Contents

7 Mechanical Springs

Springs are mechanical elements which exert forces or torques, and absorb energy. The absorbed energy is usually stored and later released. Springs are made of metal. For light loads the metal can be replaced by plastics. Some applications which require minimum spring mass use structural composites materials. Blocks of rubber can be used as springs, in bumpers and vibration isolation mountings of electric or combustion motors.

7.1 Material for Springs [20, 21]

The hot and cold working processes are used for springs manufacturing. Plain carbon steels, alloy steels, corrosion-resisting steels, or nonferrous materials can be used for spring manufacturing. Spring materials are compared by an examination of their tensile strengths which require the material, its processing, and the wire size. The tensile strength S_{ut} is a linear function of the wire diameter d, which is estimated by

$$S_{ut} = \frac{A}{d^m}, \tag{7.1}$$

where the constant A and the exponent m are presented in Table 7.1.

The torsional yield strength can be obtained by assuming that the tensile yield strength is between 60 and 90 percent of the tensile strength. Using the distortions-energy theory the torsional yield strength is

$$S_{sy} = 0.5777\, S_y, \tag{7.2}$$

and for steels it is

$$0.35 S_{ut} \leq S_{sy} \leq 0.52 S_{ut}. \tag{7.3}$$

For static application, the *maximum allowable torsional stress* τ_{all} may be used instead of S_{sy}

$$S_{sy} = \tau_{all} = \begin{cases} 0.45 S_{ut} & \text{cold-drawn carbon steel;} \\ 0.50 S_{ut} & \text{hardened and tempered carbon} \\ & \text{and low-alloy steel;} \\ 0.35 S_{ut} & \text{austenitic stainless steel} \\ & \text{and nonferrous alloys.} \end{cases} \tag{7.4}$$

Figure 7.1 shows the minimum tensile strength of commonly used spring wire materials.

7.2 Helical Extension Springs

Extension springs [Fig. 7.2(a)] are used for maintaining the torsional stress in the wire. The initial tension is the external force, F, applied to the spring. Spring manufacturers recommended that the initial tension be

$$\tau_{initial} = (0.4 \; - \; 0.8)\,\frac{S_{ut}}{C}, \tag{7.5}$$

where S_{ut} is the tensile strength in psi. The constant C is the spring index, defined by $C = \dfrac{D}{d}$, where D is the mean diameter of the coil and d is the diameter of the wire [Fig. 7.2(a)].
The bending stress, which occurs in section $A-A$, is

$$\sigma = \frac{16FD}{\pi d^3}\left(\frac{r_1}{r_3}\right), \tag{7.6}$$

and torsional stress which occurs in section $B-B$, is

$$\tau = \frac{FD}{\pi d^3}\left(\frac{r_4}{r_2}\right). \tag{7.7}$$

In practical application the radius r_4 is greater than twice the wire diameter. Hook stresses can be further reduced by winding the last few coils with a decreasing diameter D [Fig. 7.2(b)]. This lowers the nominal stress by reducing the bending and torsional moment arms.

7.3 Helical Compression Springs

The helical springs are usually made of circular cross-section wire or rod (Fig. 7.3). These springs are subjected to a torsional component and to a shear component. There is also an additional stress effect due to the curvature of the helix.

Shear stress, τ

The total shear stress, τ (psi), induced in a helical spring is

$$\tau = \frac{Tr}{J} + \frac{F}{A} = \frac{16\,T}{\pi d^3} + \frac{4\,F}{\pi d^2} = \frac{8\,F\,D}{\pi d^3} + \frac{4F}{\pi d^2}, \tag{7.8}$$

where
$T = FD/2$, is the torque, lb·in.,

$r = d/2$ is the wire radius, in.,
F is the axial load, lb,
$A = \pi d^2/4$ is cross-section area, in.^{2}, and
$J = \pi d^4/32$ is the polar second moment of inertia, in in.^{4}
The shear stress expressed in Eq. (7.8) can be rewritten as

$$\tau = K_s \frac{8FD}{\pi d^3}, \tag{7.9}$$

where K_s is the shear stress multiplication factor

$$K_s = \frac{2C+1}{2C}. \tag{7.10}$$

The spring index $C = \dfrac{D}{d}$ is in the range 6 to 12.

Curvature effect

The curvature of the wire increases the stress on the inside of the spring and decreases it on the outside. The stress equation is a function of the factor K_s which can be replaced by a correction factor K_B or K_w

$$\tau = K_B \frac{8FD}{\pi d^3}, \quad \text{or} \quad \tau = K_w \frac{8FD}{\pi d^3}, \tag{7.11}$$

where K_B is called the Bergstrasser factor (preferred factor),

$$K_B = \frac{4C+2}{4C-3}, \tag{7.12}$$

and K_w is the Wahl factor and is given by

$$K_w = \frac{4C-1}{4C-4} + \frac{0.615}{C}. \tag{7.13}$$

The factor K_B or K_w corrects both curvature and direct shear effects. The effect of the curvature alone is defined by the curvature correction factor K_c. which can be obtained as

$$K_c = \frac{K_B}{K_s}. \tag{7.14}$$

Deflection, δ

The deflection-force relations are obtained using Castigliano's theorem. The total strain energy for a helical spring is

$$U = U_t + U_s = \frac{T^2 l}{2GJ} + \frac{F^2 l}{2AG}, \tag{7.15}$$

where

$$U_t = \frac{T^2 l}{2GJ}, \tag{7.16}$$

is the torsional component of the energy, and

$$U_s = \frac{F^2 l}{2AG}, \tag{7.17}$$

is the shear component of the energy. The spring load is F, the torsion torque is T, the length of the wire is l, the second moment of inertia is J, the cross-section area of the wire is A, and the modulus of rigidity is G.
Substituting $T = FD/2$, $l = \pi DN$, $J = \pi d^4/32$, and $A = \pi d^2/4$ in Eq. (7.15), one may obtain the total strain energy as

$$U = \frac{4F^2 D^3 N}{d^4 G} + \frac{2F^2 DN}{d^2 G}, \tag{7.18}$$

where $N = N_a$ is the number of active coils.
Applying Castigliano's theorem, the deflection of the helical spring is

$$\delta = \frac{\partial U}{\partial F} = \frac{8FD^3 N}{d^4 G} + \frac{4FDN}{d^2 G}. \tag{7.19}$$

Using the spring index $C = D/d$, the deflection becomes

$$\delta = \frac{8FD^3 N}{d^4 G}\left(1 + \frac{1}{2C^2}\right) \approx \frac{8FD^3 N}{d^4 G}. \tag{7.20}$$

Spring rate

The general relationship between force and deflection can be written as

$$F = F(\delta). \tag{7.21}$$

The *spring rate* is defined as

$$k(\delta) = \lim_{\Delta\delta \to 0} \frac{\Delta F}{\Delta \delta} = \frac{dF}{d\delta}, \tag{7.22}$$

where δ must be measured in the direction of the load F and at the point of application of F. Because most of the force-deflection equations that treat the springs are linear, k is constant and is named the *spring constant*. For this reason Eq. (7.22) may be written as

$$k = \frac{F}{\delta}. \tag{7.23}$$

From Eq. (7.20), with the substitution $C = D/d$, the spring rate for a helical spring under an axial load is

$$k = \frac{G\ d^4}{8\ D^3 N} = \frac{G\ d}{8\ C^3 N}. \tag{7.24}$$

For springs in parallel having individual spring rates, k_i [Fig. 7.4(a)] the spring rate k is

$$k = k_1 + k_2 + k_3. \tag{7.25}$$

For springs in series, with individual spring rates, k_i [Fig. 7.4(b)] the spring rate k is

$$k = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2} + \dfrac{1}{k_3}}. \tag{7.26}$$

Spring ends

For helical springs the ends can be specified as shown in Fig. 7.5: (a) plain ends; (b) plain and ground ends; (c) squared ends; (d) squared and ground ends. A spring with plain ends [Fig. 7.5(a)] has a noninterupted helicoid and the ends are the same as if a long spring had been cut into sections. A spring with plain and ground ends [Fig. 7.5(b)] or squared ends [Fig. 7.5(c)] is obtained by deforming the ends to a zero-degree helix angle. Springs should always be both squared and ground [Fig. 7.5(d)] because a better transfer of the load is obtained. Table 7.2 presents the type of ends and how that affects the number of coils and the spring length. In Table 7.2, N_a is the number of active coils, and d is the wire diameter.

Stability

The springs in compression will buckle when the deflection is too large. Figure 7.6 gives the stability zones for two end conditions.

7.4 Torsion Springs

The helical torsion springs (Fig. 7.7) are used in door hinges, automobile starters, and for any application where torque is required. Torsion springs are of two general types: helical [Fig. 7.8(a)] and spiral [Fig. 7.8(b)]. The primary stress in torsion springs is bending. The bending moment $F\,a$ is applied to each end of the wire. The highest stress acting inside of the wire is

$$\sigma_i = \frac{K_i M c}{I}, \tag{7.27}$$

where the factor for inner surface stress concentration K_i is given in Fig. 7.9, and I is the moment of inertia. The distance from the neutral axis to the extreme fiber for a round solid bar is $c = d/2$, and for a rectangular bar is $c = h/2$.
For a solid round bar section $I = \pi d^4/64$, and for a rectangular bar $I = bh^3/12$.
Substituting the product $F\,a$ for bending moment and the equations for section properties of round and rectangular wire one may write,
Round wire

$$\frac{I}{c} = \frac{\pi d^3}{32}, \quad \sigma_i = \frac{32 F a}{\pi\ d^3} K_{i,round}. \tag{7.28}$$

Rectangular wire

$$\frac{I}{c} = \frac{bh^2}{6}, \quad \sigma_i = \frac{6\ F a}{bh^2} K_{i,rectangular}. \tag{7.29}$$

The angular deflection of a beam subjected to bending is

$$\theta = \frac{ML}{EI}, \tag{7.30}$$

where M is the bending moment, L the beam length, E the modulus of elasticity, and I the momentum of inertia.
Equation (7.30) can be used for helical and spiral torsion springs. Helical torsion springs and spiral springs can be made from thin rectangular wire. Round wire is often used in noncritical applications.

7.5 Torsion Bar Spring

The torsion bar spring, shown in Fig. 7.10, is used in automotive suspension. The stress, angular deflection, and spring rate equation are

$$\tau = \frac{Tr}{J}, \tag{7.31}$$

$$\theta = \frac{Tl}{JG}, \tag{7.32}$$

$$k = \frac{JG}{l}, \tag{7.33}$$

where T is the torque, $r = d/2$ is the bar radius, l is the length of the spring, G is the modulus of rigidity, and J is the second polar moment of area. For a solid round section, J is

$$J = \frac{\pi\, d^4}{32}. \tag{7.34}$$

For a solid rectangular section

$$J = \frac{bh^3}{12}. \tag{7.35}$$

For solid round rod of diameter d, Eqs. (7.31), (7.32), and (7.33), become

$$\tau = \frac{16T}{\pi d^3}, \tag{7.36}$$

$$\theta = \frac{32Tl}{\pi d^4 G}, \tag{7.37}$$

$$k = \frac{\pi d^4 G}{32l}. \tag{7.38}$$

7.6 Multi-Leaf Spring

The multi-leaf spring can be a simple cantilever [Fig. 7.11(a)] or the semi-elliptic leaf [Fig. 7.11(b)]. The design of the multi-leaf springs is based on force, F, length, L, deflection, and stress relationships. The multi-leaf spring may be considered as a triangular plate [Fig. 7.12(a)] cut into n strips of width b, or stacked in a graduated manner [Fig. 7.12(b)].

To support transverse shear N_e more extra full length leaves are added on the graduated stack, as shown in Fig. 7.13. The number N_e is always one less than the total number of full-length leaves, N.

The pre-stressed leaves have a different radius of curvature than the graduated leaves. This will leave a gap h between the extra full-length leaves and the graduated leaves before assembly (Fig. 7.14).

Bending stress, σ_e

The bending stress in the extra full-length leaves installed without initial pre-stress is

$$\sigma_e = \frac{18FL}{bt^2(3N_e + 2N_g)}, \tag{7.39}$$

where F is the total applied load at the end of the spring (lb), L is the length of the cantilever or half the length of the semi-elliptic spring (in.), b is the width of each spring leaf (in.), t is the thickness of each spring leaf (in.), N_e is the number of extra full length leaves, and N_g is the number of graduated leaves.

Bending stress, σ_g

For graduated leaves assembled with extra full-length leaves without initial pre-stress, the bending stress is

$$\sigma_g = \frac{12\,F\,L}{bt^2(3N_e + 2N_g)} = \frac{2\sigma_e}{3}. \tag{7.40}$$

Deflection of a multi-leaf spring, δ

The deflection of a multi-leaf spring with graduated and extra full-length leaves is

$$\delta = \frac{12Fl^3}{bt^2E(3N_e + 2N_g)}, \tag{7.41}$$

where E is the modulus of elasticity (psi0.

Bending stress, σ

The bending stress of multi-leaf springs without extra leaves or with extra full-length pre-stressed leaves which have the same stress after the full load has been applied is

$$\sigma = \frac{6Fl}{Nbt^2}, \tag{7.42}$$

where N is the total number of leaves.

Gap

The gap between pre-assembled graduated leaves and extra full-length leaves (Fig. 7.14) is

$$h = \frac{2FL^3}{Nbt^3E}, \tag{7.43}$$

7.7 Belleville Springs

The Belleville springs are made from tapered washers [Fig. 7.15(a)], stacked in series, parallel, or a combination of parallel-series, as shown in Fig. 7.15(b). The load-deflection and stress-deflection are

$$F = \frac{E\ \delta}{(1-\mu^2)\,(d_o/2)^2 M}[(h-\delta/2)(h-\delta)t + t^3], \tag{7.44}$$

$$\sigma = \frac{E\ \delta}{(1-\mu^2)\,(d_o/2)^2 M}[C_1(h-\delta/2) + C_2 t], \tag{7.45}$$

where F is the axial load (lb), δ is the deflection (in.), t is the thickness of the washer (in.), h, is the free height minus thickness (in.), E is the modulus of elasticity (psi), σ is the stress at inside circumference (psi.), d_o is the outside diameter of washer (in.), d_i is the inside diameter of washer (in.), and μ is the Poisson's ratio. The constants M, C_1, and C_2 are given by the equations

$$M = \frac{6}{\pi\ log_e(d_o/d_i)}\left(\frac{d_o/d_i - 1}{d_o/d_i}\right)^2,$$

$$C_1 = \frac{6}{\pi\ log_e(d_o/d_i)}\left[\frac{d_o/d_i - 1}{log_e(d_o/d_i)} - 1\right],$$

$$C_2 = \frac{6}{\pi\ log_e(d_o/d_i)}\left[\frac{d_o/d_i - 1}{2}\right].$$

7.8 Elastic Potential Energy and Virtual Work

A particle in static equilibrium position is considered. The static equilibrium position of the particle is determined by the forces that act on it. The *virtual displacement*, $\delta \mathbf{r}$, is any arbitrary small displacement away from this natural position and consistent with the system constraints. The term virtual is used to indicate that the displacement does not really exist but only is assumed to exist. The *virtual work* is the work done by any force $\mathbf{F}$ acting on the particle during the virtual displacement $\delta \mathbf{r}$:

$$\delta U = \mathbf{F} \cdot \delta \mathbf{r} = F \, \delta r \, \cos \alpha,$$

where α is the angle between $\mathbf{F}$ and $\delta \mathbf{r}$ ($|\delta \mathbf{r}| = \delta r$). The actual infinitesimal change in position $d\mathbf{r}$ can be integrated and the infinitesimal virtual or assumed movement $\delta \mathbf{r}$ cannot be integrated. Mathematically, both quantities are first-order differentials. A virtual displacement may also be a rotation $\delta\theta$ of a body. The virtual work done by a couple M during a virtual angular displacement $\delta\theta$ is $\delta U = M \, \delta\theta$. The force F or couple M remain constant during any infinitesimal virtual displacement.

Consider a particle in equilibrium position as a result of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, ... , $\mathbf{F}_n$. For an assumed virtual displacement $\delta \mathbf{r}$ of the particle away from its equilibrium position, the total virtual work done on the particle is

$$\delta U = \Sigma \mathbf{F} \cdot \delta \mathbf{r} = \Sigma F_x \, \delta x + \Sigma F_y \, \delta y + \Sigma F_z \, \delta z = 0.$$

The sum is zero, since $\Sigma \mathbf{F} = \mathbf{0}$. The equation $\delta U = 0$ is therefore an alternative statement of the equilibrium conditions for a particle. This condition of zero virtual work for equilibrium is both necessary and sufficient.

The principle of virtual work for a single particle can be extended to a rigid body treated as a system of small elements or particles rigidly attached to one another. Because the virtual work done on each particle of the body in equilibrium is zero, it results that the virtual work done on the entire rigid body is zero.

All the internal forces appear in pairs of equal, opposite, and collinear forces, and the net work done by these forces during any movement is zero. Only the virtual work done by external forces are taken into account in the evaluation of $\delta U = 0$ for the entire body.

The principle of virtual work will be extended to the equilibrium of an interconnected ideal system of rigid bodies. The *ideal systems* are systems composed of two or more rigid bodies linked together by mechanical connections which are incapable of absorbing energy through elongation or compression, and in which friction is small enough to be neglected. There are two types of forces which act in such an interconnected system:

• active forces are external forces capable of doing virtual work during possible virtual displacements;

• joint forces are forces in the connections between members. During any possible movement of the system or its parts, the net work done by the joint forces at the connections is zero, because the joint forces always exist in pairs of equal and opposite forces.

Principle of Virtual Work: The work done by external active forces on an ideal mechanical system in equilibrium is zero for any and all virtual displacements consistent with the constraints.

Mathematically, the principle can be expressed as

$$\delta U = 0. \tag{7.46}$$

The advantage of the method of virtual work is that relations between the active forces can be determined directly without reference to the joint forces. The method is useful in determining the position of equilibrium of a system under known forces. The method of virtual work cannot be applied for the system where the internal friction in a mechanical system is appreciable (the work done by internal friction should be included).

Elastic potential energy

The work done on an elastic body is stored in the body in the form of elastic potential energy, V_e. The potential energy is available to do work on some other body during the compression or extension. A spring can store and release potential energy. Consider a spring that is being compressed by a force F. The spring is elastic and linear, and the force F is directly proportional to the deflection x:

$$F = k\,x,$$

where k is the spring constant or stiffness of the spring. The work done on the spring by F during dx is

$$dU = F\,dx.$$

The elastic potential energy of the spring for a compression x is the total work done on the spring:

$$V_e = \int_0^x F\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2.$$

For an increase in the compression of the spring from x_1 to x_2, the work done on the spring equals its change in elastic potential energy:

$$\Delta V_e = \int_{x_1}^{x_2} kx\, dx = \frac{1}{2}k(x_2^2 - x_1^2).$$

During a virtual displacement δx of the spring, the virtual work done on the spring is the virtual change in elastic potential energy

$$\delta V_e = F\,\delta x = kx\,\delta x.$$

When the spring is in tension rather than compression, the work and energy relations are the same as those for compression, where x represents the elongation of the spring rather than its compression.

A torsional spring resists the rotation of a shaft or another body and the resisting moment is

$$M = K\theta,$$

where K is the torsional stiffness. The potential energy becomes

$$V_e = \int_0^{\theta} K\theta\, d\theta = \frac{1}{2}K\theta^2,$$

which is analogous to the expression for the linear extension spring.
The units of elastic potential energy are the same as those of work and are expressed in joules (J) in SI units and in foot-pounds (ft·lb) in U.S. customary units.

Gravitational potential energy

For an upward displacement δh of a body, the weight $W = mg$ does negative work, $\delta U = -mg\,\delta h$. If the body has a downward displacement δh the weight does positive work, $\delta U = +mg\,\delta h$. The gravitational potential energy V_g of a body is defined as the work done on the body by a force equal and opposite to the weight in bringing the body to the position under consideration from some arbitrary datum plane where the potential energy is defined to be zero. The potential energy is the negative of the work done by the weight. If $V_g = 0$ at $h = 0$ (datum plane) then at a height h above the datum plane, the gravitational potential energy of the body is $V_g = mgh$. If the body is a distance h below the datum plane, its gravitational potential energy is $-mgh$.

Remarks:

1. the datum plane for zero potential energy is arbitrary because only the change in potential energy matters, and this change is the same no matter where the datum plane is located;

2. the gravitational potential energy is independent of the path followed in arriving at a particular level h.

The virtual change in gravitational potential energy is

$$\delta V_g = mg\,\delta h,$$

where δh is the upward virtual displacement of the mass center of the body. The units of gravitational potential energy are the same as those for work and elastic potential energy, joules (J) in SI units and foot-pounds (ft·lb) in U.S. customary units.

Consider a linear spring attached to a body of mass m. The work done by the linear spring on the body is the negative of the change in the elastic potential energy of the spring. The work done by the gravitational force or weight mg is the negative of the change in gravitational potential energy.

The total virtual work δU is the sum of the work $\delta U'$ done by all active forces (other than spring forces and weight forces) and the work $-(\delta V_e + \delta V_g)$ done by the spring and weight forces. The virtual work equation $\delta U = 0$ becomes

$$\delta U' - (\delta V_e + \delta V_g) = 0, \quad \text{or} \quad \delta U' = \delta V, \tag{7.47}$$

where $V = V_e + V_g$ is the total potential energy of the system.

Thus, for a mechanical system with elastic members and bodies which undergo changes in position the principle of virtual work is:

For a mechanical system in equilibrium the virtual work done by all external forces (other than the gravitational and spring forces) equals the change in the elastic and potential energy of the system for any virtual displacements consistent with the constraints.

Stability of equilibrium

Consider a mechanical system where no work is done on the system by nonpotential forces. With $\delta U' = 0$, the virtual work relation becomes

$$\delta(V_e + V_g) = 0, \quad \text{or} \quad \delta V = 0. \tag{7.48}$$

Equation (7.48) expresses that the equilibrium configuration of a mechanical system is one for which the total potential energy V of the system has a stationary value.

For a system of one degree of freedom where the potential energy and its derivatives are continuous functions of the single variable, x, which describes the configuration, the equilibrium condition $\delta V = 0$ is equivalent mathematically to

$$\frac{dV}{dx} = 0. \tag{7.49}$$

Equation (7.49) states that a mechanical system is in equilibrium when the derivative of its total potential energy is zero. For systems with several degrees of freedom, the partial derivative of V with respect to each coordinate in turn must be zero for equilibrium.

There are three conditions under which Eq. (7.49) applies:

- the total potential energy is a minimum (stable equilibrium),
- the total potential energy is a maximum (unstable equilibrium),
- the total potential energy is constant (neutral equilibrium).

When a function and its derivatives are continuous, the second derivative is positive at a point of minimum value of the function and negative at a point of maximum value of the function. Thus, the mathematical conditions for equilibrium and stability of a system with a single degree of freedom x are:

$$\text{equilibrium} : \frac{dV}{dx} = 0; \quad \text{stable} : \frac{d^2V}{dx^2} > 0; \quad \text{unstable} : \frac{d^2V}{dx^2} < 0.$$

7.9 Examples

Example 7.1. A hardened and oil-tempered steel wire is used for a helical compression spring. The wire diameter is $d = 0.105$ in., and the outside diameter of the spring is $D_0 = 1.225$ in. The ends are plain and the number of total turns is $N_t = 8$. Find: a) the torsional yield strength; b) the static load corresponding to the yield strength; c) the rate of the spring; d) the deflection that would be caused by the static load found in part b); e) the solid length of the spring; f) the length of the spring so that no permanent change of the free length occurs when the spring is compressed solid and then released; g) the pitch of the spring for the free length, and h) the stability of the spring.

Solution.

a) From Eq. (7.4), the torsional yield strength for hardened and tempered carbon and low-alloy steel is

$$S_{sy} = 0.50 S_{ut}.$$

The minimum tensile strength given from Eq. (7.1) is

$$S_{ut} = \frac{A}{d^m},$$

where, from Table 7.1, the constant $A = 146$ kpsi and the exponent $m = 0.193$. The minimum tensile strength is

$$S_{ut} = \frac{A}{d^m} = \frac{146}{(0.105)^{0.193}} = 225.561 \text{ kpsi}.$$

The torsional yield strength is

$$S_{sy} = 0.50\, S_{ut} = 0.50\,(225.561) = 112.78 \text{ kpsi}.$$

b) To calculate the static load F corresponding to the yield strength it is necessary to find the spring index, C, and the shear stress correction factor, K_s. The mean diameter D is the difference between the outside diameter and the wire diameter d:

$$D = D_0 - d = 1.225 - 0.105 = 1.12 \text{ in}.$$

The spring index is

$$C = \frac{D}{d} = \frac{1.12}{0.105} = 10.666$$

From Eq. (7.10), the shear stress correction factor is

$$K_s = \frac{2C+1}{2C} = \frac{2\ (10.666)+1}{2\ (10.666)} = 1.046.$$

Using the torsional yield strength instead of shear stress, Eq. (7.8) gives the static load:

$$F = \frac{\pi d^3 S_{sy}}{8\ K_s D} = \frac{\pi\ (0.105^3)\ (112.78)(10^3)}{8\ (1.046)\ (1.12)} = 43.726 \text{ lb.}$$

c) From Table 7.2 , the number of active coils is $N_a = N_t = 8$.
For $N = N_a$, the spring rate is calculated using Eq. (7.24)

$$k = \frac{Gd}{8C^3 N_a} = \frac{(11.5)(10^6)\ (0.105)}{8\ (10.666^3)\ (8)} = 15.546 \text{ lb/in,}$$

where $G = 11.5$ Mpsi.

d) The deflection of the spring is

$$\delta = \frac{F}{k} = \frac{43.726}{15.546} = 2.812 \text{ in.}$$

e) The solid length, L_s, is calculated using Table 7.2:

$$L_s = d\,(N_t + 1) = 0.105\,(8+1) = 0.945 \text{ in.}$$

f) The free length of the spring is the solid length plus the deflection caused by the load,

$$L_0 = \delta + L_s = 2.812 + 0.945 = 3.757 \text{ in.}$$

f) From Table 7.2 the pitch, p, is calculated with the relation

$$p = \frac{L_0 - d}{N_a} = \frac{3.757 - 0.105}{8} = 0.456 \text{ in.}$$

g) Buckling is checked for the worst case of deflection:

$$\frac{\delta}{L_0} = \frac{2.812}{3.757} = 0.748 \quad \text{and} \quad \frac{\delta}{D} = \frac{2.812}{1.12} = 2.511.$$

Reference to Fig. 7.6 indicates that this spring is outside the buckling region, even if one end plate is free to tip.

Example 7.2. In a vertical plane two uniform links, each of mass m and length l, are connected and constrained as shown in Fig. 7.16(a). The spring is not stretched when the links are horizontal ($\theta = 0$). The angle θ increases with the application of the known horizontal force F. Determine the spring stiffness k which will produce equilibrium at a given angle θ.

Solution.

The ideal mechanical system has one degree of freedom. The displacement of every link can be expressed in terms of the angle θ.
The spring deflection is

$$x = 2\,l - 2\,l\,\cos\theta = 2\,l\,(1 - \cos\theta).$$

The force diagram is shown in Fig. 7.16(b). The joint forces are not included in the diagram. The elastic potential energy of the spring is

$$V_e = \frac{1}{2} k\,x^2 = 2\,k\,l^2\,(1 - \cos\theta)^2.$$

The virtual change in elastic potential energy is

$$\delta V_e = \delta\left[2\,k\,l^2\,(1 - \cos\theta)^2\right] = 2\,k\,l^2\,\delta(1 - \cos\theta)^2 = 4\,k\,l^2\,(1 - \cos\theta)\,\sin\theta\,\delta\theta.$$

The gravitational potential energy is

$$V_g = -2\,m\,g\,h = -2\,m\,g\,\left(\frac{l}{2}\sin\theta\right) = -m\,g\,l\,\sin\theta.$$

The datum for zero gravitational potential energy was taken through the support at A. The virtual change in gravitational potential energy is

$$\delta V_g = \delta(-m\,g\,l\,\sin\theta) = -m\,g\,l\,\cos\theta\,\delta\theta.$$

The virtual work done by the active external force F is

$$\delta U' = F\,\delta = F\,\delta\,[2\,l\,(1 - \cos\theta)] = 2\,F\,l\,\delta(1 - \cos\theta) = 2\,F\,l\,\sin\theta\,\delta\theta.$$

The virtual work equation $\delta U' = \delta V_e + \delta V_g$ gives

$$2\,F\,l\,\sin\theta\,\delta\theta = 4\,k\,l^2\,(1 - \cos\theta)\,\sin\theta\,\delta\theta - m\,g\,l\,\cos\theta\,\delta\theta.$$

The stiffness of the spring is

$$k = \frac{F\,\sin\theta + m\,g\,\cos\theta}{2\,k\,(1-\cos\theta)\,\sin\theta}\,.$$

Example 7.3. Figure 7.17 shows a uniform bar of mass m and length l that slides freely in the vertical and horizontal directions. The spring has the stiffness k and is not compressed when the bar is vertical. Find the equilibrium positions and examine the stability. There are no external active forces.

Solution.

The ideal mechanical system has one degree of freedom and the displacement of the bar can be expressed in term of the angle θ. The spring is undeformed when $\theta = 0$. The datum for zero gravitational potential energy is the horizontal x-axis. The spring deflection is

$$y = l - l\,\cos\theta = l\,(1-\cos\theta).$$

The elastic potential energy of the spring is

$$V_e = \frac{1}{2}k\,y^2 = \frac{1}{2}k\,l^2\,(1-\cos\theta)^2.$$

The gravitational potential energy is

$$V_g = m\,g\,h = m\,g\,\left(\frac{l}{2}\cos\theta\right) = \frac{1}{2}m\,g\,l\,\cos\theta.$$

The total potential energy is

$$V = V_e + V_g = \frac{1}{2}k\,l^2\,(1-\cos\theta)^2 + \frac{1}{2}m\,g\,l\,\cos\theta.$$

The equilibrium position is obtained by differentiating the total potential energy and setting it to zero

$$\frac{d\,V}{d\,\theta} = kl^2(1-\cos\theta)\sin\theta - \frac{mgl\sin\theta}{2} = l\sin\theta\left[kl(1-\cos\theta) - \frac{mg}{2}\right] = 0.$$

The two solutions to this equation are the equilibrium positions:

$$\sin\theta = 0 \quad \text{and} \quad \cos\theta = 1 - \frac{mg}{2kl}.$$

The sign of the second derivative of the potential energy for each of the two equilibrium positions will determine the stability of the system. The second derivative of the total potential energy is

$$\frac{d^2 V}{d\,\theta^2} = kl^2 \sin^2\theta + kl^2(1-\cos\theta)\cos\theta - \frac{mgl\cos\theta}{2}.$$

Solution 1: $\sin\theta = 0$, $\theta = 0 \implies$

$$\frac{d^2 V}{d\,\theta^2} = 0 + kl^2(1-1)(1) - \frac{mgl}{2} = -\frac{mgl}{2} < 0.$$

Equilibrium for $\theta = 0$ is never stable.

Solution 2: $\cos\theta = 1 - \dfrac{mg}{2kl} \implies$

$$\frac{d^2 V}{d\,\theta^2} = m\,g\left(k - \frac{m\,g}{4k}\right).$$

For $k > m\,g/(4k) \implies d^2V/d\,\theta^2 > 0$ the equilibrium position is stable.
For $k < m\,g/(4k) \implies d^2V/d\,\theta^2 < 0$ the equilibrium position is unstable.

7.10 Problems

7.1 A helical compression spring is made of hard-drawn spring steel. The wire diameter is 2 mm and the outside diameter is 24 mm. There are 9 total coils. The ends are plain and ground. Determine: a) the free length when the spring is compressed solid; and the stress is not greater than the yield strength; b) the force needed to compress the spring to its solid length; c) the rate of the spring; d) the stability of the spring.

7.2 A helical compression spring with squared and ground ends has an outside diameter of 1 in. and a wire diameter of 0.074 in. The solid length is 0.9 in. and the free length is 3 in. Find the spring rate and the approximate load at the solid length.

7.3 A helical compression spring has squared and ground ends. A minimum force of 50 lb is applied to compress the spring and the length cannot exceed 3 in. As the force is increased to 120 lb. the length is 0.75 in. shorter. The outside diameter of the spring is $D_0 = 1.15$ in. and the wire diameter is $d = 0.157$ in. The number of total turns is 6.38. The spring material is oil-tempered ASTM 229 wire. Find: a) the torsional yield strength; b) the static load corresponding to the yield strength; c) the rate of the spring; d) the deflection that would be caused by the static load found in part b); e) the solid length of the spring; f) the length of the spring so that no permanent change of the free length occurs when the spring is compressed solid and then released; g) the pitch of the spring for the free length; and h) the stability of the spring.

7.4 A helical compression spring made of steel with closed ends has an outside diameter of 56 mm and a wire diameter of 3 mm. The number of total coils is 13 and the free length is 100 mm. Find: a) the spring rate; b) the force required to close the spring to its solid length and the stress due to this force.

7.5 Two bars, 1 and 2, each of mass m and length l are connected and constrained as shown in Fig. 7.18. The angle θ is between the link 1 and the vertical axes. The spring of stiffness k is not stretched in the position where $\theta = 0$. Find the force F which will produce equilibrium at the angle θ.

7.6 Figure 7.19 shows a mechanism with two links, 1 and 2. Link 1 has the mass $m_1 = m$ and the length $l_1 = l$. Link 2 has the mass $m_2 = 2\,m$ and the length $l_2 = 2\,l$. The spring is unstretched in the position $\theta = 0$. A known vertical force F is applied on link 2 at D. Determine the spring stiffness k which will establish an equilibrium at a given angle θ.

7.7 For the mechanism shown in Fig. 7.20, link 1 has the mass $m_1 = 2\,m$ and the length $l_1 = 2\,l$. The link 2 has the mass $m_2 = m$ and the length $l_2 = l$. The spring has an unstretched length of L_0. Determine the spring stiffness k for an equilibrium at a given angle θ and a given force F.

7.8 The link BC shown in Fig. 7.21 has a mass m and is connected to two springs ($AB = BC = l$). Each spring has the stiffness k and the unstretched length of the two springs is L_0. Determine the spring stiffness k which will establish an equilibrium at a given angle θ. Use the following numerical application: $l = L_0$ =300 mm, m =10 kg, and $\theta = 60°$.

7.9 The mechanism shown in Fig. 7.22, has the link BC with the mass m and the length l ($AB = AC = l/2$). The spring has the stiffness k and is unstretched when $\theta = 0$. Find the equilibrium value for the coordinate θ. Use the following numerical application: l =400 mm, m = 10 kg, F =70 N, and k = 1.8 kN/m .

7.10 The link of mass m and length l is connected to two identical horizontal springs, each of stiffness k, as shown in Fig. 7.23. The initial spring compression at $\theta = 0$ is d. For a stable equilibrium position at $\theta = 0$ find the minimum value of k.

7.11 The mechanism shown in Fig. 7.24 has two identical links, 1 and 2, each of length l and negligible mass compared with the mass m of the slider 3. The two light links have a torsion spring at their common joint. The moment developed by the torsion spring is $M = K\,\theta$, where θ is the relative angle between the links at the joint. Determine the minimum value of K which will ensure the stability of the mechanism for $\theta = 0$.

7.12 Figure 7.25 shows a four-bar mechanism with $AD = l$. Each of the links has the mass m ($m_1 = m_2 = m_3 = m$) and the lenght l ($l_1 = l_2 =$

$l_3 = l$). At B a vertical force F acts on the mechanism and the spring stiffness is k. The motion is in the vertical plane. Find the equilibrium angle θ. Use the following numerical application: $l = 15$ in., $m = 10$ lb, $F = 90$ lb, and $k = 15$ lb/in.

References

[1] M. Atanasiu, *Mechanics* [Mecanica], EDP, Bucharest, 1973.

[2] A. Bedford, and W. Fowler, *Dynamics*, Addison Wesley, Menlo Park, CA, 1999.

[3] A. Bedford, and W. Fowler, *Statics*, Addison Wesley, Menlo Park, CA, 1999.

[4] A. Ertas, J. C. Jones, *The Engineering Design Process*, John Wiley & Sons, New York, 1996.

[5] A. S. Hall, A. R. Holowenko, and H. G. Laughlin, *Theory and Problems of Machine Design*, Schaum's Outline Series, McGraw-Hill, New York, 1961.

[6] B. G. Hamrock, B. Jacobson, and S. R. Schmid, *Fundamentals of Machine Elements*, McGraw-Hill, New York, 1999.

[7] R. C. Juvinall and K. M. Marshek, *Fundamentals of Machine Component Design*, 3rd ed., John Wiley & Sons, New York, 2000.

[8] D. B. Marghitu, *Mechanical Engineer's Handbook*, Academic Press, San Diego, 2001.

[9] D. B. Marghitu, M. J. Crocker, *Analytical Elements of Mechanisms*, Cambridge University Press, Cambridge, 2001.

[10] D. B. Marghitu and E. D. Stoenescu, *Kinematics and Dynamics of Machines and Machine Design*, class notes, available at www.eng.auburn.edu/users/marghitu/, 2004.

[11] C. R. Mischke, "Prediction of Stochastic Endurance Strength," *Transaction of ASME, Journal Vibration, Acoustics, Stress, and Reliability in Design*, Vol. 109 (1), pp. 113-122, 1987.

[12] R. L. Mott, *Machine Elements in Mechanical Design*, Prentice Hall, Upper Saddle River, NJ, 1999.

[13] W. A. Nash, *Strength of Materials*, Schaum's Outline Series, McGraw-Hill, New York, 1972.

[14] R. L. Norton, *Machine Design*, Prentice-Hall, Upper Saddle River, NJ, 1996.

[15] R. L. Norton, *Design of Machinery*, McGraw-Hill, New York, 1999.

[16] W. C. Orthwein, *Machine Component Design*, West Publishing Company, St. Paul, 1990.

[17] I. Popescu, *Mechanisms*, University of Craiova Press, Craiova, Romania, 1990.

[18] C. A. Rubin, *The Student Edition of Working Model*, Addison–Wesley Publishing Company, Reading, MA, 1995.

[19] I. H. Shames, *Engineering Mechanics - Statics and Dynamics*, Prentice-Hall, Upper Saddle River, NJ, 1997.

[20] J. E. Shigley and C. R. Mischke, *Mechanical Engineering Design*, McGraw-Hill, New York, 1989.

[21] J. E. Shigley, C. R. Mischke, and R. G. Budynas, *Mechanical Engineering Design*, 7th ed., McGraw-Hill, New York, 2004.

[22] J. E. Shigley and J. J. Uicker, *Theory of Machines and Mechanisms*, McGraw-Hill, New York, 1995.

[23] A. C. Ugural, *Mechanical Design*, McGraw-Hill, New York, 2004.

[24] R. Voinea, D. Voiculescu, and V. Ceausu, *Mechanics* [Mecanica], EDP, Bucharest, 1983.

[25] J. Wileman, M. Choudhury, and I. Green, "Computation of Member Stiffness in Bolted Connections," *Journal of Machine Design*, Vol. 193, pp. 432-437, 1991.

[26] S. Wolfram, *Mathematica*, Wolfram Media/Cambridge University Press, Cambridge, 1999.

[27] National Council of Examiners for Engineering and Surveying (NCEES), *Fundamentals of Engineering. Supplied-Reference Handbook*, Clemson, SC, 2001.

[28] * * * ,*The Theory of Mechanisms and Machines* [Teoria mehanizmov i masin], Vassaia scola, Minsk, 1970.

[29] * * * ,*Working Model 2D, Users Manual*, Knowledge Revolution, San Mateo, CA, 1996.

Table 7.1. Constants of tensile strengths expression.

Material	m	A	
		kpsi	MPa
Music wire	0.163	186	2060
Oil-tempered wire	0.193	146	1610
Hard-drawn wire	0.201	137	1510
Chrome vanadium	0.155	173	1790
Chrome silicom	0.091	218	1960

Source: J. E. Shigley and C. R. Mischke, *Mechanical Engineering Design*, McGraw-Hill, New York, 1989.

Table 7.2. Type of spring ends.

Term	End coils, N_e	Total coil, N_t	Free length, L_0	Solid length, L_s	Pitch, p
Plain	0	N_a	$pN_a + d$	$d(N_t + 1)$	$(L_0 - d)/N_a$
Plain and ground	1	$N_a + 1$	$p(N_a + 1)$	dN_t	$L_0/(N_a + 1)$
Squared or closed	2	$N_a + 2$	$pN_a + 3d$	$d(N_t + 1)$	$(L_0 - 3d)/N_a$
Squared and ground	2	$N_a + 2$	$pN_a + 2d$	dN_t	$(L_0 - 2d)/N_a$

Source: J. E. Shigley and C. R. Mischke, *Mechanical Engineering Design*, McGraw-Hill, New York, 1989.

Figure captions

Figure 7.1. Tensile strength of spring wire materials.
Figure 7.2. Extension springs.
Figure 7.3. Helical compression spring.
Figure 7.4. (a) Springs in parallel; (b) springs in series.
Figure 7.5. Helical springs: (a) plain ends; (b) plain and ground ends; (c) squared ends; (d) squared and ground ends.
Figure 7.6. Stability zone for springs in compression.
Figure 7.7. Helical torsion springs.
Figure 7.8. Torsion springs: (a) helical and (b) spiral.
Figure 7.9. Stress concentration factor.
Figure 7.10. Torsion bar spring.
Figure 7.11. Multi-leaf spring: (a) simple cantilever and (b) semi-elliptic.
Figure 7.12. Multi-leaf spring considered as a triangular plate.
Figure 7.13. Multi-leaf spring: extra full length leaves.
Figure 7.14. Multi-leaf spring: gap between the extra full length leaves.
Figure 7.15. Belleville springs.
Figure 7.16. (a) Mechanism and (b) force diagram for Example 7.2.
Figure 7.17. Mechanism for Example 7.3.
Figure 7.18 Mechanism for Problem 7.5.
Figure 7.19 Mechanism for Problem 7.6.
Figure 7.20. Mechanism for Problem 7.7.
Figure 7.21. Problem 7.8.
Figure 7.22 Mechanism for Problem 7.9.
Figure 7.23 Problem 7.10.
Figure 7.24 Mechanism for Problem 7.11.
Figure 7.25. Mechanism for Problem 7.12.