,(3^2 - 1.74^2) = 979.421 \text{ lb} \cdot \text{in.}$$

For $R_i = 1.74$ in. and $p_{max} = 100$ psi, the clutch is overdesigned based on the output moment by a factor of 979.421/400=2.448.

Accepting the overdesign, the clamping force is calculated from Eq. (8.4) for $M = 400$ lb·in. as

$$P = \frac{2M}{\mu\,(R_i + R_o)} = \frac{2\,(400)}{0.3\,(1.74 + 3)} = 562.588 \text{ lb.}$$

Example 8.3. Determine the force F on the handle 1 of the differential band brake [Fig. 8.5(a)] that will prevent the wheel 2 from turning on its shaft. The external moment $M = 200$ N·m is applied to the shaft. The coefficient of friction between the band and the wheel of radius $r = 100$ mm is 0.45. The following dimensions are given: $l = 500$ mm, $h = 80$ mm, and $\theta = 30°$.

Solution.

The free-body diagrams for the handle 1 and the wheel 2 are given in Fig. 8.5(b). For the band the tension T_2 is given by Eq. (8.9):

$$T_2 = T_1\, e^{\mu\,\phi} = T_1\, e^{7\pi\mu/6} = 5.203\, T_1, \tag{8.15}$$

where $\phi = \theta + \pi$.

For the wheel 2 the sum of the moments with respect to its center C gives

$$\sum M_C^{(2)} = M - r\,(T_2 - T_1) = 200 - 0.1\,(T_2 - T_1) = 0. \tag{8.16}$$

The tensions T_1 and T_2 are obtained from Eqs. (8.15) and (8.16):

$$T_1 = 475.791 \text{ N} \quad \text{and} \quad T_2 = 2475.79 \text{ N.}$$

For the link 1 the sum of the moments with respect to point O gives

$$\sum M_O^{(1)} = r\, T_2 - l\, F - h\, T_1\, \sin\theta = 0,$$

and the force F is

$$F = \frac{r\,T_2 - h\,T_1\,\sin\theta}{l} = \frac{0.1(2475.79) - 0.08(475.791)\sin 30^\circ}{0.5} = 457.095 \text{ N}.$$

Example 8.4. A 3000 rpm motor drives a machine through a V-belt with an angle $\beta = 18^\circ$ and a unit weight of 1.75 N/m (Fig. 8.6). The pulley on the motor shaft has a 0.1 m pitch radius and the angle of wrap is 170°. The maximum belt tension should be limited to 1000 N and the coefficient of friction is at least 0.3. Find the maximum power that can be transmitted by the smaller pulley of the V-belt drive.

Solution.

The speed of the belt in m/s is

$$V = \frac{\pi\,d\,n}{60} = \frac{\pi\,(0.2)\,(3000)}{60} = 31.415 \text{ m/s}, \tag{8.17}$$

where $d = 2\,r = 2\,(0.1) = 0.2$ m and $n = 3000$ rpm.

Equation (8.10) gives the tension created by the centrifugal force

$$T_c = m'\,V^2 = \left(0.178\frac{\text{kg}}{\text{m}}\right)\left(31.415\frac{\text{m}}{\text{s}}\right)^2 = 176.063 \text{ N}, \tag{8.18}$$

where m' is the mass unit length of belt:

$$m' = \frac{1.75 \text{ N/m}}{9.81 \text{ m/s}^2} = 0.178 \text{ kg/m}. \tag{8.19}$$

From Eq. (8.11), with $T_1 = T_{max} = 1000$ N, the tension T_2 is

$$\begin{aligned} T_2 &= T_c + \frac{T_1 - T_c}{e^{\mu\,\phi/\sin\beta}} = 176.063 + \frac{1000 - 176.063}{e^{0.3\,(170)\left(\frac{\pi}{180}\right)/\sin\left[18\left(\frac{\pi}{180}\right)\right]}} \\ &= 222.292 \text{ N}. \end{aligned}$$

The moment on the pulley is

$$M = (T_1 - T_2)\,r = (1000 - 222.292)\,(0.1) = 77.770 \text{ N}\cdot\text{m}. \tag{8.20}$$

The power transmitted by the pulley is

$$H = \frac{M\,n}{9549} = \frac{77.770\ (3000)}{9549} = 24.433\ \text{kW}. \tag{8.21}$$

Example 8.5. A 30 hp, 2000 rpm electric motor drives a machine through a multiple V-belt as shown in Fig. 8.7. The belts have an angle $\beta = 18°$ and a unit weight of 0.012 lb/in. The pulley on the motor shaft has a diameter of 6 in. and the angle of wrap is 165°. The maximum belt tension should be limited to 110 lb and the coefficient of friction is at least 0.2. Determine how many belts are required.

Solution.

The speed of the belt in m/s is

$$V = \frac{\pi\,d\,n}{60} = \frac{\pi\,(6)\,(2000)}{60} = 628.319\ \text{in./s}.$$

Equation (8.10) gives the tension created by the centrifugal force:

$$T_c = m'\,V^2 = \left(0.000031\frac{\text{lb}\cdot\text{s}^2}{\text{in.}^2}\right)\left(628.319\frac{\text{in.}}{\text{s}}\right)^2 = 12.260\ \text{lb},$$

where m' is the mass unit length of belt:

$$m' = \frac{0.012\ \text{lb/in.}}{(32.2\ \text{ft/s}^2)(12\ \text{in./ft})} = 0.000031\ \text{lb}\cdot\text{s}^2/\text{in}^2. \tag{8.22}$$

From Eq. (8.11), with $T_1 = T_{max} = 110$ lb, the tension T_2 is

$$\begin{aligned} T_2 &= T_c + \frac{T_1 - T_c}{e^{\mu\,\phi/\sin\beta}} = 12.260 + \frac{110 - 12.260}{e^{0.2\,(165)\left(\frac{\pi}{180}\right)/\sin\left[18\left(\frac{\pi}{180}\right)\right]}} \\ &= 27.417\ \text{lb}. \end{aligned}$$

The moment on the pulley is

$$M = (T_1 - T_2)\,d/2 = (110 - 27.417)\,(6/2) = 247.748\ \text{lb}\cdot\text{in}.$$

The power per belt transmitted by the pulley is

$$H = \frac{M\,n}{5252} = \frac{247.748\ (2000)}{5252\ (12)} = 7.862\ \text{hp/belt}.$$

The number of belts is

$$N = \frac{30}{7.862} = 3.815,$$

and 4 belts are needed.

8.4 Problems

8.1 The circular disk 1 is placed on top of disk 2 as shown in Fig. 8.8. The disk 2 is on a supporting surface 3. The diameters of 1 and 2 are 10 in. and 14 in., respectively. A compressive force of 100 lb acts on disk 1. The coefficient of friction between 1 and 2 is 0.30. Determine: a) the couple that will cause 1 to slip on 2; b) the minimum coefficient of friction between the disk 2 and the supporting surface 3 that will prevent 3 from rotating.

8.2 A shaft and a hoisting drum are used to raise the 600 kg load at constant speed as shown in Fig. 8.9. The diameter of the shaft is 40 mm and the diameter of the drum is 300 mm. The drum and shaft together have a mass of 100 kg and the coefficient of friction for the bearing is 0.3. Find the torque that must be applied to the shaft to raise the load.

8.3 The disks shown in Fig. 8.10 can be brought into contact under an axial force P. The pressure p between the disks follows the relation $p = k/r$, where k is a constant. The coefficient of friction mu is constant over the entire surface. Derive the expression for the torque M required to turn the upper disk on the fixed lower in therm of P, μ, and the inside and outside radii R_o and R_i.

8.4 The cable reel in Fig. 8.11 has a mass of 300 kg and a diameter of 600 mm and is mounted on a shaft with the diameter $d = 2\,r = 100$ mm. The coefficient of friction between the shaft and its bearing is 0.20. Find the horizontal tension T required to turn the reel.

8.5 For the V-belt in Fig. 8.3(c) derive the expression among the belt tension, the angle of contact β, and the coefficient of friction when slipping impends.

8.6 A cable supports a load of 200 kg and is subjected to a force $F = 600$ N which makes with the horizontal axis the angle θ, as shown in Fig. 8.12. The coefficient of friction between the cable and the fixed drum is 0.2. Find the minimum value of θ before the load begins to slip.

8.7 A band brake is shown in Fig. 8.13. The band itself is usually made of steel, lined with a woven friction material for flexibility. The drum has a clockwise rotation. The width of the band is b, the coefficient of friction is μ, and the angle of band contact is ϕ. Find the brake torque and the corresponding actuating force F if the maximum lining pressure is p_{max}. Use the following numerical application: $b = 80$ mm, $r = 300$ mm, $h = 150$ mm, $l = 800$ mm, $\phi = 270°$, $p_{max} = 0.6$ MPa, and $\mu = 0.3$.

8.8 Figure 8.14 shows a simple band brake operated by an applied force F of 250 N. The band is 30 mm wide and is lined with a woven material with a coefficient of friction of 0.4. The drum radius is $r = 550$ mm. Find the angle of wrap ϕ necessary to obtain a brake torque of 900 N·m and determine the corresponding maximum lining pressure.

8.9 A 25 hp, 1800 rpm electric motor drives a machine through a multiple V-belt as shown in Fig. 8.6. The belts have an angle $\beta = 18°$ and a unit weight of 0.012 lb/in. The pulley on the motor shaft has a diameter of 3.7 in. and the angle of wrap is 165°. The maximum belt tension should be limited to 200 lb and the coefficient of friction is at least 0.3. Determine how many belts are required.

8.10 A 3500 rpm motor drives a machine through a V-belt with an angle $\beta = 18°$ and a unit weight of 2.2 N/m (see Fig. 8.6). The pulley on the motor shaft has a 180 mm diameter and the angle of wrap is 160°. The maximum belt tension should be limited to 1300 N and the coefficient of friction is at least 0.33. Find the maximum power that can be transmitted by the smaller pulley of the V-belt drive.

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Figure captions

Figure 8.1. Simple disk clutch.

Figure 8.2 Disk clutch with ring friction disks.

Figure 8.3 a) Drum subjected to belt tensions; b) free-body diagram of an element of the belt; c) V-belt of angle β.

Figure 8.4. Disk brake for Example 8.1.

Figure 8.5. a) Differential band brake for Example 8.3; b) free-body diagrams.

Figure 8.6. V-belt drive for Example 8.4.

Figure 8.7. V-belt drive for Example 8.5.

Figure 8.8. Friction disks for Problem 8.1.

Figure 8.9. Hoisting drum for Problem 8.2.

Figure 8.10. Friction disks for Problem 8.3.

Figure 8.11. Cable reel for Problem 8.4.

Figure 8.12. Cable support for Problem 8.6.

Figure 8.13. Band brake for Problem 8.7.

Figure 8.14 Simple band brake for Problem 8.8.