Contents

2 PARTICLE IMPACT

2.1 Introduction

A *mechanical impact* occurs when the linear momentum $m\mathbf{v}$ has a suddenly variation or a jump. The velocity $\mathbf{v}$ has a discontinuity of first kind, i.e., different values at the beginning and at the end of the impact period. The impact period is very short. The phenomenon of impact may occur when two particles with different velocities collide. Another example is when an elastic sphere collides with a rigid wall and changes suddenly its velocity. The sphere position has a little variation during the collision.

A particle of mass m with the velocity $\mathbf{v}_0$ is subjected to an impact at the moment t_0. The impact period of time is Δ. The velocity after impact is $\mathbf{v}_1$ and corresponds to the moment $t_1 = t_0 + \Delta$. The moments t_0 and t_1 mark the beginning and the end of the impact. The effects of external forces are negligible and the total linear momentum of the particle is conserved. Newton's second law for the particle gives

$$m\,d\mathbf{v} = \mathbf{F}\,dt. \tag{2.1}$$

Integrating Eq. (2.1) for the collision time, yields

$$m\mathbf{v}_1 - m\mathbf{v}_0 = \int_{t_0}^{t_1} \mathbf{F}dt. \tag{2.2}$$

The right-hand side of Eq. (2.2) is the linear impulse of the particle and is denoted by $\mathbf{P}$

$$\mathbf{P} = \int_{t_0}^{t_1} \mathbf{F}\,dt. \tag{2.3}$$

The impulse applied to the particle during the period of impact is equal to the change of the linear momentum of the particle.

$$m\mathbf{v}_1 - m\mathbf{v}_0 = \mathbf{P}. \tag{2.4}$$

The average with respect to time of the total force acting on the particle from t_0 to t_1 is $\mathbf{F}_{av}$ and

$$\mathbf{P} = \mathbf{F}_{av}(t_1 - t_0) = \mathbf{F}_{av}\Delta.$$

In Eq. (2.4), because the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ have different values during the impact, the linear impulse $\mathbf{P}$ must be finite. The impact period Δ is very short. To obtain a finite value for $\mathbf{P}$ the force $\mathbf{F}_{av}$ must have a greater value.

During the impact, the particle displacement can be neglected. The change of the displacement during the impact has the same order as the impact period. This can be proved by integrating the relation $d\mathbf{r} = \mathbf{v}dt$ for the impact time interval (t_0, t_1)

$$\mathbf{r}_1 - \mathbf{r}_0 = \int_{t_0}^{t_1} \mathbf{v}dt. \tag{2.5}$$

Applying the average theorem for the integral in Eq. (2.5) and assuming for $\Delta = t_1 - t_0$ a very low value then, the value of difference $\mathbf{r}_1 - \mathbf{r}_0$ has the same order as Δ. For impact problems, the change in the position of the particle during the negligible impact period can be considered negligible.

2.2 Direct central impacts of two spheres

The smooth spheres 1 and 2 move along a straight line with velocities v_1 and v_2 before their impact $(v_1 > v_2)$, Fig. 2.1(a). The impact force is parallel to the line along which the spheres travel (direct central impact) [Fig. 2.1(b)]. The spheres continue to move along the same straight line after their impact [Fig. 2.1(c)]. The effects of external forces during the impact are negligible, and the total linear momentum is conserved

$$m_1\, v_1 + m_2\, v_2 = m_1\, v_1' + m_2\, v_2'. \tag{2.6}$$

Another equation is needed to determine the velocities v_1' and v_2' after the impact.

Let t_1 be the time at which 1 and 2 first come into contact. As a result of the impact, the objects will deform. At the time t_m the particles will have reached the maximum compression (period of compression, $t_1 < t < t_m$), Fig. 2.1(b). At this time the relative velocity of the particles is zero, so they have the same velocity, u. The spheres then begin to move apart and separate at a time t_2, Fig. 2.1(c). The second period, from the maximum compression to the instant at which the particles separate is term the period of restitution, $t_m < t < t_2$.

The principle of impulse and momentum is applied to 1 during the intervals of time from t_1 to the time of closest approach t_m and also from t_m to t_2

$$\int_{t_1}^{t_m} -F_c\, dt = m_1 u - m_1 v_1, \tag{2.7}$$

$$\int_{t_m}^{t_2} -F_r\, dt = m_1 v_1' - m u, \tag{2.8}$$

where F_c is the magnitude of the contact force during compression phase and F_r is the magnitude of the contact force during restitution phase. Then the principle of impulse and momentum is applied to 2 for the same intervals of time

$$\int_{t_1}^{t_m} F_c\, dt = m_2 u - m_2 v_2, \tag{2.9}$$

$$\int_{t_m}^{t_2} F_r \, dt = m_2 v_2' - m_2 u. \tag{2.10}$$

As a result of the impact, part of the kinetic energy of the particles can be lost (due to a permanent deformation, generation of heat and sound, etc.). The impulse during the restitution phase of the impact from t_m to t_2 is in general smaller than the impulse during the compression phase t_1 to t_m.

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The ratio of these impulses is called the *coefficient of restitution* (this definition was introduced by Poisson)

$$e = \frac{\int_{t_m}^{t_2} F_r \, dt}{\int_{t_1}^{t_m} F_c \, dt}. \tag{2.11}$$

The value of the coefficient of restitution depends on the properties of the objects as well as their velocities and orientations when they collide, and it can be determined by experiment or by a detailed analysis of the deformations of the objects during the impact.

If Eq. (2.8) is divided by Eq. (2.7) and Eq. (2.10) is divided by Eq. (2.9), the resulting equations are

$$(u - v_1)e = v_1' - u,$$
$$(u - v_2)e = v_2' - u.$$

Subtracting the first equation from the second one the coefficient of restitution is

$$e = \frac{v_2' - v_1'}{v_1 - v_2}. \tag{2.12}$$

Thus the coefficient of restitution is related to the relative velocities of the objects before and after the impact (this is the kinematic definition of e introduced by Newton). If the coefficient of restitution e is known, Eq. (2.12) together with the equation of conservation of linear momentum, Eq. (2.6), may be used to determine v_1' and v_2'.

If $e = 0$, then $v_1' = v_2'$ and the objects remain together after the impact. The impact is perfectly plastic.

If $e = 1$, the total kinetic energy is the same before and after the impact

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 (v_1')^2 + \frac{1}{2}m_2 (v_2')^2.$$

The impact in which kinetic energy is conserved is called *perfectly elastic.*

If $0 < e < 1$, then the impact is elasto-plastic. The spheres have deformations after impact. This case is usually found in practice. The velocities v_1' and v_2' are determined from the the system of equations

$$\begin{aligned} m_1 v_1 + m_2 v_2 &= m_1 v_1' + m_2 v_2', \\ e &= \frac{v_2' - v_1'}{v_1 - v_2}. \end{aligned} \tag{2.13}$$

Then,

$$\begin{aligned} v_1' &= v_1 - \frac{(v_1 - v_2)(1+e)}{1 + \dfrac{m_1}{m_2}}, \\ v_2' &= v_2 + \frac{(v_1 - v_2)(1+e)}{1 + \dfrac{m_2}{m_1}}. \end{aligned} \tag{2.14}$$

If T_i and T_f denote the kinetic energies at the beginning and at the end of the impact then, the loss of the kinetic energy during the impact is

$$\Delta T = T_i - T_f = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{1}{2}m_1 ({v'}_1)^2 - \frac{1}{2}m_2 ({v'}_2)^2. \tag{2.15}$$

Substituting the velocities ${v'}_1$ and ${v'}_2$ expressed by Eq. (2.14) in Eq. (2.15) one can obtain

$$\Delta T = \frac{m_1 m_2}{2(m_1 + m_2)}(v_1 - v_2)^2 (1 - e^2). \tag{2.16}$$

Remark

The energy loss ΔT is null for the elastic impact ($e = 1$) and has a maximum value for perfectly plastic impact ($e = 0$).

Consider now the case of the perfectly plastic impact of two bodies, one of them having no initial velocity ($v_2 = 0$). The energy loss ΔT is

$$\Delta T = \frac{m_1 m_2}{2(m_1 + m_2)} v_1^2. \tag{2.17}$$

The ratio $\Delta T / T_i$ is

$$\frac{\Delta T}{T_i} = \frac{\dfrac{m_1 m_2}{2(m_1 + m_2)} v_1^2}{\dfrac{1}{2} m_1 v_1^2} = \frac{m_2}{m_1 + m_2} = \frac{1}{1 + \dfrac{m_1}{m_2}}. \tag{2.18}$$

Two important situations are described.

i) The ratio m_1/m_2 has a larger value. In this case, the ratio $\Delta T/T_i$ has a smaller value. A small part of the kinetic energy is dissipated. The hitting with a weighty hammer of a nail of mass m_2 in a wall having the mass m_1 is a good practical example (the hammer must have the mass much greater than the nail).

ii) The ratio m_1/m_2 has a smaller value. Then, $\Delta T/T_i$ is close to unit. Almost all the kinetic energy is dissipated into deformation energy. This is the case of forging of a work piece. The piece that must be manufactured is placed on a heavy support with mass m_2, much bigger than the mass m_1 of the forging tool.

2.3 Oblique central impacts

Assume that two small spheres O_1 and O_2 with the masses m_1 and m_2 collide and the directions of their velocities before the impact have different angles [Fig. 2.2(a)]. The angles α_1, α_2 are between the line O_1O_2 that passes through the centers of masses and the velocities $\mathbf{w_1}$, $\mathbf{w_2}$. It is considered that the impact involve only forces along the line O_1O_2 (there is no friction) and the linear impulse has the direction O_1O_2. The velocities $\mathbf{w_1}$, $\mathbf{w_2}$ are projected on the direction O_1O_2 and on the direction perpendicular to O_1O_2. The components $\mathbf{v_1}$, $\mathbf{u_1}$ are obtained for $\mathbf{w_1}$ and the components $\mathbf{v_2}$, $\mathbf{u_2}$ are obtained for $\mathbf{w_2}$ with the following magnitudes

$$\begin{aligned} v_1 &= w_1 \cos\alpha_1, \quad u_1 = w_1 \sin\alpha_1, \\ v_2 &= w_2 \cos\alpha_2, \quad u_2 = w_2 \sin\alpha_2. \end{aligned}$$

Because there is no friction, the components u_1, u_2 are identical after the impact. The components v_1 and v_2 become v'_1 and v'_2 with the values given by Eqs. (2.14) deduced for the central impact. Finally, the velocities after impact $\mathbf{w}'_1$ and $\mathbf{w}'_2$ have the magnitudes

$$w'_1 = \sqrt{u_1^2 + (v'_1)^2} \quad \text{and} \quad w'_2 = \sqrt{u_2^2 + (v'_2)^2}, \tag{2.19}$$

and their angles with O_1O_2 are β_1 and β_2 [Fig. 2.2(b)]

$$\tan\beta_1 = \frac{u_1}{v'_1} \quad \text{and} \quad \tan\beta_2 = \frac{u_2}{v'_2}. \tag{2.20}$$

Consequently, the velocities after impact can be found.

2.4 Oblique impact of a sphere with a wall

Suppose that a sphere hits a wall with a velocity $\mathbf{v}$. The angle between the velocity of the sphere and the normal to the wall at the impact point is α ([Fig. 2.3]). After impact, the component of $\mathbf{v}$ along the wall direction does not modify, but the normal component of $\mathbf{v}$ changes its direction and its magnitude becomes $e\,v\,\cos\alpha$. Thus, after impact the velocity $\mathbf{w}$ has the components $v\sin\alpha$ and $-e\,v\,\cos\alpha$. The magnitude of the rebound velocity $\mathbf{w}$ is

$$w = \sqrt{v^2\sin^2\alpha + e^2v^2\cos^2\alpha},$$

or,

$$w = v\sqrt{\sin^2\alpha + e^2\cos^2\alpha}. \tag{2.21}$$

Remarks

Referring to Eq. (2.21), it can be observed that $|\mathbf{w}| \leq |\mathbf{v}|$.

The angle β between the direction of the velocity $\mathbf{w}$ and the normal direction to the wall is

$$\tan\beta = \frac{v\,\sin\alpha}{R\,v\,\cos\alpha}. \tag{2.22}$$

Then,

$$\tan\beta = \frac{1}{e}\tan\alpha.$$

Because $e \leq 1$, yields $\beta \geq \alpha$. In this case, after impact the velocity decreases and its direction deviates with respect to the normal direction.

If $e = 1$, then $w = v$ and $\alpha = \beta$. This is a similar situation to the reflection law of a light beam into a mirror. The incidence angle is equal to the reflexion angle.

2.5 General laws of impact

Suppose that a system of elementary particles takes part in an impact phenomenon. Separating from the system a particle A_i with the mass m_i,

$$m_i\left(\mathbf{v}_{i1}-\mathbf{v}_{i0}\right)=\mathbf{P}_i, \tag{2.23}$$

where $\mathbf{P}_i$ is the total linear impulse acting upon the particle, $\mathbf{v}_{i0}$ and $\mathbf{v}_{i1}$ are the velocities before and after the impact. Writing Eq. (2.23) for all the particles of the system and adding them, yields

$$\sum_{i=1}^{n} m_i\mathbf{v}_{i1}-\sum_{i=1}^{n} m_i\mathbf{v}_{i0}=\sum_{i=1}^{n}\mathbf{P}_i, \tag{2.24}$$

where n is the number of the particles of the system. The term $\sum_{i=1}^{n} m_i\mathbf{v}_{i0}=\mathbf{L}_0$ is the linear momentum of the system before the impact and the term $\sum_{i=1}^{n} m_i\mathbf{v}_{i1}=\mathbf{L}_1$ is the linear momentum of the system after impact. Because the internal forces and the linear impulses are always in equilibrium, the resultant force and the resultant momentum of internal forces are null. Denoting by $\Delta\mathbf{L}=\mathbf{L}_1-\mathbf{L}_0$ the variation of the total linear momentum, the following relation is obtained

$$\Delta\mathbf{L}=\mathbf{L}_1-\mathbf{L}_0=\sum_{i=1}^{n}\mathbf{P}_i. \tag{2.25}$$

Equation (2.25) is the first law of impact:

"The variation of the total momentum of a system in the case of an impact is equal to the resultant of the linear impulses that act upon the system."

Consider now a reference point. The position vectors of the particles are expressed with respect to that reference point. The position vector of an arbitrary particle A_i is the vector $\mathbf{r}_i$. Cross multiplying Eq. (2.24) by the position vector $\mathbf{r}_i$ and adding the equations for all of the particles of the system,

$$\sum_{i=1}^{n}\mathbf{r}_i\times\left(m_i\mathbf{v}_{i1}\right)-\sum_{i=1}^{n}\mathbf{r}_i\times\left(m_i\mathbf{v}_{i0}\right)=\sum_{i=1}^{n}\mathbf{r}_i\times\mathbf{P}_i.$$

But, $\mathbf{H}_0 = \sum_{i=1}^{n} \mathbf{r}_i \times (m_i \mathbf{v}_{i0})$ is the angular momentum of the system about the reference point at the beginning of the impact and $\mathbf{H}_1 = \sum_{i=1}^{n} \mathbf{r}_i \times (m_i \mathbf{v}_{i1})$ is the angular momentum of the system about the reference point at the end of impact. Denoting by $\Delta\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_0$ the variation of the angular momentum of the system is

$$\Delta\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_0 = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{P}_i. \tag{2.26}$$

Equation (2.26) is the second law of impact:

"The variation of the angular momentum of a system about a reference point in the case of an impact, is equal to the resultant angular momentum of the linear impulses about the same reference point."

2.6 Carnot's theorem

Consider that upon a system of particles act only ideal constraints. Separating from the system a particle with the mass m_i that has the velocities $\mathbf{v}_{i0}$ and $\mathbf{v}_{i1}$ at the beginning and at the end of the impact, Eq. (2.4) becomes

$$m_i(\mathbf{v}_{i1} - \mathbf{v}_{i0}) = \mathbf{P}_i, \tag{2.27}$$

where $\mathbf{P}_i$ is the linear impulse acting on the particle. Multiplying Eq. (2.27) by $\mathbf{v}_{i1}$, yields

$$m_i \mathbf{v}_{i1} \cdot (\mathbf{v}_{i1} - \mathbf{v}_{i0}) = \mathbf{P}_i \cdot \mathbf{v}_{i1},$$

or

$$\frac{1}{2} m_i (v_{i1}^2 - v_{i0}^2) + \frac{1}{2} m_i (\mathbf{v}_{i1} - \mathbf{v}_{i0})^2 = \mathbf{P}_i \cdot \mathbf{v}_{i1}. \tag{2.28}$$

For all particles of the system, Eq. (2.28) takes the form

$$\sum_{i=1}^{n} \frac{m_i v_{i1}^2}{2} - \sum_{i=1}^{n} \frac{m_i v_{i0}^2}{2} + \sum_{i=1}^{n} \frac{m_i}{2} (\mathbf{v}_{i1} - \mathbf{v}_{i0})^2 = \sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{v}_{i1}. \tag{2.29}$$

Notice that

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{v}_{i1} = 0. \tag{2.30}$$

Denote by E_{c0} and E_{c1} the kinetic energies of the system at the beginning and at the end of impact. For every particle the energy loss E'_c is defined as

$$E'_c = \sum_{i=1}^{n} \frac{m_i}{2} (\mathbf{v}_{i1} - \mathbf{v}_{i0})^2, \tag{2.31}$$

and Eq. (2.29) in this case becomes

$$E_{c0} - E_{c1} = E'_c. \tag{2.32}$$

Equation (2.32) represents the Carnot's theorem.

2.7 Problems

Problem 2.1

Consider the case of a sphere falling on an horizontal surface. This case is assumed to be a boundary case for the impact of two spheres, where the surface is considered a sphere with the mass $m_2 \to \infty$ and being at rest ($v_2 = 0$). Suppose that the sphere with the mass m_1 falls perpendicularly on the surface and rebounds to a height H_1 (Fig. P2.1). The velocity of the sphere before the impact is

$$v_1 = \sqrt{2gH}. \tag{2.33}$$

Substituting Eq. (2.14) for $m_1/m_2 \to 0$ and $m_2/m_1 \to \infty$ into Eq. (2.33), yields

$$\begin{aligned} v'_1 &= v_1 - (1+e)v_1 = -ev_1, \qquad (2.34) \\ v'_2 &= 0, \end{aligned}$$

where e is the coefficient of restitution. If v'_1 is expressed as a function of the height H_1 at which the sphere rebounds,

$$ev_1 = \sqrt{2gH_1},$$

then,

$$e = \sqrt{\frac{H_1}{H}}. \tag{2.35}$$

In Fig. P2.1, the height H from which the sphere falls is known, and the height H_1 at which the sphere rebounds, can be measured. Now, it must be said that the sphere will perform few rebounds with different heights H_i. The goal is to compute the total displacement h of the sphere and the time T until the rest of the sphere. If $H_1, H_2, H_3, \ldots, H_n$ denote the further heights of the sphere then,

$$\begin{aligned} H_1 &= e^2 H \\ H_2 &= e^2 H_1 = e^4 H \\ \cdots & \quad \cdots\cdots \\ H_n &= e^{2n} H. \qquad (2.36) \end{aligned}$$

The total distance h covered by the sphere until the rest is

$$h = H + 2\left(\sum_{i=1}^{n} H_i\right). \tag{2.37}$$

Taking into account Eq. (2.36), it can be observed that $\sum_{i=1}^{\infty} H_i$ is the sum of a geometric progression with the first term $e^2 H$ and the common ratio e^2. Therefore,

$$h = \frac{1+e^2}{1-e^2}H. \tag{2.38}$$

The time t_0 necessary to cover the distance H is

$$t_0 = \frac{v_1}{g} = \frac{\sqrt{2gH}}{g} = \sqrt{\frac{2H}{g}}, \tag{2.39}$$

and for the distance H_1 the time t_1 is

$$t_1 = \sqrt{\frac{2H_1}{g}}.$$

The ratio t_1/t_0 is

$$\frac{t_1}{t_0} = \sqrt{\frac{H_1}{H}} = e,$$

and then

$$t_1 = et_0. \tag{2.40}$$

Similarly,

$$\begin{aligned} t_2 &= et_1 = e^2 t_0 \\ t_3 &= et_2 = e^3 t_0 \\ \cdots & \quad \cdots\cdots \\ t_n &= e^n t_0. \end{aligned} \tag{2.41}$$

Denoting by T the time necessary for the sphere to cover the total distance h, then

$$T = t_0 + 2\left(\sum_{i=1}^{n} t_i\right). \tag{2.42}$$

In the right hand side of Eq.(2.42) appears the sum of a geometric progression with the first term Rt_0 and the common ratio R. The sum of the progression is

$$T = t_0 + 2t_0\frac{e}{1-e} = \frac{1+e}{1-e}t_0,$$

or

$$T = \frac{1+e}{1-e}\sqrt{\frac{2H}{g}}. \tag{2.43}$$

Problem 2.2

Two smooth spheres of equal mass collide as shown in Fig. P2.2. The velocity vectors of the mass centers of the spheres, prior to impact, are given as $\mathbf{v}_1 = v_{1x}\,\imath + v_{1y}\,\jmath$, and $\mathbf{v}_2 = v_{2x}\,\imath + v_{2y}\,\jmath$. Assuming the coefficient of restitution to equal e determine the postimpact velocities of the spheres. Numerical application: v_{1x}=30 m/s, v_{1y}=10 m/s, v_{2x}=-10 m/s, v_{1y}=-30 m/s, e=0.5.

Solution

Considering the system consisting of the two spheres, the conservation of linear momentum equation and the definition of the coefficient of restitution give

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x} \quad \text{and} \quad e = \frac{v'_{2x} - v'_{1x}}{v_{1x} - v_{2x}}.$$

For two spheres 1 and 2 of identical masses, the postimpact velocities of the spheres along the impact axis, y-axis, are

$$v'_{1x} = v_{1x} - \frac{(v_{1x} - v_{2x})(1+e)}{1 + \dfrac{m_1}{m_2}} = v_{1x} - \frac{(v_{1x} - v_{2x})(1+e)}{2},$$

$$v'_{2x} = v_{2x} + \frac{(v_{1x} - v_{2x})(1+e)}{1 + \dfrac{m_2}{m_1}} = v_{2x} + \frac{(v_{1x} - v_{2x})(1+e)}{2}.$$

The postimpact velocities of the spheres along the normal axis, y-axis, are

$$v'_{1y} = v_{1y} \quad \text{and} \quad v'_{2y} = v_{2y}.$$

The final, or postimpact, velocities are

$$\mathbf{v}'_1 = \left[v_{1x} - \frac{(v_{1x} - v_{2x})(1+e)}{2} \right] \imath + v_{1y}\,\jmath,$$

$$\mathbf{v}'_2 = \left[v_{2x} + \frac{(v_{1x} - v_{2x})(1+e)}{2} \right] \imath + v_{2y}\,\jmath,$$

or numerically

$$\mathbf{v}'_1 = 10\,\jmath \quad \text{and} \quad \mathbf{v}'_2 = 20\,\imath - 30\,\jmath.$$

Problem 2.3

The spheres 1 and 2 of equal mass are shown in Fig. P2.3. Sphere 1 is released from rest at an angle α with the vertical axis. The coefficient of restitution between the spheres is e. Find the maximum angle β of the sphere 2 after the impact.

Solution

To determine the velocity of sphere 1 right before impact, v_{1i}, the work-energy theorem is applied to sphere 1

$$T_I^{(1)} + U_{I\to II}^{(1)} = T_{II}^{(1)},$$

or

$$0 + m\,g\,[l\,(1-\cos\alpha)] = \frac{m\,v_{1i}^2}{2},$$

where $U_{I\to II}^{(1)} = m\,g\,[l\,(1-\cos\alpha)]$ is the work done by 1 from the initial position I to the final position II and $T_{II}^{(1)} = \dfrac{m\,v_{1i}^2}{2}$ is the the kinetic energy of 1 at II. The velocity of sphere 1 before impact is

$$v_{1i} = \sqrt{2\,l\,g\,(1-\cos\alpha)}. \qquad (2.44)$$

The conservation of linear momentum for the system consisting of the two spheres gives

$$m\,v_{1i} + m\,v_{2i} = m\,v_1' + m\,v_2', \quad \text{or}$$
$$v_{1i} + 0 = v_1' + v_2'. \qquad (2.45)$$

The definition of the coefficient of restitution for the impact of the spheres gives

$$e = \frac{v_2' - v_1'}{v_{1i} - v_{2i}} = \frac{v_2' - v_1'}{v_{1i}}. \qquad (2.46)$$

From Eqs. (2.45) and (2.46) the post-impact velocity of the sphere 2 is

$$v_2' = 0.5\,(1+e)\,v_{1i} = 0.5\,(1+e)\,\sqrt{2\,l\,g\,(1-\cos\alpha)}. \qquad (2.47)$$

After the impact the motion of sphere 2 is analyzed again using the work-kinetic energy principle. The kinetic energy of 2 at position I is

$$T_I^{(1)} = \frac{m\,v_{2II}^2}{2} = \frac{m\,(v_2')^2}{2} = \frac{m\,l\,g\,(1+e)^2\,(1-\cos\alpha)}{4}.$$

The kinetic energy of 2 at position II is $T_{II}^{(2)} = \dfrac{m\, v_{2II}^2}{2} = 0$ because v_{2II}=0. The work done by 2 from I to II is $U_{I \to II}^{(2)} = -m\, g\, [l\, (1 - \cos\beta)]$. The vertical displacement is upward and the gravity force is downward and that is why the work $U_{I \to II}^{(2)}$ carries a minus sign. The work-kinetic energy principle for sphere 2 is

$$T_I^{(2)} + U_{I \to II}^{(2)} = T_{II}^{(2)},$$

or

$$\frac{m\, l\, g\, (1+e)^2\, (1 - \cos\alpha)}{4} - m\, g\, l\, (1 - \cos\beta) = 0.$$

The angle β is

$$\beta = \arccos\left[1 - \frac{(1+e)^2\, (1 - \cos\alpha)}{4}\right].$$

Problem 2.4

Three bodies 1, 2, and 3 of equal mass, m, are shown in Fig. P2.4(a). The bodies 2 and 3 are Initially stationary and the spring of elastic constant k is unstressed. Body 1 moves horizontally with the velocity v collides inelastically with body 2. Determine: a) the velocities of the bodies immediately after impact; b) the maximum kinetic energy of 3; c) the minimum kinetic energy of 2; d) the maximum compression of the spring; e) the final motion of 1.

Solution

No external forces act on the system in the direction of motion and therefore linear momentum is conserved. Because e=0, inelastic impact, the bodies 1 and 2 will move at the same velocity, v_2, immediately after impact. The principle of conservation of linear momentum gives

$$m\,v + m\,0 = (m+m)\,v_f,$$

and the velocity of the two bodies 1 and 2 after impact is $v_f = v/2$. After impact the resulting equivalent system in shown in Fig. P2.4(b). Body 3 does not move at the time of impact because the spring is initially unstressed and therefore 1 and 2 must move through a finite displacement before any force acts on 3. At all times after the impact, the total energy is conserved. It is equal to the kinetic energy just after impact, since no potential energy is stored in the spring at this time. The kinetic energy immediately after impact is

$$T = \frac{(m+m)\,v_f^2}{2} = \frac{2\,m\,(v/2)^2}{2} = \frac{m\,v^2}{4}.$$

Writing the general expression for the total energy and setting it equal to the kinetic energy just after impact, it is obtained

$$\frac{(m+m)\,v_{12}^2}{2} + \frac{m\,v_3^2}{2} + \frac{1}{2}\,k\,(x_2 - x_3) = \frac{m\,v^2}{4}, \tag{2.48}$$

where v_{12} is the velocity of bodies 1 and 2, and v_3 is the velocity of body 3. The principle of conservation of linear momentum for 1 and 2 moving together and for body 3 is

$$(m+m)\,v_f = (m+m)\,v_{12} + m\,v_3 \quad \text{or} \quad m\,v = 2\,v_{12} + m\,v_3 \quad \text{or}$$

$$v_{12} = \frac{v - v_3}{2}. \tag{2.49}$$

Because the total energy is conserved, the total kinetic energy will be maximum when the potential energy is minimum, that is, zero. This occurs when $x_2 = x_3$ corresponding to an unstressed spring. So, setting $x_2 = x_3$ and substituting for v_{12} in the energy equation, results

$$\frac{(v - v_3)^2}{4} + \frac{v_3^2}{2} = \frac{v^2}{4} \quad \text{or} \quad v_3\,(3\,v_3 - 2\,v) = 0,$$

from which

$$v_3 = 0 \quad \text{and} \quad v_3 = \frac{2\,v}{3}.$$

To find the extreme values of the kinetic energy associated with the individual particles, these values occur when the individual velocities reach extreme values, that is, when the accelerations are zero. But this occurs for a zero spring force, or $x_2 = x_3$. Therefore, it can be seen that the extreme values of the kinetic energy of the individual particles occur when the total kinetic energy is maximum. It is clear, then, that the maximum kinetic energy of 3 occurs for $v_3 = 2\,v/3$ that is, for the largest root

$$(T_3)_{\max} = \frac{m\,(2\,v/3)^2}{2} = \frac{2\,m\,v^2}{9}.$$

The linear momentum is conserved and v_{12} is minimum at the same time when v_3 is maximum

$$m\,v = 2\,v_{12} + m\,v_3 \quad \text{or} \quad m\,v = 2\,(v_{12})_{\min} + m\,(v_3)_{\max} \quad \text{or} \quad m\,v = 2\,(v_{12})_{\min} + 2\,m\,v/3,$$

or

$$(v_{12})_{\min} = \frac{v}{6}.$$

The corresponding kinetic energy of 2 is

$$(T_2)_{\min} = \frac{m\,(v/6)^2}{2} = \frac{m\,v^2}{72}.$$

The maximum compression of the spring occurs when the relative velocity of its two ends is zero, in which case all three particles are moving with the same velocity. The conservation of linear momentum is used to solve for the common velocity at this time

$$2\,m\,v_{12} + m\,v_3 = m\,v \quad \text{and} \quad v_{12} = v_3 \quad \Longrightarrow \quad v_{12} = v_3 = v/3.$$

Substituting these values into the general energy equation it results

$$\frac{(m+m)\,(v/3)^2}{2} + \frac{m\,(v/3)^2}{2} + \frac{1}{2}\,k\,(x_2 - x_3) = \frac{m\,v^2}{4}, \tag{2.50}$$

from which the maximum spring compression is

$$(x_2 - x_3)_{\max} = v\,\sqrt{m/(6\,k)}.$$

In order to solve for the final motion of 1 we recall that 1 hit 2 inelastically, but this does not imply that they move together indefinitely. In fact, 1 cannot accelerate to the right, since we assume that 2 cannot exert a pull on it. So, at the first instant in which 2 accelerates to the right, 1 and 2 will separate. This occurs when 2 accelerates from its minimum velocity $v/6$. From this time onward, 1 moves at a constant velocity $v/6$. Meanwhile, the center of mass of 2 and 3 translates uniformly at a velocity $5\,v/12$, as can be seen from momentum considerations. So 1 is left behind permanently. Masses 1 and 2 continue to oscillate relative to their center of mass, since the initial separation occurs at the moment of $(v_{12})_{\min}$ and $(v_3)_{\max}$, these extremes are never exceeded.

2.8 Problem Set

Problem Set 2.1

The spheres A and B of masses m_A and m_B are shown in Fig. PS2.1. Sphere A is released from rest at an angle α with the vertical axis. After impact the angle of the sphere B with the vertical axis is β. Find the coefficient of restitution between the spheres.

Problem Set 2.2

Mass m_1, moving along the x-axis with velocity v, hits m_2 and sticks to it (Fig. PS2.2). If all three particles are of equal mass m, and if m_2 and m_3 are connected by a rigid, massless rod, as shown, find the motion of the particles after impact. All particles can move without friction on the horizontal xy plane.

Problem Set 2.3

Solve Problem Set 2.2 for the case of perfectly elastic impact between m_1 and m_2. We assume that the spheres m_1 and m_2 are very small and perfectly smooth.