Discontinuity in mechanics

In classical mechanics it was admitted the continuity of velocities, of accelerations and forces. The mechanical phenomena are continuous in time. There are cases when for very small time intervals there is a very large variation of velocity, but the displacement in space of the particle is very small.

If a sphere falls on a vertical spring, the sphere makes contact with the sphere. The spring compresses under the weight of the sphere. The compression phase ends when the velocity of the sphere is zero. Next phase is the restitution phase when the spring expanding and the sphere is moving upward. At the end of the restitution phase there is the separation of the sphere.

The x-axis is selected downward.

Assume that at moment $t = 0$ the sphere is in contact with the spring. At that moment the velocity of the sphere is

$$\mathbf{v}(t = 0) = \mathbf{v}_0 = v_0\, \mathbf{\imath}.$$

The equation of motion for the sphere in contact with the spring is

$$m\, \ddot{x} = m\, g - k\, x, \tag{1}$$

where k is the spring constant ($k > 0$).
The contact force due to the elastic force is

$$\mathbf{P} = -k\, x\, \mathbf{\imath}.$$

The initial conditions are

$$x(0) = 0,\ \dot{x}(0) = v_0.$$

The following notation is introduced

$$\frac{k}{m} = \omega^2, (\omega > 0),$$

and Eq. (1) becomes

$$\ddot{x} + \omega^2\, x = g. \tag{2}$$

Assume that the solution of Eq. (2) has the following form

$$x = a\, \cos(\omega\, t - \varphi_0) + b, \tag{3}$$

then

$$\dot{x} = -a\,\omega\,\sin(\omega\,t - \varphi_0),\;\; \ddot{x} = -a\,\omega^2\,\cos(\omega\,t - \varphi_0). \tag{4}$$

Substituting Eq. (4) in Eq. (2),

$$-a\,\omega^2\,\cos(\omega\,t - \varphi_0) + a\,\omega^2\,\cos(\omega\,t - \varphi_0) + b\,\omega^2 = g. \tag{5}$$

Hence,

$$b = \frac{g}{\omega^2}. \tag{6}$$

With the trigonometric transformation of $\cos(\omega\,t - \varphi_0)$ Eq. (3) becomes

$$x = a\,\cos\varphi_0\,\cos\,\omega t + a\,\sin\varphi_0\,\sin\,\omega t + b. \tag{7}$$

From initial conditions ($x(0) = 0$, $\dot{x}(0) = v_0$)

$$x(0) = a\,\cos\varphi_0 + b = 0, \dot{x}(0) = a\,\omega\sin\varphi_0 = v_0.$$

Hence,

$$a\,\cos\varphi_0 = -b = -\frac{g}{\omega^2},\;\; a\,\sin\varphi_0 = \frac{v_0}{\omega}. \tag{8}$$

Introducing Eq. (6) and Eq. (8) in Eq. (7), the solution of Eq. (1) is

$$x - \frac{g}{\omega^2} = -\frac{g}{\omega^2}\,\cos\,\omega\,t + \frac{v_0}{\omega}\sin\,\omega\,t.$$

With Eq. (6), Eq. (3) becomes

$$x - \frac{g}{\omega^2} = a\,\cos(\omega\,t - \varphi_0), \tag{9}$$

where from Eq. (8)

$$a^2\,(\cos^2\varphi_0 + \sin^2\varphi_0) = \frac{g^2}{\omega^4} + \frac{v_0^2}{\omega^2},\quad \frac{\sin\varphi_0}{\cos\varphi_0} = -\frac{v_0\,\omega}{g},$$

$$a = \sqrt{\frac{g^2}{\omega^4} + \frac{v_0^2}{\omega^2}},\;\; \tan\,\varphi_0 = -\frac{v_0\,\omega}{g},\;\; \varphi_0 = -\arctan\frac{v_0\,\omega}{g}. \tag{10}$$

If the sphere would be connected to the spring, it would oscillate around the position $x = \dfrac{g}{\omega^2}$. The sphere reaches its maximum position on x-axis at $t = t_1$ when $\dot{x}(t_1) = 0$. The derivative of Eq. (9) gives

$$\dot{x}(t_1) = -a\,\omega\,\sin(\omega\,t - \varphi_0) = 0. \tag{11}$$

Furthermore,

$$\omega\,t_1 - \varphi_0 = \pi \quad \text{or} \quad t_1 = \frac{\pi}{\omega} - \frac{1}{\omega}\,\varphi_0. \tag{12}$$

Substituting φ_0 from Eq. (10) in Eq. (12), the time t_1 is

$$t_1 = \frac{\pi}{\omega} - \frac{1}{\omega}\,\text{arctan}\,\frac{v_0\,\omega}{g}.$$

At the moment $t = t_2 = 2\,t_1$, the sphere attains again the reference O and $x(t)$ is minimum. At this moment, the sphere separates itself and moves upward, and the spring compresses. The velocity of the sphere at $t = t_2$ is

$$\dot{x}(t_2) = a\,\omega\,\sin(\omega\,t_2 - \varphi_0) = -v_0. \tag{13}$$

The contact time between the sphere and the spring is

$$t_2 = 2\,t_1 = \frac{2\,\pi}{\omega} - \frac{2}{\omega}\,\text{arctan}\,\frac{v_0\,\omega}{g}. \tag{14}$$

Using the initial condition $\dot{x}(0) = v_0$ and Eq. (13), the jump in velocity between the two moments is

$$\begin{aligned} &\Delta v = \dot{x}(0) - \dot{x}(t_2) = \\ &v_0 - (-v_0) = 2\,v_0. \end{aligned} \tag{15}$$

The relative displacement is

$$\lambda = x(0) - x(t_1). \tag{16}$$

Because

$$\dot{x}(t_1) = 0, \tag{17}$$

from Eq. (11),

$$
\begin{aligned}
&-a\,\sin(\omega\,t_1 - \varphi_0) = 0 \;\Rightarrow \\
&\omega\,t_1 - \varphi_0 = 0
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\cos(\omega\,t_1 - \varphi_0) = 1 \;\Rightarrow \\
&x(t_1) = a\,\cos(\omega\,t_1 - \varphi_0) + b = a + b,
\end{aligned}
$$

Substituting b from Eq. (6),

$$
\begin{aligned}
&x(t_1) = \frac{g}{\omega^2} + a. \;\Rightarrow \\
&\lambda = 0 - x(t_1) = -\left(\frac{g}{\omega^2} + a\right) = -\left(\frac{g}{\omega^2} + \sqrt{\frac{g^2}{\omega^4} + \frac{v_0^2}{\omega^2}}\right). \qquad (18)
\end{aligned}
$$

Numerical Example

The mass of the spring is m=10 kg, the elastic constant is k=294×10^3 N/m, and the sphere falls from h=1 m on the spring. The initial velocity is v_0=$\sqrt{2\,g\,h}$=4.42945 m/s. The total time of contact t_2 is calculated with Eq. (14) t_2=0.0184728 s. The relative displacement is $|\lambda|$=0.0261689 m. The jump in velocity between the two moments is calculated with Eq. (15), and is Δ=8.85889 m/s. The maximum elastic force in this case is

$$
P_{max} = k\,x(t_1) = k\,|\lambda| = 7693.65\text{N},
$$

76 greater then the weight of the sphere. In the mechanical process studied, the displacement of the sphere is very small, almost null, while the velocity jump is big.

Figure 2. represents the dependence of the displacement of the sphere with respect to time calculated with Eq. (9). Figure 3. shows the variation in time of the velocity of the sphere in contact with the spring. At $t = 0$ the sphere gets in contact with the spring, and at $t = t_2$, the sphere separates from the spring. Note that the initial velocity is equal with the absolute value of the final velocity. Figure 4 shows the variation of the elastic force in time, with its maximum at the time of maximum compression of the spring.