6 Dynamics of a Double Pendulum

A two link kinematic chain (double pendulum) is considered, Fig. 6.1. The links 1 and 2 have the masses m_1 and m_2 and the lengths L_1 and L_2. The system is free to move in a vertical plane. The local acceleration of gravity is g. Numerical application: $m_1 = m_2 = m$=1 kg, $L_1 = L_2 = L = 1$ m, and g=9.8 m/s^2.

Equations of Motion

The plane of motion is xy plane with the y-axis vertical, with the positive sense directed upward. The origin of the reference frame is at A The mass centers of the links are designated by $C_1(x_{C1}, y_{C1}, 0)$ and $C_2(x_{C2}, y_{C2}, 0)$.

Generalized coordinates

The number of degrees of freedom for the mechanism can be computed using the relation

$$M = 3n - 2c_5 - c_4,$$

where n is the number of moving links, c_5 is the number of pin joints or slider joints with one degree of freedom, and c_4 is the number of pin joints or slider joints with two degrees of freedom.

In our case, $n = 2,\ c_5 = 2,\ c_4 = 0$, and the mechanism has two degrees of freedom, $M = 2$, and two generalized coordinates. One can choose the angles $q_1(t)$ and $q_2(t)$ as the generalized coordinates (Fig. 6.1).

Kinematics

The position vector of the center of the mass C_1 of the link 1 is

$$\mathbf{r}_{C_1} = x_{C_1}\boldsymbol{\imath} + y_{C_1}\boldsymbol{\jmath}, \tag{6.1}$$

where x_{C_1} and y_{C_1} are the coordinates of C_1

$$x_{C_1} = \frac{L_1}{2}\cos q_1,\ \ y_{C_1} = \frac{L_1}{2}\sin q_1. \tag{6.2}$$

The position vector of the center of the mass C_2 of the link 2 is

$$\mathbf{r}_{C_2} = x_{C_2}\boldsymbol{\imath} + y_{C_2}\boldsymbol{\jmath}, \tag{6.3}$$

where x_{C_2} and y_{C_2} are the coordinates of C_2

$$x_{C_2} = L_1\cos q_1 + \frac{L_2}{2}\cos q_2,\ \ y_{C_2} = L_1\sin q_1 + \frac{L_2}{2}\sin q_2. \tag{6.4}$$

The velocity vector of C_1 is the derivative with respect to time of the position vector of C_1

$$\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1}\boldsymbol{\imath} + \dot{y}_{C_1}\boldsymbol{\jmath}, \tag{6.5}$$

where

$$\dot{x}_{C_1} = -\frac{L_1}{2}\dot{q}_1 \sin q_1,\ \dot{y}_{C_1} = \frac{L_1}{2}\dot{q}_1 \cos q_1. \tag{6.6}$$

The velocity vector of C_2 is the derivative with respect to time of the position vector of C_2

$$\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2}\boldsymbol{\imath} + \dot{y}_{C_2}\boldsymbol{\jmath}, \tag{6.7}$$

where

$$\begin{aligned} \dot{x}_{C_2} &= -L_1\dot{q}_1 \sin q_1 - \frac{L_2}{2}\dot{q}_2 \sin q_2, \\ \dot{y}_{C_2} &= L_1\dot{q}_1 \cos q_1 + \frac{L_2}{2}\dot{q}_2 \cos q_2. \end{aligned} \tag{6.8}$$

The acceleration vector of C_1 is the double derivative with respect to time of the position vector of C_1

$$\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1}\boldsymbol{\imath} + \ddot{y}_{C_1}\boldsymbol{\jmath}, \tag{6.9}$$

where

$$\begin{aligned} \ddot{x}_{C_1} &= -\frac{L_1}{2}\ddot{q}_1 \sin q_1 - \frac{L_1}{2}\dot{q}_1^2 \cos q_1, \\ \ddot{y}_{C_1} &= \frac{L_1}{2}\ddot{q}_1 \cos q_1 - \frac{L_1}{2}\dot{q}_1^2 \sin q_1. \end{aligned} \tag{6.10}$$

The acceleration vector of C_2 is the double derivative with respect to time of the position vector of C_2

$$\mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2}\boldsymbol{\imath} + \ddot{y}_{C_2}\boldsymbol{\jmath}, \tag{6.11}$$

where

$$\begin{aligned} \ddot{x}_{C_2} &= -L_1\ddot{q}_1 \sin q_1 - L_1\dot{q}_1^2 \cos q_1 - \frac{L_2}{2}\ddot{q}_2 \sin q_2 - \frac{L_2}{2}\dot{q}_2^2 \cos q_2, \\ \ddot{y}_{C_2} &= L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2. \end{aligned} \tag{6.12}$$

The angular velocity vectors of the links 1 and 2 are

$$\boldsymbol{\omega}_1 = \dot{q}_1\mathbf{k},\ \boldsymbol{\omega}_2 = \dot{q}_2\mathbf{k}. \tag{6.13}$$

The angular acceleration vectors of the links 1 and 2 are

$$\boldsymbol{\alpha}_1 = \ddot{q}_1\mathbf{k},\ \boldsymbol{\alpha}_2 = \ddot{q}_2\mathbf{k}. \tag{6.14}$$

Force analysis

The gravitational forces of the links 1 and 2 are

$$\mathbf{G}_1 = -m_1 g\boldsymbol{\jmath},\ \mathbf{G}_2 = -m_2 g\boldsymbol{\jmath}. \tag{6.15}$$

The mass moments of inertia of the link 1 with respect to the center of mass C_1 is

$$I_{C_1} = \frac{m_1 L_1^2}{12}.$$

The mass moments of inertia of the link 1 with respect to the fixed point of rotation A is

$$I_A = I_{C_1} + m_1\left(\frac{L_1}{2}\right)^2 = \frac{m_1 L_1^2}{3}.$$

The mass moment of inertia of the link 2 with respect to the center of mass C_2 is

$$I_{C_2} = \frac{m_2 L_2^2}{12}.$$

Newton-Euler equations of motion

In this section the equation of motion for the mechanism is solved using the Newton-Euler method. There are two rigid bodies in the system and one can write the Newton-Euler equations for each link using the free body diagrams shown in Fig. 6.2 .

a. Link 1

The Newton-Euler equations for the link 1 are

$$m_1\mathbf{a}_{C_1} = \mathbf{F}_{01} + \mathbf{F}_{21} + \mathbf{G}_1, \tag{6.16}$$

$$I_{C_1}\boldsymbol{\alpha} = \mathbf{r}_{C_1A} \times \mathbf{F}_{01} + \mathbf{r}_{C_1B} \times \mathbf{F}_{21}, \tag{6.17}$$

where $\mathbf{F}_{01}$ is the joint reaction of the ground 0 on the link 1 at point A, and $\mathbf{F}_{21}$ is the joint reaction of the link 2 on the link 1 at point B

$$\mathbf{F}_{01} = F_{01x}\boldsymbol{\imath} + F_{01y}\boldsymbol{\jmath},\quad \mathbf{F}_{21} = F_{21x}\boldsymbol{\imath} + F_{21y}\boldsymbol{\jmath}.$$

Since the link 1 has a fixed point of rotation at A the moment sum about the fixed point must be equal to to the product of the rod mass moment of inertia about that point and the rod angular acceleration. Thus

$$\begin{aligned}
&I_A\boldsymbol{\alpha} = \mathbf{r}_{AC_1} \times \mathbf{G}_1 + \mathbf{r}_{AB} \times \mathbf{F}_{21}, \quad \text{or} \qquad (6.18)\\
&\frac{mL_1^2}{3}\,\ddot{q}_1\mathbf{k} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_{C_1} & y_{C_1} & 0 \\ 0 & -m_1 g & 0 \end{vmatrix} + \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_B & y_B & 0 \\ F_{21x} & F_{21y} & 0 \end{vmatrix}, \quad \text{or}\\
&\frac{mL_1^2}{3}\,\ddot{q}_1\mathbf{k} = (-m_1 g x_{C_1} + F_{21y}x_B - F_{21x}y_B)\mathbf{k}.
\end{aligned}$$

The equation of motion for link 1 is

$$\frac{mL_1^2}{3}\,\ddot{q}_1 = \left(-m_1 g\frac{L_1}{2}\cos q_1 + F_{21y}L_1\cos q_1 - F_{21x}L_1\sin q_1\right). \tag{6.19}$$

b. Link 2

The Newton-Euler equations for the link 2 are

$$\begin{aligned}
m_2\mathbf{a}_{C_2} &= \mathbf{F}_{12} + \mathbf{G}_2,\\
I_{C_2}\boldsymbol{\alpha}_2 &= \mathbf{r}_{C_2B} \times \mathbf{F}_{12},
\end{aligned} \tag{6.20}$$

where $\mathbf{F}_{12} = -\mathbf{F}_{21}$ is the joint reaction of the link 1 on the link 2 at B. Equation (6.20 can be written as

$$\begin{aligned}
m_2\ddot{x}_{C_2} &= -F_{21x},\\
m_2\ddot{y}_{C_2} &= -F_{21y} - m_2\, g,\\
\frac{mL_2^2}{12}\,\ddot{q}_2\mathbf{k} &= \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_B - x_{C_2} & y_B - y_{C_2} & 0 \\ -F_{21x} & -F_{21y} & 0 \end{vmatrix},
\end{aligned} \tag{6.21}$$

or

$$\begin{aligned}
&m_2\left(-L_1\ddot{q}_1\sin q_1 - L_1\dot{q}_1^2\cos q_1 - \frac{L_2}{2}\ddot{q}_2\sin q_2 - \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)\\
&= -F_{21x}, \qquad (6.22)\\
&m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2\right)\\
&= -F_{21y} - m_2\, g, \qquad (6.23)\\
&\frac{mL_2^2}{12}\,\ddot{q}_2 = \frac{L_2}{2}\left(-F_{21y}\cos q_2 + F_{21x}\sin q_2\right). \qquad (6.24)
\end{aligned}$$

The reaction components F_{21x} and F_{21x} are obtained from Eqs. (6.22)(6.23)

$$F_{21x} = m_2\left(L_1\ddot{q}_1 \sin q_1 + L_1\dot{q}_1^2 \cos q_1 + \frac{L_2}{2}\ddot{q}_2 \sin q_2 + \frac{L_2}{2}\dot{q}_2^2 \cos q_2\right), \tag{6.25}$$

$$F_{21y} = -m_2\left(L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2\right) + m_2\, g. \tag{6.26}$$

The equations of motion are obtained substituting F_{21x} and F_{21x} in Eq. (6.19) and Eq. (6.24)

$$\begin{aligned}\frac{mL_1^2}{3}\,\ddot{q}_1 &= -m_1 g\frac{L_1}{2}\cos q_1 - \\ &m_2\left(L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2 - g\right) L_1 \cos q_1 - \\ &m_2\left(L_1\ddot{q}_1 \sin q_1 + L_1\dot{q}_1^2 \cos q_1 + \frac{L_2}{2}\ddot{q}_2 \sin q_2 + \frac{L_2}{2}\dot{q}_2^2 \cos q_2\right) L_1 \sin q_1,\end{aligned} \tag{6.27}$$

$$\begin{aligned}\frac{mL_2^2}{12}\,\ddot{q}_2 &= \\ &\frac{m_2\, L_2}{2}\left(L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2 - g\right)\cos q_2 + \\ &\frac{m_2\, L_2}{2}\left(L_1\ddot{q}_1 \sin q_1 + L_1\dot{q}_1^2 \cos q_1 + \frac{L_2}{2}\ddot{q}_2 \sin q_2 + \frac{L_2}{2}\dot{q}_2^2 \cos q_2\right)\sin q_2.\end{aligned} \tag{6.28}$$

The eqautions of motion represent two nonlinear differential equations. The initial conditions (Cauchy problem) are necessary to solve the equations At $t = 0$ the initial conditions are

$$q_1(0) = q_{10},\ q_2(0) = q_{20},\ \dot{q}_1(0) = \dot{q}_{10},\ \dot{q}_2(0) = \dot{q}_{20}$$