2 Direct Dynamics Newton-Euler Equations of Motion

The Newton-Euler equations of motion for a rigid body in plane motion are

$$m\ddot{\mathbf{r}}_C = \sum \mathbf{F} \quad \text{and} \quad I_{Czz}\,\boldsymbol{\alpha} = \sum \mathbf{M}_C,$$

or using the cartesian components

$$m\ddot{x}_C = \sum F_x, \;\; m\ddot{y}_C = \sum F_y, \;\; \text{and} \quad I_{Czz}\ddot{\theta} = \sum M_C.$$

The forces and moments are known and the differential equations are solved for the motion of the rigid body (direct dynamics).

2.1 Double Pendulum

A two-link planar chain (double pendulum) is considered, Fig. 1. The links 1 and 2 have the masses m_1 and m_2 and the lengths $AB = L_1$ and $BD = L_2$. The system is free to move in a vertical plane. The local acceleration of gravity is g. Numerical application: $m_1 = m_2 = 1$ kg, $L_1 = 1$ m, $L_2 = 1$ m, and $g = 9.807$ m/s^2. Find and solve the equations of motion.

Solution

The plane of motion is xy plane with the y-axis vertical, with the positive sense directed downward. The origin of the reference frame is at A. The mass centers of the links are designated by $C_1(x_{C_1}, y_{C_1}, 0)$ and $C_2(x_{C_2}, y_{C_2}, 0)$. The number of degrees of freedom are computed using the relation

$$M = 3n - 2c_5 - c_4,$$

where n is the number of moving links, c_5 is the number of one degree of freedom joints, and c_4 is the number of two degrees of freedom joints. For the double pendulum $n = 2,\; c_5 = 2,\; c_4 = 0$, and the system has two degrees of freedom, $M = 2$, and two generalized coordinates. The angles $q_1(t)$ and $q_2(t)$ are selected as the generalized coordinates as shown in Fig. 1.

Kinematics

The position vector of the center of the mass C_1 of the link 1 is

$$\mathbf{r}_{C_1} = x_{C_1}\mathbf{\imath} + y_{C_1}\mathbf{\jmath},$$

where x_{C_1} and y_{C_1} are the coordinates of C_1

$$x_{C_1} = \frac{L_1}{2}\cos q_1,\ \ y_{C_1} = \frac{L_1}{2}\sin q_1.$$

The position vector of the center of the mass C_2 of the link 2 is

$$\mathbf{r}_{C_2} = x_{C_2}\boldsymbol{\imath} + y_{C_2}\boldsymbol{\jmath},$$

where x_{C_2} and y_{C_2} are the coordinates of C_2

$$x_{C_2} = L_1\cos q_1 + \frac{L_2}{2}\cos q_2 \quad \text{and} \quad y_{C_2} = L_1\sin q_1 + \frac{L_2}{2}\sin q_2.$$

The velocity vector of C_1 is the derivative with respect to time of the position vector of C_1

$$\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1}\boldsymbol{\imath} + \dot{y}_{C_1}\boldsymbol{\jmath},$$

where

$$\dot{x}_{C_1} = -\frac{L_1}{2}\dot{q}_1\sin q_1 \quad \text{and} \quad \dot{y}_{C_1} = \frac{L_1}{2}\dot{q}_1\cos q_1.$$

The velocity vector of C_2 is the derivative with respect to time of the position vector of C_2

$$\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2}\boldsymbol{\imath} + \dot{y}_{C_2}\boldsymbol{\jmath},$$

where

$$\begin{aligned}\dot{x}_{C_2} &= -L_1\dot{q}_1\sin q_1 - \frac{L_2}{2}\dot{q}_2\sin q_2,\\ \dot{y}_{C_2} &= L_1\dot{q}_1\cos q_1 + \frac{L_2}{2}\dot{q}_2\cos q_2.\end{aligned}$$

The acceleration vector of C_1 is the double derivative with respect to time of the position vector of C_1

$$\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1}\boldsymbol{\imath} + \ddot{y}_{C_1}\boldsymbol{\jmath},$$

where

$$\begin{aligned}\ddot{x}_{C_1} &= -\frac{L_1}{2}\ddot{q}_1\sin q_1 - \frac{L_1}{2}\dot{q}_1^2\cos q_1,\\ \ddot{y}_{C_1} &= \frac{L_1}{2}\ddot{q}_1\cos q_1 - \frac{L_1}{2}\dot{q}_1^2\sin q_1.\end{aligned}$$

The acceleration vector of C_2 is the double derivative with respect to time of the position vector of C_2

$$\mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2}\boldsymbol{\imath} + \ddot{y}_{C_2}\boldsymbol{\jmath},$$

where

$$\ddot{x}_{C_2} = -L_1\ddot{q}_1 \sin q_1 - L_1\dot{q}_1^2 \cos q_1 - \frac{L_2}{2}\ddot{q}_2 \sin q_2 - \frac{L_2}{2}\dot{q}_2^2 \cos q_2,$$
$$\ddot{y}_{C_2} = L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2.$$

The MATLAB commands for the linear accelerations of the mass centers C_1 and C_2 are

```
L1 = 1; L2 = 1; m1 = 1; m2 = 1; g = 9.807;
t = sym('t','real');
xB = L1*cos(sym('q1(t)'));
yB = L1*sin(sym('q1(t)'));
rB = [xB yB 0];
rC1 = rB/2;
vC1 = diff(rC1,t);
aC1 = diff(vC1,t);
xD = xB + L2*cos(sym('q2(t)'));
yD = yB + L2*sin(sym('q2(t)'));
rD = [xD yD 0];
rC2 = (rB + rD)/2;
vC2 = diff(rC2,t);
aC2 = diff(vC2,t);
```

The angular velocity vectors of the links 1 and 2 are

$$\boldsymbol{\omega}_1 = \dot{q}_1\mathbf{k} \quad \text{and} \quad \boldsymbol{\omega}_2 = \dot{q}_2\mathbf{k}.$$

The angular acceleration vectors of the links 1 and 2 are

$$\boldsymbol{\alpha}_1 = \ddot{q}_1\mathbf{k} \quad \text{and} \quad \boldsymbol{\alpha}_2 = \ddot{q}_2\mathbf{k}.$$

The MATLAB commands for the angular accelerations of the links 1 and 2 are

```
omega1 = [0 0 diff('q1(t)',t)];
alpha1 = diff(omega1,t);
omega2 = [0 0 diff('q2(t)',t)];
alpha2 = diff(omega2,t);
```

Newton-Euler equations of motion The weight forces on the links 1 and 2 are

$$\mathbf{G}_1 = m_1 g \mathbf{J} \quad \text{and} \quad \mathbf{G}_2 = m_2 g \mathbf{J},$$

and in MATLAB

```
G1 = [0 m1*g 0];
G2 = [0 m2*g 0];
```

The mass moment of inertia of the link 1 with respect to the center of mass C_1 is

$$I_{C_1} = \frac{m_1 L_1^2}{12}.$$

The mass moment of inertia of the link 1 with respect to the fixed point of rotation A is

$$I_A = I_{C_1} + m_1 \left(\frac{L_1}{2}\right)^2 = \frac{m_1 L_1^2}{3}.$$

The mass moment of inertia of the link 2 with respect to the center of mass C_2 is

$$I_{C_2} = \frac{m_2 L_2^2}{12}.$$

The MATLAB commands for the mass moments of inertia are

```
IC1 = m1*L1^2/12;
IA = IC1 + m1*(L1/2)^2;
IC2 = m2*L2^2/12;
```

The equations of motion of the pendulum are inferred using the Newton-Euler method. There are two rigid bodies in the system and the Newton-Euler equations are written for each link.

Link 1

The Newton-Euler equations for the link 1 are

$$m_1\mathbf{a}_{C_1} = \mathbf{F}_{01} + \mathbf{F}_{21} + \mathbf{G}_1,$$
$$I_{C_1}\boldsymbol{\alpha}_1 = \mathbf{r}_{C_1A} \times \mathbf{F}_{01} + \mathbf{r}_{C_1B} \times \mathbf{F}_{21},$$

where $\mathbf{F}_{01}$ is the joint reaction of the ground 0 on the link 1 at point A, and $\mathbf{F}_{21}$ is the joint reaction of the link 2 on the link 1 at point B

$$\mathbf{F}_{01} = F_{01x}\mathbf{\imath} + F_{01y}\mathbf{\jmath} \quad \text{and} \quad \mathbf{F}_{21} = F_{21x}\mathbf{\imath} + F_{21y}\mathbf{\jmath}.$$

Since the link 1 has a fixed point of rotation at A the moment sum about the fixed point must be equal to to the product of the link mass moment of inertia about that point and the link angular acceleration. Thus

$$I_A\boldsymbol{\alpha}_1 = \mathbf{r}_{AC_1} \times \mathbf{G}_1 + \mathbf{r}_{AB} \times \mathbf{F}_{21}, \tag{2.1}$$

or

$$\frac{m_1L_1^2}{3}\,\ddot{q}_1\mathbf{k} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_{C_1} & y_{C_1} & 0 \\ 0 & -m_1g & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_B & y_B & 0 \\ F_{21x} & F_{21y} & 0 \end{vmatrix}, \quad \text{or}$$
$$\frac{m_1L_1^2}{3}\,\ddot{q}_1\mathbf{k} = (-m_1gx_{C_1} + F_{21y}x_B - F_{21x}y_B)\mathbf{k}.$$

The equation of motion for link 1 is

$$\frac{m_1L_1^2}{3}\,\ddot{q}_1 = \left(-m_1g\frac{L_1}{2}\cos q_1 + F_{21y}L_1\cos q_1 - F_{21x}L_1\sin q_1\right). \tag{2.2}$$

Link 2

The Newton-Euler equations for the link 2 are

$$m_2\mathbf{a}_{C_2} = \mathbf{F}_{12} + \mathbf{G}_2, \tag{2.3}$$
$$I_{C_2}\boldsymbol{\alpha}_2 = \mathbf{r}_{C_2B} \times \mathbf{F}_{12}, \tag{2.4}$$

where $\mathbf{F}_{12} = -\mathbf{F}_{21}$ is the joint reaction of the link 1 on the link 2 at B. Equation (2.4) becomes

$$m_2\ddot{x}_{C_2} = -F_{21x},$$
$$m_2\ddot{y}_{C_2} = -F_{21y} - m_2\,g,$$
$$\frac{mL_2^2}{12}\,\ddot{q}_2\mathbf{k} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_B - x_{C_2} & y_B - y_{C_2} & 0 \\ -F_{21x} & -F_{21y} & 0 \end{vmatrix}, \tag{2.5}$$

or

$$m_2\left(-L_1\ddot{q}_1\sin q_1 - L_1\dot{q}_1^2\cos q_1 - \frac{L_2}{2}\ddot{q}_2\sin q_2 - \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)$$
$$= -F_{21x}, \tag{2.6}$$
$$m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2\right)$$
$$= -F_{21y} - m_2\,g, \tag{2.7}$$
$$\frac{m_2L_2^2}{12}\,\ddot{q}_2 = \frac{L_2}{2}\left(-F_{21y}\cos q_2 + F_{21x}\sin q_2\right). \tag{2.8}$$

The reaction components F_{21x} and F_{21y} are obtained from Eqs. (2.6)(2.7)

$$F_{21x} = m_2\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right),$$
$$F_{21y} = -m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2\right) + m_2\,g. \tag{2.9}$$

The equations of motion are obtained substituting F_{21x} and F_{21y} in Eq. (2.2) and Eq. (2.8)

$$\frac{m_2L_1^2}{3}\,\ddot{q}_1 = -m_1g\frac{L_1}{2}\cos q_1 -$$
$$m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2 - g\right)L_1\cos q_1 -$$
$$m_2\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)L_1\sin q_1, \tag{2.10}$$

$$\frac{m_2L_2^2}{12}\,\ddot{q}_2 =$$
$$\frac{m_2\,L_2}{2}\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2 - g\right)\cos q_2 +$$
$$\frac{m_2\,L_2}{2}\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)\sin q_2. \tag{2.11}$$

The eqautions of motion represent two nonlinear differential equations. The initial conditions (Cauchy problem) are necessary to solve the equations. At $t = 0$ the initial conditions are

$$q_1(0) = q_{10},\ \dot{q}_1(0) = \omega_{10},\ q_2(0) = q_{20},\ \text{and}\ \dot{q}_2(0) = \omega_{20}.$$

The equations of motion for the mechanical system will be solved using MATLAB. First the reaction joint force $\mathbf{F}_{21}$ is calculated from Eq. (2.3)

```
F21 = -m2*aC2 + G2;
```

The moment equations for each link, Eqs. (2.1) and (2.4), using MATLAB are

```
EqA = -IA*alpha1 + cross(rB, F21) + cross(rC1, G1);
Eq2 = -IC2*alpha2 + cross(rB - rC2, -F21);
```

Two lists `slist` and `nlist` are created

```
slist={diff('q1(t)',t,2),diff('q2(t)',t,2),...
       diff('q1(t)',t),diff('q2(t)',t),'q1(t)','q2(t)'};
nlist = {'ddq1', 'ddq2', 'x(2)', 'x(4)', 'x(1)','x(3)';
% diff('q1(t)',t,2) will be replaced by 'ddq1'
% diff('q2(t)',t,2) will be replaced by 'ddq2'
% diff('q1(t)',t) will be replaced by 'x(2)'
% diff('q2(t)',t) will be replaced by 'x(4)'
% 'q1(t)' will be replaced by 'x(1)'
% 'q2(t)' will be replaced by 'x(3)'
```

In the equations of motion `EqA` and `Eq2` the symbolical variables in `slist` are replaced with the symbolical variables in `nlist`

```
eq1 = subs(EqA(3),slist,nlist);
eq2 = subs(Eq2(3),slist,nlist);
```

The previous equations are solved in terms of `'ddq1'` and `'ddq2'`

```
sol = solve(eq1,eq2,'ddq1, ddq2');
```

The second order ODE system of two equations has to be rewritten your as a first order system.
Let `x(1)`$=q_1(t)$, `x(2)`$=\dot{q}_1(t)$, `x(3)`$=q_2(t)$, and `x(4)`$=\dot{q}_2(t)$, this gives the first order system
`d[x(1)]/dt = x(2),`

```
d[x(2)]/dt = ddq1,
d[x(3)]/dt = x(4),
d[x(4)]/dt = ddq2.
```

The MATLAB commands for the first order ODE system are

```
dx1 = sym('x(2)');
dx2 = sol.ddq1;
dx3 = sym('x(4)');
dx4 = sol.ddq2;

dx1dt = char(dx1);
dx2dt = char(dx2);
dx3dt = char(dx3);
dx4dt = char(dx4);
```

The `inline` function `g` is defined for the right hand side of the first order system

```
g = inline(sprintf('[%s;%s;%s;%s]', dx1dt, dx2dt, dx3dt, dx4dt),'t','x');
```

The time `t` is going from an initial value `t0` to a final value `f`

```
t0 = 0; tf = 5; time = [0 tf];
```

The initial conditions at $t_0 = 0$ are $q_1(0) = \pi/4$ rad, $\dot{q}_1(0) = 0$ rad/s, $q_2(0) = \pi/3$ rad, $\dot{q}_2(0) = 0$ rad/s, or in MATLAB

```
x0 = [pi/4; 0; pi/3; 0]; % define initial conditions
```

The numerical solution of all the components of the solution for `t` going from `t0` to `f` is obtained using the command

```
[t,xs] = ode45(g, time, x0);
```

where `x0` is the initial value vector at the starting point `t0`.

The plot of the solution curves q_1 and q_2 are obtained using the commands

```
x1 = xs(:,1);
```

```
x3 = xs(:,3);
subplot(2,1,1),plot(t,x1*180/pi,'r'),...
xlabel('t (s)'),ylabel('q1 (deg)'),grid,...
subplot(2,1,2),plot(t,x3*180/pi,'b'),...
xlabel('t (s)'),ylabel('q2 (deg)'),grid
```

Instead of using the `inline` function `g` the system of differential equations can be solved numerically by m-file functions. The function file, `RR.m` is created using the statements

```
......
sol = solve(eq1,eq2,'ddq1, ddq2');
dx2 = sol.ddq1; dx4 = sol.ddq2;
dx2dt = char(dx2); dx4dt = char(dx4);

% create the function file RR.m

fid = fopen('RR.m','w+');
fprintf(fid,'function dx = RR(t,x)\n');
fprintf(fid,'dx = zeros(4,1);\n');
fprintf(fid,'dx(1) = x(2);\n');
fprintf(fid,'dx(2) = ');
fprintf(fid,dx2dt);
fprintf(fid,';\n');
fprintf(fid,'dx(3) = x(4);\n');
fprintf(fid,'dx(4) = ');
fprintf(fid,dx4dt);
fprintf(fid,';');
fclose(fid);
cd(pwd);
```

The terms `dx2dt` and `dx4dt` are calculated symbolically from the previous program. The MATLAB command `fid = fopen(file,perm)` opens the file `file` in the mode specified by `perm`. The mode `'w+'` deletes the contents of an existing file, or creates a new file, and opens it for reading and writing. The statement `fclose(fid)` closes the file associated with file identifier `fid`.

The `ode45` solver is used for the system of differential equations

```
t0 = 0; tf = 5; time = [0 tf];
x0 = [pi/4 0 pi/3 0];
[t,xs] = ode45(@RR, time, x0);
```

The computing time for solving the system of differential equations is shorter using the function file `RR.m`.

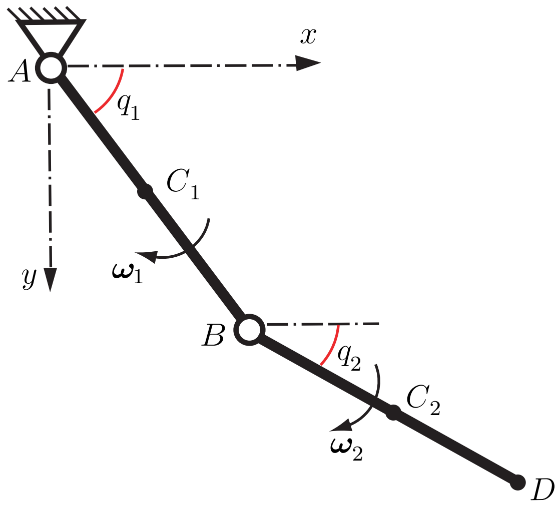

Figure 1

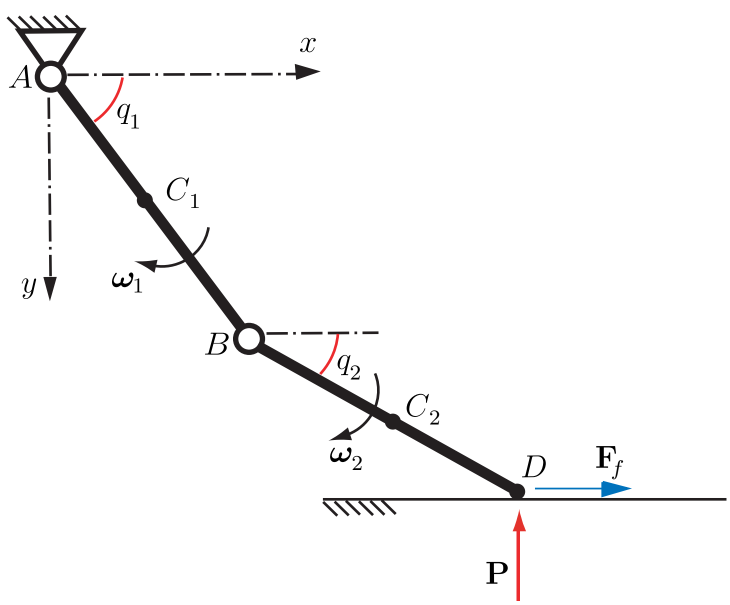

Figure 2