Impact of a sphere with a flat surface

A sphere of mass m and radius R is impacting a flat rigid surface with the initial velocity v_0. The sphere has the elastic modulus E_1 and the rigid flat surface has the elastic modulus $E_2 = E_1 = E$. The equivalent elastic modulus E' is calculated from

$$\frac{1}{E'} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \quad \text{or} \quad E' = \frac{E}{2(1-\nu^2)}\,, \tag{1}$$

where ν_1 and ν_2 are Poisson's ratios of the two materials in contact $\nu_1 = \nu_2 = \nu$ (same material). The equivalent radius for two spheres in contact is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{2}$$

For the the collision of the sphere with the radius R_1 against the rigid flat surface with the radius $R_2 \to \infty$ the equivalent radius is $R = R_1$.
An effective mass for two bodies in contact can be obtained from the definition

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{3}$$

For the the collision of the sphere with the the mass m_1 against the rigid flat surface with the mass $m_2 \to \infty$ the effective mass is $m = m_1$.

The interference, x, is the distance the sphere is displaced normally into the rigid flat. The Hertz solution assumes that the interference is small enough such that the geometry does not change signifiable.

The motion of the impact point during the collision can be specified by one of the following three cases:

I. Elastic compression phase

This case starts with the contact moment (the contact force P is zero, $P = 0$) and ends when the contact force reaches the known value of the critical force ($P = P_c$). For the critical force the deformation is the critical deformation ($x = x_c$).

During this case there are only elastic deformations $x = x_e$ ($x_e \le x_c$) and the Hertz law is applied

$$P = k_1 x^{3/2}, \tag{4}$$

where

$$k_1 = \frac{4}{3} E' \sqrt{R} = \frac{2\,E\,\sqrt{R}}{3(1-\nu^2)}. \tag{5}$$

The equation of motion of the sphere is

$$m\,\ddot{x} = m\,g - P \quad \text{or} \quad m\,\ddot{x} = m\,g - k_1\,x^{3/2}. \tag{6}$$

The initial conditions are at t=0, $x(0)$=0, and $\dot{x}(0)$=v_0.

As the contact force P increases the stresses also increase. These stresses eventually cause the material within the sphere to yield. The distance at this initial point of yielding is known as the critical interference, x_c. This critical deformation obtained by Chang et al. [1] is

$$x_c = \left(\frac{\pi\,K\,H}{2\,E'}\right)^2 R, \tag{7}$$

where

H characterizes the plastic property of the material and can be approximate with the Brinell hardness,

K is a hardness factor and is given by $K = 0.454 + 0.41\nu$.

The critical value of the impact force P_c, can be expressed in terms of critical deformation x_c,

$$P_c = k_1\,x_c^{3/2}. \tag{8}$$

Jackson and Green [2] derives this critical interference analytically using the von Mises yield criterion

$$x_c = \left(\frac{\pi\,C\,S_y}{2\,E'}\right)^2 R, \tag{9}$$

where

S_y is the yield strength and

C is a critical yield stress coefficient given by

$$C = 1.295\,e^{0.736\,\nu}. \tag{10}$$

The Poissons ratio, ν, and yield strength, S_y, to be used in Eq. (10) is that of the material which yields first. To determine which material yields, one has to use the combination of $C\,S_y$, such that $C\,S_y = \min[C(\nu_1)\,S_{y1},\, C(\nu_2)\,S_{y2}]$, accounts for the possibility of two different material properties [3]. The critical force, P_c is calculated from the critical interference, x_c, by substituting it into Hertz theory. The resulting critical contact force at initial yielding is

$$P_c = \frac{4}{3}\left(\frac{R}{E'}\right)^2 \left(\frac{\pi\,C\,S_y}{2}\right)^3 . \tag{11}$$

The Jackson and Green (JG) model predicts the contact force between an elastic perfectly plastic hemisphere and a flat. At $0 \leq x/x_c \leq 1.9$ the JG single asperity model effectively coincides with the Hertzian solution (the Hertz solution for elastic contact is then used), even though the onset of plastic deformation occurs at x/x_c=1.

The equation of motion $m\,\ddot{x} = m\,g - k_1 x^{3/2}$ will be applied for the interval $0 \leq x \leq 1.9\,x_c$. At the end of this phase $x = 1.9\,x_c$ the time is t_I and the velocity is v_I. The time t_I and the velocity v_I are calculated numerically from the equation of motion. The initial conditions for the next phase are the final results of the elastic compression phase

$$t = t_I,\ x(t_I) = 1.9\,x_c,\ \text{and}\ \dot{x}(t_I) = v_I.$$

II. Elasto-plastic compression phase

At displacements (x/x_c) larger than 1.9 the formulation below is used as the current point contact model for modeling elasto-plastic impact

$$P_{EP} = P_c \left\{ \left[e^{-\frac{1}{4}\left(\frac{x}{x_c}\right)^{5/12}} \right] \left(\frac{x}{x_c}\right)^{3/2} + \frac{4}{C}\left(\frac{H_G}{S_y}\right) \left[1 - e^{-\frac{1}{25}\left(\frac{x}{x_c}\right)^{5/9}} \right] \frac{x}{x_c} \right\}, \tag{12}$$

where
$\dfrac{H_G}{S_y} = 2.84 - 0.92\,[1 - \cos(\pi\,a/R)]$, H_G is the hardness geometric limit.
The contact radius a, $0 < a/R < 1$, is calculated from

$$A = \pi\,a^2 = \pi\,R\,x\left(\frac{x}{1.9\,x_c}\right)^B ,$$

where

B is th econtact area material property coefficient,

$B = 0.14\, e^{23\, \epsilon_y}$, and

ϵ_y is the yield strength to elastic modulus ratio, $\epsilon_y = S_y/E'$,

The equation of motion of the sphere for the elasto-plastic compression phase is

$$m\, \ddot{x} = m\, g - P_{EP}, \tag{13}$$

with the initial conditions at t=t_I, $x(t_I)$=$1.9\, x_c$, and $\dot{x}(t_I)$=v_I. This phase will end when the velocity is zero (maximum compression). For the maximum compression the contact force is maximum P_m and the deformation is maximum x_m.

For this stage the current deformation is $1.9\, x_c \leq x \leq x_m$.

The time t_m and the displacement x_m are calculated numerically from the equation of motion given by Eq. (13). The initial conditions for the next phase are the final results of the elasto-plastic compression phase

$$t = t_m,\ x(t_m) = x_m,\ \text{and}\ \dot{x}(t_m) = 0.$$

III. Restitution phase

The last phase is the restitution phase. The contact force decreases from the maximum value, P_m, to zero. The maximum plastic deformation x_p remains the same and the elastic deformation decreases according to Hertz law. It is assumed that the sphere recovers in a completely elastic manner, then the Hertz solution can be used to model the contact force as the sphere rebounds. The sphere will not fully recover to its original shape so the radius of curvature will change to R_r and the surface will be compressed permanently by a residual interference, x_r.

There are two ways by which x_r and R_r can be obtained. First from Etsion et al. [4]

$$\frac{x_r}{x_m} = 1 - \frac{3\, P_m}{4\, E'\, a_m\, x_m} \quad \text{and} \quad R_r = \frac{4\, E'\, a_m^3}{3\, P_m}, \tag{14}$$

where a_m is the contact radius at the maximum interference, x_m.

The second way is by fitting an equation to the finite element results of [2]

$$\frac{x_r}{x_m} = 1.02\left[1 - \left(\frac{x_m/x_c + 5.9}{6.9}\right)^{-0.54}\right] \quad \text{and} \quad R_r = \frac{1}{(x_m - x_r)^3}\left(\frac{3\,P_m}{4\,E'}\right)^2. \tag{15}$$

The rebound phase ends when the deformation x is equal to maximum plastic deformation x_r.

The equation of motion of the sphere for the restitution phase is

$$m\,\ddot{x} = m\,g - \left(\frac{4}{3}E'\sqrt{R_r}\right)\,(x - x_r)^{3/2}, \tag{16}$$

with the initial conditions at $t{=}t_m$, $x(t_m){=}x_m$, and $\dot{x}(t_m){=}0$. This phase will end when $x = x_r$. At the end of restitution the rebound velocity v_f is obtained. The coefficient of restitution is calculated as $e = v_f/v_0$.

Numerical Data

steel
E=200 GPa
S_y=1.12 GPa
ν=0.33
R=0.01 m
ϵ_y=0.005
m=0.0327 kg

E(aluminum)=68 GPa
E(aluminum oxide)=370 GPa
ν(aluminum)=0.33
ν(aluminum oxide)=0.22
S_y(aluminum)=0.38 GPa
R=0.0025 m
ϵ_y=0.006
ρ(aluminum)=2700 kg/m^3

References

[1] Chang, W.R., Etsion, I., and Bogy, D.B., “An Elastic-plastic Model for Contact of Rough Surfaces,” ASME Journal of Tribology, Vol. 109, pp. 257-263,1987.

[2] Jackson, R.L. and Green, I., “A Finite Element Study of Elasto-Plastic hemispherical Contact, ” ASME Journal of Tribology, Vol. 127, pp. 343-354, 2005.

[3] Green, I., “Poisson ratio effects and critical values in spherical and cylindrical Hertzian contacts,” International Journal of Applied Mechanics and Engineering, Vol. 10(3), pp. 451-462, 2005.

[4] Etsion, I., Kligerman, Y., and Kadin, Y., “Unloading of an elastic-plastic loaded spherical contact,” Int. J. Solids and Structures, Vol. 42(13) pp. 3716-3729, 2005.

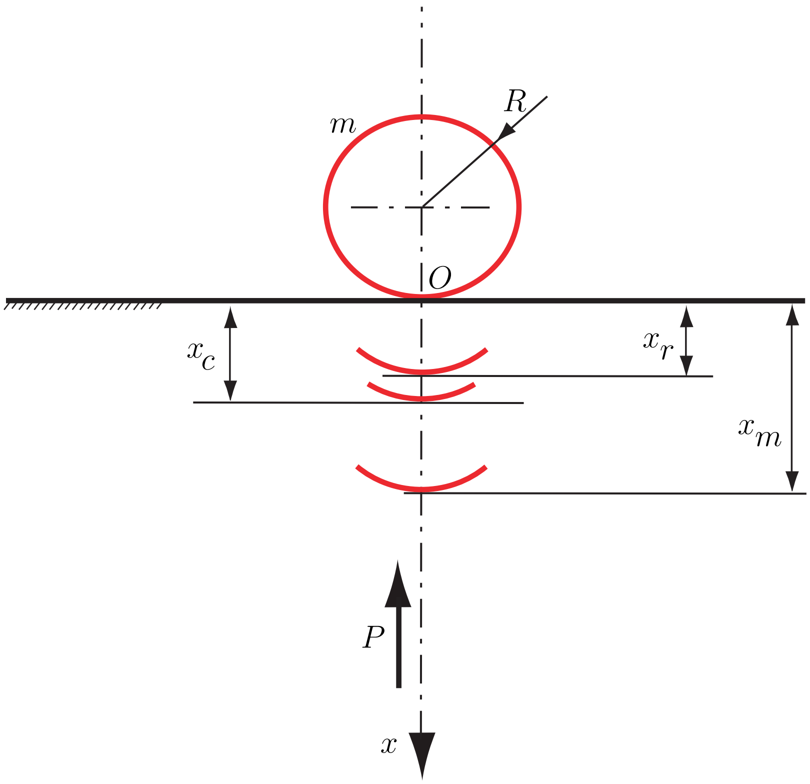

Figure 1