Contents

6 Analytical Dynamics

6.1 Kane's Dynamical Equations

A two-link kinematic chain is considered in Fig. 5.7. The bars 1 and 2 are homogenuos and have the lengths $L_1 = L_2 = L$. The masses of the rigid links are $m_1 = m_2 = m$ and the gravitational acceleration is g.

The plane of motion is xy plane with the y-axis vertical, with the positive sense directed downward. The origin of the reference frame is at A. The system has two degrees of freedom. To characterize the instantaneous configuration of the system, two generalized coordinates $q_1(t)$ and $q_2(t)$ are employed. The generalized coordinates q_1 and q_2 denote the radian measure of the angles between the link 1 and 2 and the horizontal x-axis.

There are two *generalized speeds* defined as

$$u_1 = \dot{q}_1 \ \text{ and } \ u_2 = \dot{q}_2. \tag{6.1}$$

The mass centers of the links are designated by $C_1(x_{C_1}, y_{C_1}, 0)$ and $C_2(x_{C_2}, y_{C_2}, 0)$.

Kinematics

The position vector of the center of the mass C_1 of the link 1 is

$$\mathbf{r}_{C_1} = x_{C_1}\boldsymbol{\imath} + y_{C_1}\boldsymbol{\jmath},$$

where x_{C_1} and y_{C_1} are the coordinates of C_1

$$x_{C_1} = \frac{L_1}{2}\cos q_1, \ \ y_{C_1} = \frac{L_1}{2}\sin q_1.$$

The velocity vector of C_1 is the derivative with respect to time of the position vector of C_1

$$\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1}\boldsymbol{\imath} + \dot{y}_{C_1}\boldsymbol{\jmath},$$

where

$$\dot{x}_{C_1} = -\frac{L_1}{2}\dot{q}_1 \sin q_1 \quad \text{and} \quad \dot{y}_{C_1} = \frac{L_1}{2}\dot{q}_1 \cos q_1,$$

or

$$\mathbf{v}_{C_1} = -\frac{L_1}{2}u_1 \sin q_1 \ \boldsymbol{\imath} + \frac{L_1}{2}u_1 \cos q_1 \ \boldsymbol{\jmath}.$$

The acceleration vector of C_1 is the double derivative with respect to time of the position vector of C_1

$$\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1}\boldsymbol{\imath} + \ddot{y}_{C_1}\boldsymbol{\jmath},$$

where

$$\ddot{x}_{C_1} = -\frac{L_1}{2}\ddot{q}_1 \sin q_1 - \frac{L_1}{2}\dot{q}_1^2 \cos q_1,$$
$$\ddot{y}_{C_1} = \frac{L_1}{2}\ddot{q}_1 \cos q_1 - \frac{L_1}{2}\dot{q}_1^2 \sin q_1,$$

or

$$\mathbf{a}_{C_1} = (-\frac{L_1}{2}\dot{u}_1 \sin q_1 - \frac{L_1}{2}u_1^2 \cos q_1)\ \mathbf{\imath} + (\frac{L_1}{2}\dot{u}_1 \cos q_1 - \frac{L_1}{2}u_1^2 \sin q_1)\ \mathbf{\jmath}.$$

The position vector of the center of the mass C_2 of the link 2 is

$$\mathbf{r}_{C_2} = x_{C_2}\mathbf{\imath} + y_{C_2}\mathbf{\jmath},$$

where x_{C_2} and y_{C_2} are the coordinates of C_2

$$x_{C_2} = L_1 \cos q_1 + \frac{L_2}{2}\cos q_2 \quad \text{and} \quad y_{C_2} = L_1 \sin q_1 + \frac{L_2}{2}\sin q_2.$$

The velocity vector of C_2 is the derivative with respect to time of the position vector of C_2

$$\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2}\mathbf{\imath} + \dot{y}_{C_2}\mathbf{\jmath},$$

where

$$\dot{x}_{C_2} = -L_1\dot{q}_1 \sin q_1 - \frac{L_2}{2}\dot{q}_2 \sin q_2,$$
$$\dot{y}_{C_2} = L_1\dot{q}_1 \cos q_1 + \frac{L_2}{2}\dot{q}_2 \cos q_2,$$

or

$$\mathbf{v}_{C_2} = (-L_1 u_1 \sin q_1 - \frac{L_2}{2}u_2 \sin q_2)\ \mathbf{\imath} + (L_1 u_1 \cos q_1 + \frac{L_2}{2}u_2 \cos q_2)\ \mathbf{\jmath}.$$

The acceleration vector of C_2 is the double derivative with respect to time of the position vector of C_2

$$\mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2}\mathbf{\imath} + \ddot{y}_{C_2}\mathbf{\jmath},$$

where

$$\ddot{x}_{C_2} = -L_1\ddot{q}_1 \sin q_1 - L_1\dot{q}_1^2 \cos q_1 - \frac{L_2}{2}\ddot{q}_2 \sin q_2 - \frac{L_2}{2}\dot{q}_2^2 \cos q_2,$$
$$\ddot{y}_{C_2} = L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2,$$

or

$$\begin{aligned}\mathbf{a}_{C_2} = {} & \left(-L_1\dot{u}_1\sin q_1 - L_1u_1^2\cos q_1 - \frac{L_2}{2}\dot{u}_2\sin q_2 - \frac{L_2}{2}u_2^2\cos q_2\right)\boldsymbol{\imath} + \\ & \left(L_1\dot{u}_1\cos q_1 - L_1u_1^2\sin q_1 + \frac{L_2}{2}\dot{u}_2\cos q_2 - \frac{L_2}{2}u_2^2\sin q_2\right)\boldsymbol{\jmath}.\end{aligned}$$

The position vector of the end point D is

$$\mathbf{r}_D = x_D\boldsymbol{\imath} + y_D\boldsymbol{\jmath},$$

where

$$x_D = L_1\cos q_1 + L_2\cos q_2 \quad \text{and} \quad y_D = L_1\sin q_1 + L_2\sin q_2.$$

The velocity of the end point D is

$$\mathbf{v}_D = \dot{\mathbf{r}}_D = \dot{x}_D\boldsymbol{\imath} + \dot{y}_D\boldsymbol{\jmath},$$

where

$$\begin{aligned}\dot{x}_D &= -L_1\dot{q}_1\sin q_1 - L_2\dot{q}_2\sin q_2,\\ \dot{y}_D &= L_1\dot{q}_1\cos q_1 + L_2\dot{q}_2\cos q_2,\end{aligned}$$

or

$$\mathbf{v}_D = \left(-L_1u_1\sin q_1 - L_2u_2\sin q_2\right)\boldsymbol{\imath} + \left(L_1u_1\cos q_1 + L_2u_2\cos q_2\right)\boldsymbol{\jmath}.$$

The acceleration of the end point D is

$$\mathbf{a}_D = \ddot{\mathbf{r}}_D = \ddot{x}_D\boldsymbol{\imath} + \ddot{y}_D\boldsymbol{\jmath},$$

where

$$\begin{aligned}\ddot{x}_D &= -L_1\ddot{q}_1\sin q_1 - L_1\dot{q}_1^2\cos q_1 - L_2\ddot{q}_2\sin q_2 - L_2\dot{q}_2^2\cos q_2,\\ \ddot{y}_D &= L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + L_2\ddot{q}_2\cos q_2 - L_2\dot{q}_2^2\sin q_2,\end{aligned}$$

or

$$\begin{aligned}\mathbf{a}_D = {} & \left(-L_1\dot{u}_1\sin q_1 - L_1u_1^2\cos q_1 - L_2\dot{u}_2\sin q_2 - L_2u_2^2\cos q_2\right)\boldsymbol{\imath} + \\ & \left(L_1\dot{u}_1\cos q_1 - L_1u_1^2\sin q_1 + L_2\dot{u}_2\cos q_2 - L_2u_2^2\sin q_2\right)\boldsymbol{\jmath}.\end{aligned}$$

The angular velocity vectors of the links 1 and 2 are

$$\boldsymbol{\omega}_1 = \dot{q}_1\mathbf{k} = u_1\mathbf{k} \quad \text{and} \quad \boldsymbol{\omega}_2 = \dot{q}_2\mathbf{k} = u_2\mathbf{k}.$$

The angular acceleration vectors of the links 1 and 2 are

$$\boldsymbol{\alpha}_1 = \ddot{q}_1\mathbf{k} = \dot{u}_1\mathbf{k} \quad \text{and} \quad \boldsymbol{\alpha}_2 = \ddot{q}_2\mathbf{k} = \dot{u}_2\mathbf{k}.$$

The mass moment of inertia of the link 1 with respect to the center of mass C_1 is

$$I_{C_1} = \frac{m_1 L_1^2}{12}.$$

The mass moment of inertia of the link 1 with respect to the fixed point of rotation A is

$$I_A = I_{C_1} + m_1 \left(\frac{L_1}{2}\right)^2 = \frac{m_1 L_1^2}{3}.$$

The mass moment of inertia of the link 2 with respect to the center of mass C_2 is

$$I_{C_2} = \frac{m_2 L_2^2}{12}.$$

Generalized inertia forces

The generalized inertia forces for a rigid body RB are

$$K_{in} = \frac{\partial \mathbf{v}_{CG}}{\partial u_r} \cdot \mathbf{F}_{in} + \frac{\partial \boldsymbol{\omega}}{\partial u_r} \cdot \mathbf{T}_{in}, \tag{6.2}$$

where $\mathbf{v}_{CG}$ is the velocity of the mass center RB, and $\boldsymbol{\omega} = \omega_x\boldsymbol{\imath} + \omega_y\boldsymbol{\jmath} + \omega_z\mathbf{k}$ is the angular velocity of RB.

The inertia force for the rigid body RB is

$$\mathbf{F}_{in} = -M\,\mathbf{a}_{CG}, \tag{6.3}$$

where M is the mass of RB, and $\mathbf{a}_{CG}$ is the acceleration of the mass center of RB.

The inertia torque $\mathbf{T}_{in}$ for RB is

$$\mathbf{T}_{in} \;=\; -\boldsymbol{\alpha} \cdot \bar{I} - \boldsymbol{\omega} \times (\bar{I} \cdot \boldsymbol{\omega}), \tag{6.4}$$

where $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \alpha_x\boldsymbol{\imath} + \alpha_y\boldsymbol{\jmath} + \alpha_z\mathbf{k}$ is the angular acceleration of RB, and $\bar{I} = (I_x\boldsymbol{\imath})\boldsymbol{\imath} + (I_y\boldsymbol{\jmath})\boldsymbol{\jmath} + (I_z\mathbf{k})\mathbf{k}$ is the central inertia dyadic of RB. The central principal axes of RB are parallel to $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ and the associated moments of inertia have the values I_x, I_y, I_z, respectively.

- Link 1:

$$\begin{aligned}&\mathbf{F}_{in_1} = -m_1 \mathbf{a}_{C_1} = \\ &-m_1 \frac{L_1}{2} \left[\left(-\dot{u}_1 \sin q_1 - u_1^2 \cos q_1\right) \boldsymbol{\imath} + \left(\dot{u}_1 \cos q_1 - u_1^2 \sin q_1\right) \boldsymbol{\jmath} \right], \\ &\mathbf{T}_{in_1} = -\boldsymbol{\alpha}_{10} \cdot \bar{I}_1 = -\boldsymbol{\alpha}_1 I_{C_1} = -\frac{m_1 L_1^2}{12} \dot{u}_1 \mathbf{k}. \qquad (6.5)\end{aligned}$$

- Link 2:

$$\begin{aligned}&\mathbf{F}_{in_2} = -m_2 \mathbf{a}_{C_2} = \\ &-m_2 \left(-L_1 \dot{u}_1 \sin q_1 - L_1 u_1^2 \cos q_1 - \frac{L_2}{2} \dot{u}_2 \sin q_2 - \frac{L_2}{2} u_2^2 \cos q_2\right) \boldsymbol{\imath} \\ &-m_2 \left(L_1 \dot{u}_1 \cos q_1 - L_1 u_1^2 \sin q_1 + \frac{L_2}{2} \dot{u}_2 \cos q_2 - \frac{L_2}{2} u_2^2 \sin q_2\right) \boldsymbol{\jmath} \\ &\mathbf{T}_{in_2} = -\boldsymbol{\alpha}_2 \cdot \bar{I}_2 = -\boldsymbol{\alpha}_2 I_{C_2} = -\frac{m_2 L_2^2}{12} \dot{u}_2 \mathbf{k}. \qquad (6.6)\end{aligned}$$

The generalized inertia forces associated to q_1 and q_2 are

$$\begin{aligned}K_{in_1} &= \frac{\partial \mathbf{v}_{C_1}}{\partial u_1} \cdot \mathbf{F}_{in_1} + \frac{\partial \boldsymbol{\omega}_1}{\partial u_1} \cdot \mathbf{T}_{in_1} + \frac{\partial \mathbf{v}_{C_2}}{\partial u_1} \cdot \mathbf{F}_{in_2} + \frac{\partial \boldsymbol{\omega}_2}{\partial u_1} \cdot \mathbf{T}_{in_2}, \\ K_{in_2} &= \frac{\partial \mathbf{v}_{C_1}}{\partial u_2} \cdot \mathbf{F}_{in_1} + \frac{\partial \boldsymbol{\omega}_1}{\partial u_2} \cdot \mathbf{T}_{in_1} + \frac{\partial \mathbf{v}_{C_2}}{\partial u_2} \cdot \mathbf{F}_{in_2} + \frac{\partial \boldsymbol{\omega}_2}{\partial u_2} \cdot \mathbf{T}_{in_2}. \qquad (6.7)\end{aligned}$$

Generalized active forces

The weight forces on the links 1 and 2 are

$$\begin{aligned}\mathbf{G}_1 &= m_1 g\, \jmath, \ \text{ acts at } C_1,\\ \mathbf{G}_2 &= m_2 g\, \jmath, \ \text{ acts at } C_2.\end{aligned}$$

The impact force act at the end point D

$$\mathbf{P} = -\text{sign}(\mathbf{v}_D \cdot \imath)\, \mu\, F\, \imath + F\, \jmath,$$

where F is the normal impulsive force and μ is the coefficient of friction.

The generalized active forces associated to q_1 and q_2 are

$$\begin{aligned} Q_1 &= \frac{\partial \mathbf{v}_{C_1}}{\partial u_1} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial u_1} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_D}{\partial u_1} \cdot \mathbf{P},\\ Q_2 &= \frac{\partial \mathbf{v}_{C_1}}{\partial u_2} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial u_2} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_D}{\partial u_2} \cdot \mathbf{P}.\end{aligned}$$

The Kane's dynamical equations are

$$K_{in_r} + Q_r = 0, \quad r = 1, 2. \tag{6.8}$$

The solution of the system is obtained from Kane's dynamical relations Eq. (6.8) and from kinematical relations Eq. (6.1) with the initial conditions $q_{10} = q_1(0)$, $q_{20} = q_2(0)$, $u_{10} = u_1(0)$, and $u_{20} = u_2(0)$.

6.2 Lgrange's Equations of Motion

Kinetic energy
The kinetic energy of the link 1 which is in rotational motion is

$$T_1 = \frac{1}{2}\, I_A\, \boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_1 = \frac{1}{2}\, I_A\, \dot{q}_1^2 = \frac{1}{2}\, \frac{mL^2}{3}\, \dot{q}_1^2 = \frac{mL^2}{6}\, \dot{q}_1^2,$$

where I_A is the mass moment of inertia about the center of rotation A, $I_A = mL^2/3$.
The kinetic energy of the bar 2 is due to the translation and rotation and can be expressed as

$$T_2 = \frac{1}{2} I_{C_2}\, \boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_1 + \frac{1}{2} m_2\, \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} = \frac{1}{2} I_{C_2} \dot{q}_2^2 + \frac{1}{2} m_2\, \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2},$$

where I_{C_2} is the mass moment of inertia about the center of mass C_2 and

$$\mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} = \mathbf{v}_{C_2}^2 = L^2\, \dot{q}_1^2 + \frac{1}{4}\, L^2\, \dot{q}_2^2 + L^2\, \dot{q}_1\, \dot{q}_2 \cos(q_2 - q_1).$$

Equation (6.9) becomes

$$T_2 = \frac{1}{2}\, \frac{mL^2}{12}\, \dot{q}_2^2 + \frac{1}{2} mL^2 \left[\dot{q}_1^2 + \frac{1}{4}\, \dot{q}_2^2 + \dot{q}_1\, \dot{q}_2 \cos(q_2 - q_1) \right].$$

The total kinetic energy of the system is

$$T = T_1 + T_2 = \frac{mL^2}{6}\, \left[4\dot{q}_1^2 + 3\, \dot{q}_1\, \dot{q}_2\, \cos(q_2 - q_1) + \dot{q}_2^2 \right].$$

The left hand sides of Lagrange's equations $\partial T / \partial \dot{q}_i,\ i = 1, 2$ are

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_1} &= \frac{mL^2}{6}\, \left[8\dot{q}_1 + 3\dot{q}_2\, \cos(q_2 - q_1) \right], \\ \frac{\partial T}{\partial \dot{q}_2} &= \frac{mL^2}{6}\, \left[3\dot{q}_1\, \cos(q_2 - q_1) + 2\dot{q}_2 \right]. \end{aligned}$$

The two Lagrange's equations are

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} &= Q_1, \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} &= Q_2, \end{aligned} \tag{6.9}$$

where the generalized active forces associated to q_1 and q_2 are

$$\begin{aligned}
Q_1 &= \frac{\partial \mathbf{r}_{C_1}}{\partial q_1} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{r}_{C_2}}{\partial q_1} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{r}_D}{\partial q_1} \cdot \mathbf{P}, \\
Q_2 &= \frac{\partial \mathbf{r}_{C_1}}{\partial q_2} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{r}_{C_2}}{\partial q_2} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{r}_D}{\partial q_2} \cdot \mathbf{P}.
\end{aligned}$$