

<i>4 Velocity and Acceleration Analysis</i>	0
---	---

Contents

4 Velocity and Acceleration Analysis	1
4.1 Introduction	1
4.2 Velocity Field for a Rigid Body	2
4.3 Acceleration Field for a Rigid Body	5
4.4 Motion of a Point that Moves Relative to a Rigid Body	10
4.5 Problems	13

4 Velocity and Acceleration Analysis

4.1 Introduction

The motion of a rigid body (RB) is defined when the position vector, velocity and acceleration of all points of the rigid body are defined as functions of time with respect to a fixed reference frame with the origin at O_0 .

Let $\mathbf{i}_0, \mathbf{j}_0$ and \mathbf{k}_0 , be the constant unit vectors of a fixed orthogonal Cartesian reference frame $x_0y_0z_0$ and \mathbf{i}, \mathbf{j} and \mathbf{k} be the unit vectors of a body fixed (mobile or rotating) orthogonal Cartesian reference frame xyz (Fig. 4.1). The unit vectors $\mathbf{i}_0, \mathbf{j}_0$, and \mathbf{k}_0 of the primary reference frame are constant with respect to time.

A reference frame that moves with the rigid body is a *body fixed* (or mobile) reference frame. The unit vectors \mathbf{i}, \mathbf{j} , and \mathbf{k} of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame. The location of the point O is arbitrary.

The position vector of a point M ($M \in (RB)$), with respect to the fixed reference frame $x_0y_0z_0$ is denoted by $\mathbf{r}_1 = \mathbf{r}_{O_0M}$ and with respect to the mobile reference frame $Oxyz$ is denoted by $\mathbf{r} = \mathbf{r}_{OM}$. The location of the origin O of the mobile reference frame with respect to the fixed point O_0 is defined by the position vector $\mathbf{r}_O = \mathbf{r}_{O_0O}$.

Then the relation between the vectors \mathbf{r}_1, \mathbf{r} and \mathbf{r}_O is given by

$$\mathbf{r}_1 = \mathbf{r}_O + \mathbf{r} = \mathbf{r}_O + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad (4.1)$$

where x, y and z represent the projections of the vector $\mathbf{r} = \mathbf{r}_{OM}$ on the mobile reference frame: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

The magnitude of the vector $\mathbf{r} = \mathbf{r}_{OM}$ is a constant as the distance between the points O and M is constant ($O \in (RB)$ and $M \in (RB)$). Thus, the x, y and z components of the vector \mathbf{r} with respect to the mobile reference frame are constant. The unit vectors \mathbf{i}, \mathbf{j} and \mathbf{k} are time-dependent vector functions.

The vectors \mathbf{i}, \mathbf{j} and \mathbf{k} are the unit vector of an orthogonal Cartesian reference frame, thus one can write

$$\mathbf{i} \cdot \mathbf{i} = 1, \quad \mathbf{j} \cdot \mathbf{j} = 1, \quad \mathbf{k} \cdot \mathbf{k} = 1, \quad (4.2)$$

$$\mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{j} \cdot \mathbf{k} = 0, \quad \mathbf{k} \cdot \mathbf{i} = 0. \quad (4.3)$$

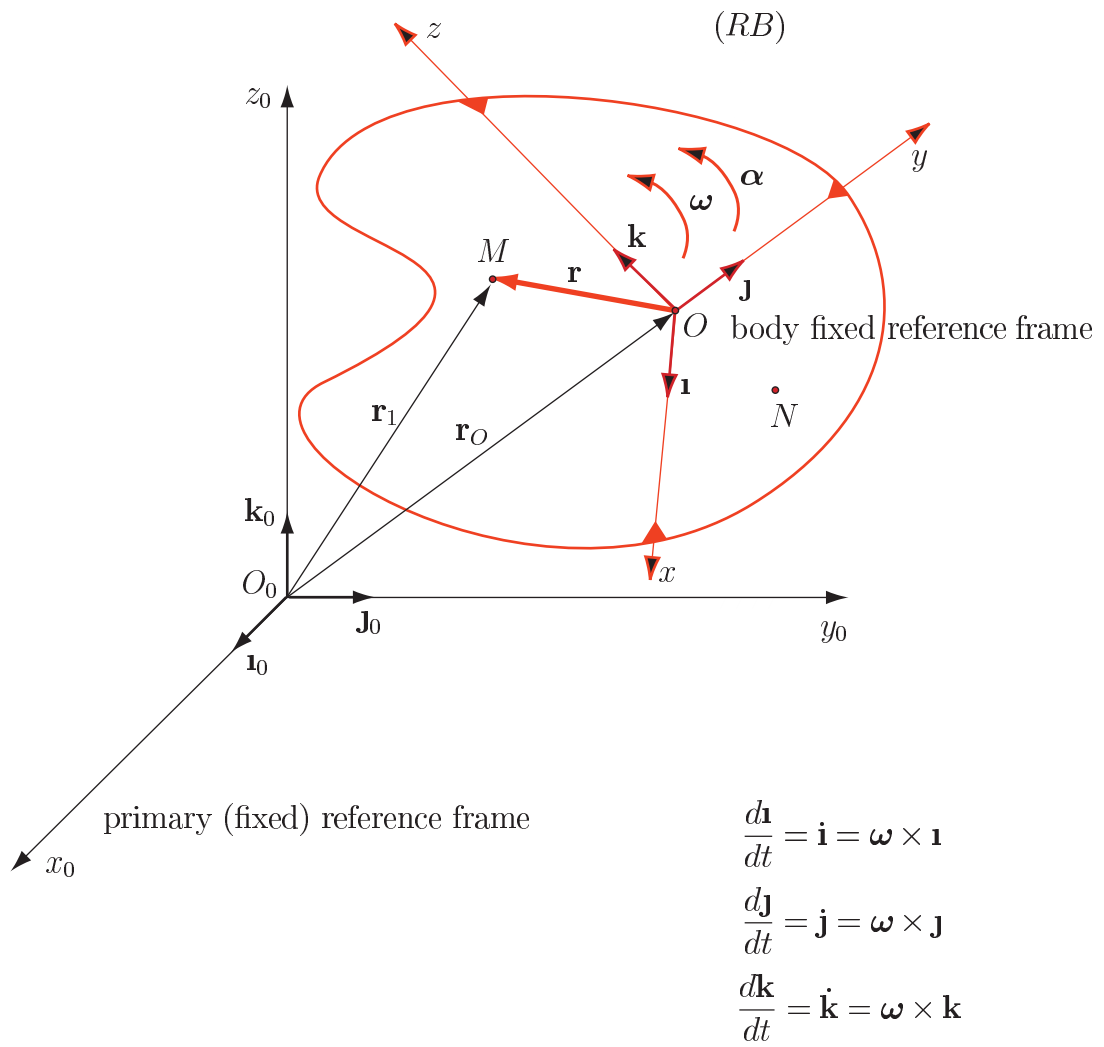


Figure 4.1

4.2 Velocity Field for a Rigid Body

The velocity of an arbitrary point M of the rigid body with respect to the fixed reference frame $x_0y_0z_0$, is the derivative with respect to time of the position vector \mathbf{r}_1

$$\mathbf{v} = \frac{d\mathbf{r}_1}{dt} = \frac{d\mathbf{r}_{O_0M}}{dt} = \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_O + \dot{\mathbf{r}} = \mathbf{v}_O + x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}} + \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}, \quad (4.4)$$

where $\mathbf{v}_O = \dot{\mathbf{r}}_O$ represent the velocity of the origin of the mobile reference frame $O_1x_1y_1z_1$ with respect to the fixed reference frame $Oxyz$. Because all the points in the rigid body maintain their relative position, their velocity relative to the mobile reference frame xyz is zero, i.e., $\dot{x} = \dot{y} = \dot{z} = 0$.

The velocity of point M is

$$\mathbf{v} = \mathbf{v}_O + x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}}.$$

The derivative of the Eqs.(4.2) and (4.3) with respect to time gives

$$\dot{\mathbf{i}} \cdot \mathbf{i} = 0, \quad \dot{\mathbf{j}} \cdot \mathbf{j} = 0, \quad \dot{\mathbf{k}} \cdot \mathbf{k} = 0, \quad (4.5)$$

and

$$\dot{\mathbf{i}} \cdot \mathbf{j} + \mathbf{j} \cdot \dot{\mathbf{i}} = 0, \quad \dot{\mathbf{j}} \cdot \mathbf{k} + \dot{\mathbf{k}} \cdot \mathbf{j} = 0, \quad \dot{\mathbf{k}} \cdot \mathbf{i} + \dot{\mathbf{i}} \cdot \mathbf{k} = 0. \quad (4.6)$$

For Eq.(4.6) one can introduce the convention

$$\begin{aligned} \dot{\mathbf{i}} \cdot \mathbf{j} &= -\mathbf{j} \cdot \dot{\mathbf{i}} = \omega_z, \\ \dot{\mathbf{j}} \cdot \mathbf{k} &= -\dot{\mathbf{k}} \cdot \mathbf{j} = \omega_x, \\ \dot{\mathbf{k}} \cdot \mathbf{i} &= -\dot{\mathbf{i}} \cdot \mathbf{k} = \omega_y, \end{aligned} \quad (4.7)$$

where ω_x , ω_y and ω_z may be considered as the projections of a vector $\boldsymbol{\omega}$, $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$.

To calculate $\dot{\mathbf{i}}$, $\dot{\mathbf{j}}$, $\dot{\mathbf{k}}$ one can use the relation for an arbitrary vector \mathbf{v}

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k}. \quad (4.8)$$

Using Eq.(4.8) and the results from Eqs.(4.5) and (4.6) one can write

$$\begin{aligned} \dot{\mathbf{i}} &= (\dot{\mathbf{i}} \cdot \mathbf{i})\mathbf{i} + (\dot{\mathbf{i}} \cdot \mathbf{j})\mathbf{j} + (\dot{\mathbf{i}} \cdot \mathbf{k})\mathbf{k} \\ &= (0)\mathbf{i} + (\omega_z)\mathbf{j} - (\omega_y)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{i}, \end{aligned}$$

$$\begin{aligned}
\mathbf{j} &= (\mathbf{j} \cdot \mathbf{i}) \mathbf{i} + (\mathbf{j} \cdot \mathbf{J}) \mathbf{J} + (\mathbf{j} \cdot \mathbf{k}) \mathbf{k} \\
&= (-\omega_z) \mathbf{i} + (0) \mathbf{J} + (\omega_x) \mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{J} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 1 & 0 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{J},
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{k}} &= (\dot{\mathbf{k}} \cdot \mathbf{i}) \mathbf{i} + (\dot{\mathbf{k}} \cdot \mathbf{J}) \mathbf{J} + (\dot{\mathbf{k}} \cdot \mathbf{k}) \mathbf{k} \\
&= (\omega_y) \mathbf{i} - (\omega_x) \mathbf{J} + (0) \mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{J} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 0 & 1 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{k}.
\end{aligned} \tag{4.9}$$

The relations

$$\mathbf{i} = \boldsymbol{\omega} \times \mathbf{i}, \quad \mathbf{j} = \boldsymbol{\omega} \times \mathbf{J}, \quad \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k}. \tag{4.10}$$

are known as *Poisson formulas*.

Using Eqs.(4.4) and (4.10) one can obtain

$$\mathbf{v} = \mathbf{v}_O + x\boldsymbol{\omega} \times \mathbf{i} + y\boldsymbol{\omega} \times \mathbf{J} + z\boldsymbol{\omega} \times \mathbf{k} = \mathbf{v}_O + \boldsymbol{\omega} \times (x\mathbf{i} + y\mathbf{J} + z\mathbf{k}),$$

or

$$\mathbf{v} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}. \tag{4.11}$$

Combining Eqs.(4.4) and (4.11) it results

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}. \tag{4.12}$$

Using Eq.(4.11) one can write the components of the velocity as

$$\begin{aligned}
v_x &= v_{Ox} + z\omega_y - y\omega_z, \\
v_y &= v_{Oy} + x\omega_z - z\omega_x, \\
v_z &= v_{Oz} + y\omega_x - x\omega_y.
\end{aligned}$$

The relation between the velocities \mathbf{v}_M and \mathbf{v}_O of two points M and O on the rigid body is

$$\mathbf{v}_M = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OM}, \tag{4.13}$$

or

$$\mathbf{v}_M = \mathbf{v}_O + \mathbf{v}_{MO}^{rel}, \tag{4.14}$$

where \mathbf{v}_{MO}^{rel} is the relative velocity, for rotational motion, of M with respect to O and is given by

$$\mathbf{v}_{MO}^{rel} = \mathbf{v}_{MO} = \boldsymbol{\omega} \times \mathbf{r}_{OM}. \quad (4.15)$$

The relative velocity \mathbf{v}_{MO} is perpendicular to the position vector \mathbf{r}_{OM} , $\mathbf{v}_{MO} \perp \mathbf{r}_{OM}$, and has the direction given by the angular velocity vector $\boldsymbol{\omega}$. The magnitude of the relative velocity is $|\mathbf{v}_{MO}| = v_{MO} = \omega r_{OM}$.

4.3 Acceleration Field for a Rigid Body

The acceleration of an arbitrary point $M \in (RB)$ with respect to a fixed reference frame $O_0x_0y_0z_0$, represents the double derivative with respect to time of the position vector \mathbf{r}_1

$$\mathbf{a} = \ddot{\mathbf{r}}_1 = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}) = \frac{d}{dt}\mathbf{v}_O + \frac{d}{dt}\boldsymbol{\omega} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d}{dt}\mathbf{r} = \dot{\mathbf{v}}_O + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}. \quad (4.16)$$

The acceleration of the point O with respect to the fixed reference frame $O_0x_0y_0z_0$ is

$$\mathbf{a}_O = \dot{\mathbf{v}}_O = \ddot{\mathbf{r}}_O. \quad (4.17)$$

The derivative of the vector $\boldsymbol{\omega}$ with respect to the time is the vector $\boldsymbol{\alpha}$ given by

$$\begin{aligned} \boldsymbol{\alpha} &= \dot{\boldsymbol{\omega}} = \dot{\omega}_x \mathbf{i} + \dot{\omega}_y \mathbf{j} + \dot{\omega}_z \mathbf{k} + \omega_x \dot{\mathbf{i}} + \omega_y \dot{\mathbf{j}} + \omega_z \dot{\mathbf{k}} \\ &= \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} + \omega_x \boldsymbol{\omega} \times \mathbf{i} + \omega_y \boldsymbol{\omega} \times \mathbf{j} + \omega_z \boldsymbol{\omega} \times \mathbf{k} \\ &= \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k}. \end{aligned} \quad (4.18)$$

where $\alpha_x = \dot{\omega}_x$, $\alpha_y = \dot{\omega}_y$, and $\alpha_z = \dot{\omega}_z$.

In the previous expression the Poisson formulas

$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i}, \quad \dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j}, \quad \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k},$$

have been used.

Using Eqs.(4.16), (4.17) and (4.18) one can write the acceleration of the point M as

$$\mathbf{a} = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (4.19)$$

Using Eq.(4.19) one can write the components of the acceleration as

$$\begin{aligned} a_x &= a_{Ox} + (z\alpha_y - y\alpha_z) + \omega_y(y\omega_x - x\omega_y) + \omega_z(x\omega_x - x\omega_z), \\ a_y &= a_{Oy} + (x\alpha_z - z\alpha_x) + \omega_z(z\omega_y - y\omega_z) + \omega_x(x\omega_y - y\omega_x), \\ a_z &= a_{Oz} + (y\alpha_x - x\alpha_y) + \omega_x(x\omega_z - z\omega_x) + \omega_y(y\omega_z - z\omega_y). \end{aligned}$$

The relation between the accelerations \mathbf{a}_M and \mathbf{a}_O of two points M and O on the rigid body is

$$\mathbf{a}_M = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{OM}). \quad (4.20)$$

In the case of planar motion

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{OM}) = -\omega^2 \mathbf{r}_{OM},$$

and Eq. (4.20) becomes

$$\mathbf{a}_M = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OM} - \omega^2 \mathbf{r}_{OM}. \quad (4.21)$$

Equation (4.22) can be written as

$$\mathbf{a}_M = \mathbf{a}_O + \mathbf{a}_{MO}^{rel}, \quad (4.22)$$

where \mathbf{a}_{MO}^{rel} is the relative acceleration, for rotational motion, of M with respect to O and is given by

$$\mathbf{a}_{MO}^{rel} = \mathbf{a}_{MO} = \mathbf{a}_{MO}^n + \mathbf{a}_{MO}^t. \quad (4.23)$$

The normal relative acceleration of M with respect to O is

$$\mathbf{a}_{MO}^n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{OM}), \quad (4.24)$$

is parallel to the position vector \mathbf{r}_{OM} , $\mathbf{a}_{MO}^n \parallel \mathbf{r}_{OM}$, and has the direction towards the center of rotation, from M to O . The magnitude of the normal relative acceleration is

$$|\mathbf{a}_{MO}^n| = a_{MO}^n = \omega^2 r_{OM} = \frac{v_{MO}^2}{r_{OM}}.$$

The tangential relative acceleration of M with respect to O is

$$\mathbf{a}_{MO}^t = \boldsymbol{\alpha} \times \mathbf{r}_{OM}, \quad (4.25)$$

is perpendicular to the position vector \mathbf{r}_{OM} , $\mathbf{a}_{MO}^t \perp \mathbf{r}_{OM}$, and has the direction given by the angular velocity $\boldsymbol{\alpha}$. The magnitude of the normal relative acceleration is

$$|\mathbf{a}_{MO}^t| = a_{MO}^t = \alpha r_{OM}.$$

REMARKS

1. If the orientation of a rigid body RB in a reference frame RF_0 depends on only a single scalar variable ζ , there exists for each value of ζ a vector $\boldsymbol{\omega}$ such that the derivative with respect to ζ in RF_0 of every vector \mathbf{c} fixed in rigid body RB is given by

$$\frac{d\mathbf{c}}{d\zeta} = \boldsymbol{\omega} \times \mathbf{c}, \quad (4.26)$$

where the vector $\boldsymbol{\omega}$ is the rate of change of orientation of the rigid body RB in the reference frame RF_0 with respect to ζ . The vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \frac{\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}}, \quad (4.27)$$

where \mathbf{a} and \mathbf{b} are any two nonparallel vectors fixed in the rigid body RB .

The vector $\boldsymbol{\omega}$ is a free vector, i.e. is not associated with any particular point. With the help of $\boldsymbol{\omega}$ one can replace the process of differentiation with that of cross multiplication.

The vector $\boldsymbol{\omega}$ may be expressed in a symmetrical relation in \mathbf{a} and \mathbf{b}

$$\boldsymbol{\omega} = \frac{1}{2} \left(\frac{\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} + \frac{\frac{d\mathbf{b}}{d\zeta} \times \frac{d\mathbf{a}}{d\zeta}}{\frac{d\mathbf{b}}{d\zeta} \cdot \mathbf{a}} \right). \quad (4.28)$$

2. The first derivatives of a vector \mathbf{p} with respect to a scalar variable ζ in two reference frames RF_i and RF_j are related as follows

$$\frac{{}^{(j)}d\mathbf{p}}{d\zeta} = \frac{{}^{(i)}d\mathbf{p}}{d\zeta} + \boldsymbol{\omega}_{ij} \times \mathbf{p}, \quad (4.29)$$

where $\boldsymbol{\omega}_{ij}$ is the rate of change of orientation of RF_i in RF_j with respect to ζ and $\frac{{}^{(j)}d\mathbf{p}}{d\zeta}$ is the total derivative of \mathbf{p} with respect to ζ in RF_j .

Proof

The vector \mathbf{p} can be expressed as

$$\mathbf{p} = p_1 \mathbf{1}_1 + p_2 \mathbf{1}_2 + p_3 \mathbf{1}_3,$$

where $\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3$ are three units vectors not parallel to the same plane fixed in RF_i , and p_x, p_y, p_z are the scalar measure numbers of \mathbf{p} . Differentiating in RF_j

$$\begin{aligned} \frac{{}^{(j)}d\mathbf{p}}{d\zeta} &= \frac{{}^{(j)}d}{d\zeta} (p_1 \mathbf{1}_1 + p_2 \mathbf{1}_2 + p_3 \mathbf{1}_3) \\ &= \frac{{}^{(j)}d p_1}{d\zeta} \mathbf{1}_1 + \frac{{}^{(j)}d p_2}{d\zeta} \mathbf{1}_2 + \frac{{}^{(j)}d p_3}{d\zeta} \mathbf{1}_3 + p_1 \frac{{}^{(j)}d \mathbf{1}_1}{d\zeta} + p_2 \frac{{}^{(j)}d \mathbf{1}_2}{d\zeta} + p_3 \frac{{}^{(j)}d \mathbf{1}_3}{d\zeta} \\ &= \frac{d p_1}{d\zeta} \mathbf{1}_1 + \frac{d p_2}{d\zeta} \mathbf{1}_2 + \frac{d p_3}{d\zeta} \mathbf{1}_3 + p_1 \boldsymbol{\omega}_{ij} \times \mathbf{1}_1 + p_2 \boldsymbol{\omega}_{ij} \times \mathbf{1}_2 + p_3 \boldsymbol{\omega}_{ij} \times \mathbf{1}_3 \\ &= \frac{{}^{(i)}d p_1}{d\zeta} \mathbf{1}_1 + \frac{{}^{(i)}d p_2}{d\zeta} \mathbf{1}_2 + \frac{{}^{(i)}d p_3}{d\zeta} \mathbf{1}_3 + \boldsymbol{\omega}_{ij} \times (p_1 \mathbf{1}_1 + p_2 \mathbf{1}_2 + p_3 \mathbf{1}_3) \\ &= \frac{{}^{(i)}d\mathbf{p}}{d\zeta} + \boldsymbol{\omega}_{ij} \times \mathbf{p}. \end{aligned} \tag{4.30}$$

3. The angular velocity of a rigid body RB in a reference frame RF_0 is the rate of change of orientation with respect to the time t

$$\boldsymbol{\omega} = \frac{1}{2} \left(\frac{d\mathbf{a}}{dt} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{b}}{dt} \times \frac{d\mathbf{a}}{dt} \right) = \frac{1}{2} \left(\frac{\dot{\mathbf{a}} \times \dot{\mathbf{b}}}{\dot{\mathbf{a}} \cdot \mathbf{b}} + \frac{\dot{\mathbf{b}} \times \dot{\mathbf{a}}}{\dot{\mathbf{b}} \cdot \mathbf{a}} \right).$$

The direction of $\boldsymbol{\omega}$ is related to the direction of the rotation of the rigid body through a right-hand rule.

4. Let RF_i , $i = 1, 2, \dots, n$ be n reference frames. The angular velocity of a rigid body r in the reference frame RF_n , can be expressed as

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r1} + \boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{23} + \dots + \boldsymbol{\omega}_{r,n-1}.$$

Proof

Let \mathbf{p} be any vector fixed in the rigid body. Then

$$\begin{aligned} \frac{{}^{(i)}d\mathbf{p}}{dt} &= \boldsymbol{\omega}_{ri} \times \mathbf{p} \\ \frac{{}^{(i-1)}d\mathbf{p}}{dt} &= \boldsymbol{\omega}_{r,i-1} \times \mathbf{p}. \end{aligned}$$

On the other hand

$$\frac{{}^{(i)}d\mathbf{p}}{dt} = \frac{{}^{(i-1)}d\mathbf{p}}{dt} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p}.$$

Hence

$$\boldsymbol{\omega}_{ri} \times \mathbf{p} = \boldsymbol{\omega}_{r,i-1} \times \mathbf{p} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p},$$

as this equation is satisfied for all \mathbf{p} fixed in the rigid body

$$\boldsymbol{\omega}_{ri} = \boldsymbol{\omega}_{r,i-1} + \boldsymbol{\omega}_{i,i-1}. \quad (4.31)$$

With $i = n$, Eq. (4.31) gives

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-1} + \boldsymbol{\omega}_{n,n-1}. \quad (4.32)$$

With $i = n - 1$, Eq. (4.31) gives

$$\boldsymbol{\omega}_{r,n-1} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2}. \quad (4.33)$$

Substitute Eq. (4.33) into Eq. (4.32)

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2} + \boldsymbol{\omega}_{n,n-1}.$$

Next use Eq. (4.31) with $i = n - 2$, then with $i = n - 3$, and so forth.

4.4 Motion of a Point that Moves Relative to a Rigid Body

A reference frame that moves with the rigid body is a body fixed reference frame. Figure 4.2 shows a rigid body (RB), in motion relative to a primary reference frame with its origin at point $O_0, x_0y_0z_0$. The primary reference frame is a fixed reference frame or an earth fixed reference frame. The unit vectors $\mathbf{i}_0, \mathbf{j}_0$, and \mathbf{k}_0 of the primary reference frame are constant.

The body fixed reference frame, xyz , has its origin at a point O of the rigid body ($O \in (RB)$), and is a moving reference frame relative to the primary reference. The unit vectors \mathbf{i}, \mathbf{j} , and \mathbf{k} of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame.

The position vector of a point P of the rigid body ($P \in (RB)$) relative to the origin, O , of the body fixed reference frame is the vector \mathbf{r}_{OP} . The velocity of P relative to O is

$$\frac{d\mathbf{r}_{OP}}{dt} = \mathbf{v}_{PO}^{rel} = \boldsymbol{\omega} \times \mathbf{r}_{OP},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body.

The position vector of a point A (the point A is not assumed to be a point of the rigid body, $A \notin (RB)$), relative to the origin O_0 of the primary reference frame is, Fig. 4.2

$$\mathbf{r}_A = \mathbf{r}_O + \mathbf{r},$$

where

$$\mathbf{r} = \mathbf{r}_{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

is the position vector of A relative to the origin O , of the body fixed reference frame, and x, y , and z are the coordinates of A in terms of the body fixed reference frame. The velocity of the point A is the time derivative of the position vector \mathbf{r}_A

$$\begin{aligned} \mathbf{v}_A &= \frac{d\mathbf{r}_O}{dt} + \frac{d\mathbf{r}}{dt} = \mathbf{v}_O + \mathbf{v}_{AO}^{rel} = \\ &= \mathbf{v}_O + \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt} + \frac{dy}{dt}\mathbf{j} + y\frac{d\mathbf{j}}{dt} + \frac{dz}{dt}\mathbf{k} + z\frac{d\mathbf{k}}{dt}. \end{aligned}$$

Using Poisson formulas, the total derivative of the the position vector \mathbf{r} is

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} + \boldsymbol{\omega} \times \mathbf{r}.$$

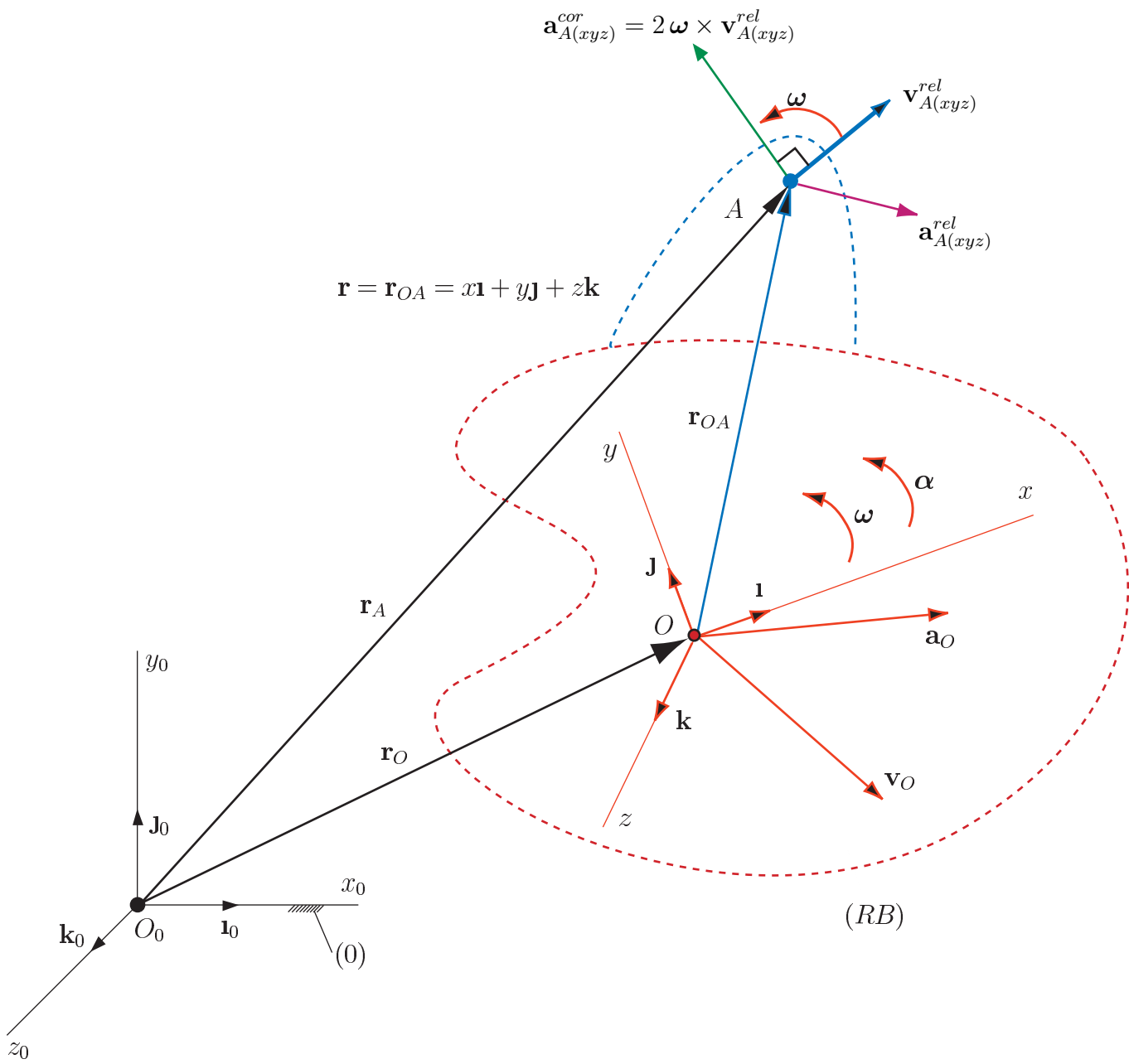


Figure 4.2

The velocity of A relative to the body fixed reference frame is a derivative in the body fixed reference frame

$$\mathbf{v}_{A(xyz)}^{rel} = \frac{{}^{(xyz)}d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}, \quad (4.34)$$

A general formula for the total derivative of a moving vector \mathbf{r} may be written as

$$\frac{d\mathbf{r}}{dt} = \frac{{}^{(xyz)}d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}, \quad (4.35)$$

where $\frac{d\mathbf{r}}{dt} = \frac{{}^{(0)}d\mathbf{r}}{dt}$ is the derivative in the fixed (primary) reference frame (0) ($x_0y_0z_0$), and $\frac{{}^{(xyz)}d\mathbf{r}}{dt}$ is the derivative in the mobile (rotating or body fixed) reference frame (xyz).

The velocity of the point A relative to the primary reference frame is

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A(xyz)}^{rel} + \boldsymbol{\omega} \times \mathbf{r}, \quad (4.36)$$

Equation (4.36) expresses the velocity of a point A as the sum of three terms:

- the velocity of a point O of the rigid body,
- the velocity $\mathbf{v}_{A(xyz)}^{rel}$ of A relative to the rigid body, and
- the velocity $\boldsymbol{\omega} \times \mathbf{r}$ of A relative to O due to the rotation of the rigid body.

The acceleration of the point A relative to the primary reference frame is obtained by taking the time derivative of Eq. (4.36)

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO}, \\ &= \mathbf{a}_O + \mathbf{a}_{A(xyz)}^{rel} + 2\boldsymbol{\omega} \times \mathbf{v}_{A(xyz)}^{rel} + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \end{aligned} \quad (4.37)$$

where

$$\mathbf{a}_{A(xyz)}^{rel} = \frac{{}^{(xyz)}d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}, \quad (4.38)$$

is the acceleration of A relative to the body fixed reference frame or relative to the rigid body. The term

$$\mathbf{a}_{A(xyz)}^{cor} = 2\boldsymbol{\omega} \times \mathbf{v}_{A(xyz)}^{rel}.$$

is called the Coriolis acceleration. The direction of the Coriolis acceleration is obtained by rotating the linear relative velocity $\mathbf{v}_{A(xyz)}^{rel}$ through 90° in the direction of rotation given by $\boldsymbol{\omega}$.

In the case of planar motion, Eq. (4.37) becomes

$$\begin{aligned}\mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{OA}, \\ &= \mathbf{a}_O + \mathbf{a}_{A(xyz)}^{rel} + 2\boldsymbol{\omega} \times \mathbf{v}_{A(xyz)}^{rel} + \boldsymbol{\alpha} \times \mathbf{r} - \boldsymbol{\omega}^2 \mathbf{r},\end{aligned}\quad (4.39)$$

The motion of the rigid body (RB) is described relative to the primary reference frame. The velocity \mathbf{v}_A and the acceleration \mathbf{a}_A of a point A are relative to the primary reference frame. The terms $\mathbf{v}_{A(xyz)}^{rel}$ and $\mathbf{a}_{A(xyz)}^{rel}$ are the velocity and acceleration of point A relative to the body fixed reference frame i.e., they are the velocity and acceleration measured by an observer moving with the rigid body, Fig. 4.2.

If A is a point of the rigid body, $A \in RB$, $\mathbf{v}_{Arel} = \mathbf{0}$ and $\mathbf{a}_{Arel} = \mathbf{0}$.

Motion of a point relative to a moving reference frame

The velocity and acceleration of an arbitrary point A relative to a point O of a rigid body, in terms of the body fixed reference frame, are given by Eqs. (4.36) and (4.37)

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{AO}^{rel} + \boldsymbol{\omega} \times \mathbf{r}_{OA}, \quad (4.40)$$

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{AO}^{rel} + 2\boldsymbol{\omega} \times \mathbf{v}_{AO}^{rel} + \boldsymbol{\alpha} \times \mathbf{r}_{OA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{OA}). \quad (4.41)$$

These results apply to any reference frame having a moving origin O and rotating with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ relative to a primary reference frame (Fig. 4.2). The terms \mathbf{v}_A and \mathbf{a}_A are the velocity and acceleration of an arbitrary point A relative to the primary reference frame. The terms \mathbf{v}_{AO}^{rel} and \mathbf{a}_{AO}^{rel} are the velocity and acceleration of A relative to the secondary moving reference frame i.e., they are the velocity and acceleration measured by an observer moving with the secondary reference frame. The Coriolis acceleration is $\mathbf{a}_{AO}^{cor} = 2\boldsymbol{\omega} \times \mathbf{v}_{AO}^{rel}$.

4.5 Problems

Problem 4.1: four-bar mechanism

The four-bar mechanism shown in Fig. P4.1 has the dimensions: $AB=CD=0.04$ m and $AD=BC=0.09$ m. The driver link AB rotates with a constant angular speed of 120 rpm. Find the velocities and the accelerations of the four-bar mechanism for the case when the angle of the driver link AB with the horizontal axis is $\phi=30^\circ$. For $\phi=30^\circ$ the position of the mechanism is given by: $x_B=0.034641$ m, $y_B=0.02$ m, $x_C=0.103859$ m, $y_C=-0.0375222$ m, $\phi_2=-39.7274^\circ$, $\phi_3=-69.7274^\circ$.

Results

$$\begin{aligned} \dot{x}_B &= -0.251327 \text{ m/s}, \quad \dot{y}_B = 0.435312 \text{ m/s}, \quad \ddot{x}_B = -5.47029 \text{ m/s}^2, \quad \ddot{y}_B = -3.15827 \text{ m/s}^2, \\ \dot{x}_C &= -0.884619 \text{ m/s}, \quad \dot{y}_C = -0.32675 \text{ m/s}, \quad \ddot{x}_C = 3.84741 \text{ m/s}^2, \quad \ddot{y}_C = 25.1222 \text{ m/s}^2, \\ \omega_2 &= \dot{\phi}_2 = -11.0095 \text{ rad/s}, \quad \alpha_2 = \ddot{\phi}_2 = 307.84 \text{ rad/s}^2, \\ \omega_3 &= \dot{\phi}_3 = -23.5759 \text{ rad/s}, \quad \alpha_3 = \ddot{\phi}_3 = 307.84 \text{ rad/s}^2. \end{aligned}$$

Problem 4.2: slider crank mechanism

The slider crank mechanism shown in Fig. P4.2 has the dimensions: $AB = 0.1$ m and $BC = 0.2$ m. The driver link 1 rotates with a constant angular speed of $n = 60$ rpm. Find the velocity and acceleration of the slider 3 when the angle of the driver link with the horizontal axis is $\phi = 45^\circ$. For $\phi=45^\circ$ the position of the mechanism is given by: $x_B=0.0707107$ m, $y_B=0.0707107$ m, $x_C=0.257794$ m, $y_C=0$ m, $\phi_2=-0.361367$ rad.

Results

$$\begin{aligned} \dot{x}_B &= -0.444288 \text{ m/s}, \quad \dot{y}_B = 0.444288 \text{ m/s}, \quad \ddot{x}_B = -2.79155 \text{ m/s}^2, \quad \ddot{y}_B = -2.79155 \text{ m/s}^2, \\ \dot{x}_C &= -0.612213 \text{ m/s}, \quad \dot{y}_C = 0 \text{ m/s}, \quad \ddot{x}_C = -2.94227 \text{ m/s}^2, \quad \ddot{y}_C = 0 \text{ m/s}^2, \\ \omega_2 &= \dot{\phi}_2 = -2.37482 \text{ rad/s}, \quad \alpha_2 = \ddot{\phi}_2 = 12.7898 \text{ rad/s}^2. \end{aligned}$$

Problem 4.3: R-RRR-RRT mechanism

A planar mechanism is shown in Fig. P4.3. The following data are given: $AB=0.150$ m, $BC=0.400$ m, $CD=0.370$ m, $CE=0.230$ m, $EF=CE$, $L_a=0.300$ m, $L_b=0.450$ m, and $L_c=CD$. The constant angular speed of the driver link 1 is 60 rpm. Find the velocities and the accelerations of the mechanism for $\phi=\phi_1=30^\circ$. For $\phi=30^\circ$ the position of the mechanism is given by: $x_B=0.129904$ m, $y_B=0.075$ m, $x_C=-0.0689445$ m, $y_C=0.422073$ m, $x_E=-0.298288$ m, $y_E=0.404712$ m, $x_F=-0.37$ m, $y_F=0.186177$ m, $\phi_2=-1.05052$ rad, $\phi_3=0.0755515$ rad, $\phi_4=1.25372$ rad.

Results

$$\begin{aligned}
\dot{x}_B &= -0.471239 \text{ m/s}, \quad \dot{y}_B = 0.81621 \text{ m/s}, \quad \ddot{x}_B = -5.1284 \text{ m/s}^2, \quad \ddot{y}_B = -2.96088 \text{ m/s}^2, \\
\dot{x}_C &= -0.0788027 \text{ m/s}, \quad \dot{y}_C = 1.04105 \text{ m/s}, \quad \ddot{x}_C = 2.87595 \text{ m/s}^2, \quad \ddot{y}_C = 1.03567 \text{ m/s}^2, \\
\dot{x}_E &= -0.127788 \text{ m/s}, \quad \dot{y}_E = 1.68819 \text{ m/s}, \quad \ddot{x}_E = 4.66371 \text{ m/s}^2, \quad \ddot{y}_E = 1.67947 \text{ m/s}^2, \\
\dot{x}_F &= 0 \text{ m/s}, \quad \dot{y}_F = 1.64625 \text{ m/s}, \quad \ddot{x}_F = 0 \text{ m/s}^2, \quad \ddot{y}_F = 3.29262 \text{ m/s}^2, \\
\omega_2 &= \dot{\phi}_2 = -1.1307 \text{ rad/s}, \quad \alpha_2 = \ddot{\phi}_2 = -22.33 \text{ rad/s}^2, \\
\omega_3 &= \dot{\phi}_3 = -2.82169 \text{ rad/s}, \quad \alpha_3 = \ddot{\phi}_3 = -2.20443 \text{ rad/s}^2, \\
\omega_4 &= \dot{\phi}_3 = 0.58475 \text{ rad/s}, \quad \alpha_4 = \ddot{\phi}_3 = -21.453 \text{ rad/s}^2.
\end{aligned}$$

Problem 4.4: R-RRR-RTT mechanism

The R-RRR-RTT mechanism is shown in Fig. P4.4. The following data are given: $AB=0.080$ m, $BC=0.350$ m, $CE=0.200$ m, $CD=0.150$ m, $L_a=0.200$ m, $L_b=0.350$ m, and $L_c=0.040$ m. The driver link 1 rotates with a constant angular speed of $n = 300$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link with the horizontal axis is $\phi=155^\circ$. For $\phi=155^\circ$ the position of the mechanism is given by: $x_B=-0.0725046$ m, $y_B=0.0338095$ m, $x_C=0.254847$ m, $y_C=0.157668$ m, $x_D=0.295983$ m, $y_D=0.0134181$ m, $\phi_2=0.361716$ rad, $\phi_3=-1.293$ rad.

Results

$$\begin{aligned}
\dot{x}_B &= -1.06216 \text{ m/s}, \quad \dot{y}_B = -2.2778 \text{ m/s}, \quad \ddot{x}_B = 71.5592 \text{ m/s}^2, \quad \ddot{y}_B = -33.3686 \text{ m/s}^2, \\
\dot{x}_C &= -1.73662 \text{ m/s}, \quad \dot{y}_C = -0.495228 \text{ m/s}, \quad \ddot{x}_C = 37.3878 \text{ m/s}^2, \quad \ddot{y}_C = 27.6173 \text{ m/s}^2, \\
\dot{x}_D &= -3.03908 \text{ m/s}, \quad \dot{y}_D = -0.866649 \text{ m/s}, \quad \ddot{x}_D = 65.4287 \text{ m/s}^2, \quad \ddot{y}_D = 48.3303 \text{ m/s}^2, \\
\omega_2 &= \dot{\phi}_2 = 5.44543 \text{ rad/s}, \quad \alpha_2 = \ddot{\phi}_2 = 197.52 \text{ rad/s}^2, \\
\omega_3 &= \dot{\phi}_3 = -9.0292 \text{ rad/s}, \quad \alpha_3 = \ddot{\phi}_3 = 217.641 \text{ rad/s}^2.
\end{aligned}$$

Problem 4.5

The mechanism in Fig. P4.5 has the dimensions: $AB=200$ mm, $AC=600$ mm, $BD=1000$ mm, $L_a=150$ mm, and $L_b=250$ mm. The driver link 1 rotates with a constant angular speed of $n=60$ rpm. Find the velocities and the accelerations of the mechanism for $\phi=\phi_1=120^\circ$. For $\phi=120^\circ$ the position of the mechanism is given by: $x_B=-0.1$ m, $y_B=0.173205$ m, $x_C=-0.6$ m, $y_C=0$ m, $x_D=-1.04491$ m, $y_D=-0.154122$ m, $\phi_2=0.333473$ rad, $\phi_5=0.940376$ rad.

Results

$$\begin{aligned}
\dot{x}_{B_1} &= -1.08828 \text{ m/s}, \quad \dot{y}_{B_1} = -0.628319 \text{ m/s}, \quad \ddot{x}_{B_1} = 3.94784 \text{ m/s}^2, \quad \ddot{y}_{B_1} = -6.83786 \text{ m/s}^2, \\
v_{C_2C_3} &= -1.23399 \text{ m/s}, \quad \omega_2 = -0.448799 \text{ rad/s}, \quad \dot{x}_{D_2} = -1.23518 \text{ m/s}, \quad \dot{y}_{D_2} = -0.204243 \text{ m/s}, \\
a_{C_2C_3x}^{cor} &= -0.362557 \text{ m/s}^2, \quad a_{C_2C_3y}^{cor} = 1.04661 \text{ m/s}^2, \quad a_{C_2C_3} = 1.59873 \text{ m/s}^2, \quad \alpha_3 = - \\
&= 16.7458 \text{ rad/s}^2, \quad \ddot{x}_{D_2} = -1.34318 \text{ m/s}^2, \quad \ddot{y}_{D_2} = 9.05135 \text{ m/s}^2, \quad v_{D_5D_4} = 0.893105 \text{ m/s}, \\
\omega_5 &= -1.75371 \text{ rad/s}, \quad a_{D_5D_4x}^{cor} = 2.53037 \text{ m/s}^2, \quad a_{D_5D_4y}^{cor} = -1.84656 \text{ m/s}^2, \quad a_{D_5D_4} = - \\
&= 4.98108 \text{ m/s}^2, \quad \alpha_5 = -6.57248 \text{ rad/s}^2.
\end{aligned}$$

Problem 4.6

The mechanism in Fig. P4.6 has the dimensions: $AB=150$ mm, $AC=350$ mm, $BD=530$ mm, $DE=300$ mm, $EF=200$ mm, $L_a=55$ mm, and $L_b=125$ mm. The constant angular speed of the driver link 1 is $n=30$ rpm. Find the velocities and the accelerations of the mechanism for $\phi=\phi_1=120^\circ$. For $\phi=120^\circ$ the position of the mechanism is given by: $x_B=-0.075$ m, $y_B=0.129904$ m, $x_D=-0.554223$ m, $y_D=-0.0964704$ m, $x_E=-0.482421$ m, $y_E=0.19481$ m, $\phi_2=0.441306$ rad, $\phi_4=1.32911$ rad, $\phi_5=-0.356559$ rad.

Results

$\dot{x}_B=-0.408105$ m/s, $\dot{y}_B=-0.235619$ m/s, $\ddot{x}_B=0.74022$ m/s², $\ddot{y}_B=-1.2821$ m/s²,
 $v_{C_3C_2x}=0.42465$ m/s, $v_{C_3C_2y}=0.200595$ m/s, $\omega_2=-0.127362$ rad/s, $\omega_4=-1.16943$ rad/s,
 $\omega_5=1.37953$ rad/s, $\dot{x}_E=-0.0963053$ m/s, $\dot{y}_E=-0.258552$ m/s, $a_{C_3C_2x}^{cor}=0.0510963$ m/s²,
 $a_{C_3C_2y}^{cor}=-0.108168$ m/s², $\alpha_2=-5.24453$ rad/s², $a_{C_3C_2x}=-0.114494$ m/s², $a_{C_3C_2y}=-$
 0.0540842 m/s², $\ddot{x}_D=-0.439231$ m/s², $\ddot{y}_D=1.23487$ m/s², $\alpha_4=-3.94679$ rad/s²,
 $\alpha_5=-3.66019$ rad/s², $\ddot{x}_E=0.612199$ m/s², $\ddot{y}_E=0.553139$ m/s².

Problem 4.7

The dimensions of the mechanism shown in Fig. P4.7 are: $AB=200$ mm, $AC=300$ mm, $CD=500$ mm, $DE=250$ mm, and $L_a=400$ mm. The constant angular speed of the driver link 1 is $n=40$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi=60^\circ$. For $\phi=60^\circ$ the position of the mechanism is given by: $x_B=0.1$ m, $y_B=0.173205$ m, $x_D=-0.758831$ m, $y_D=-0.19868$ m, $x_E=-0.7$ m, $y_E=0.0442993$ m, $\phi_2=0.408638$ rad, $\phi_4=1.33324$ rad.

Results

$\dot{x}_{B_1}=-0.72552$ m/s, $\dot{y}_{B_1}=0.418879$ m/s, $\ddot{x}_{B_1}=-1.7546$ m/s², $\ddot{y}_{B_1}=-3.03905$ m/s²,
 $\dot{x}_D=0.30661$ m/s, $\dot{y}_D=-0.708086$ m/s, $\ddot{x}_D=0.841861$ m/s², $\ddot{y}_D=1.05257$ m/s²,
 $\dot{x}_E=0$ m/s, $\dot{y}_E=-0.633848$ m/s, $\ddot{x}_E=0$ m/s², $\ddot{y}_E=0.846817$ m/s², $\omega_2=\dot{\phi}_2=1.54324$ rad/s,
 $\alpha_2=\ddot{\phi}_2=-1.26276$ rad/s², $\omega_4=\dot{\phi}_3=1.26188$ rad/s, $\alpha_4=\ddot{\phi}_3=3.0792$ rad/s².

Problem 4.8

The dimensions of the mechanism shown in Fig. P4.8 are: $AB=180$ mm, $AC=90$ mm, and $CD=200$ mm. The constant angular speed of the driver link 1 is $n=180$ rpm. Find the velocities and the accelerations of the mechanism for $\phi=\phi_1=60^\circ$. For $\phi=60^\circ$ the position of the mechanism is given by: $x_B=0.09$ m, $y_B=0.155885$ m, $x_D=-0.16138$ m, $y_D=-0.19868$ m, $x_E=-0.7$ m, $y_E=-0.0281381$ m, $\phi_2=0.631914$ rad, $\phi_4=0.172624$ rad.

Results

$$\begin{aligned} \dot{x}_{B_1} &= -2.93835 \text{ m/s}, \dot{y}_{B_1} = 1.69646 \text{ m/s}, \ddot{x}_{B_1} = -31.9775 \text{ m/s}^2, \ddot{y}_{B_1} = -55.3867 \text{ m/s}^2, \\ \dot{x}_{D_3} &= 3.28823 \text{ m/s}, \dot{y}_{D_3} = -4.4918 \text{ m/s}, \ddot{x}_{D_3} = 178.405 \text{ m/s}^2, \ddot{y}_{D_3} = 18.6037 \text{ m/s}^2, \\ \omega_2 &= \dot{\phi}_2 = 27.8338 \text{ rad/s}, \alpha_2 = \ddot{\phi}_2 = 451.854 \text{ rad/s}^2, \omega_4 = \dot{\phi}_3 = 30.4604 \text{ rad/s}, \alpha_4 = \ddot{\phi}_3 = 992.942 \text{ rad/s}^2. \end{aligned}$$

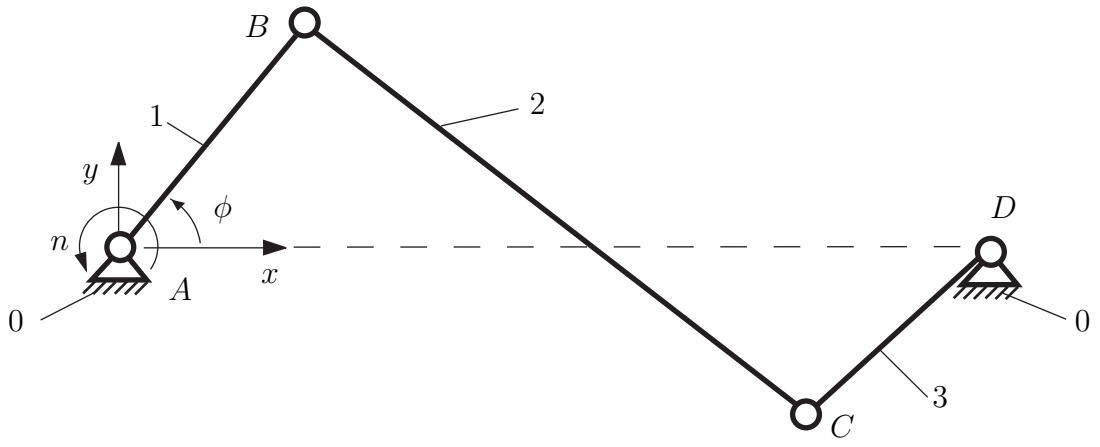


Figure P4.1

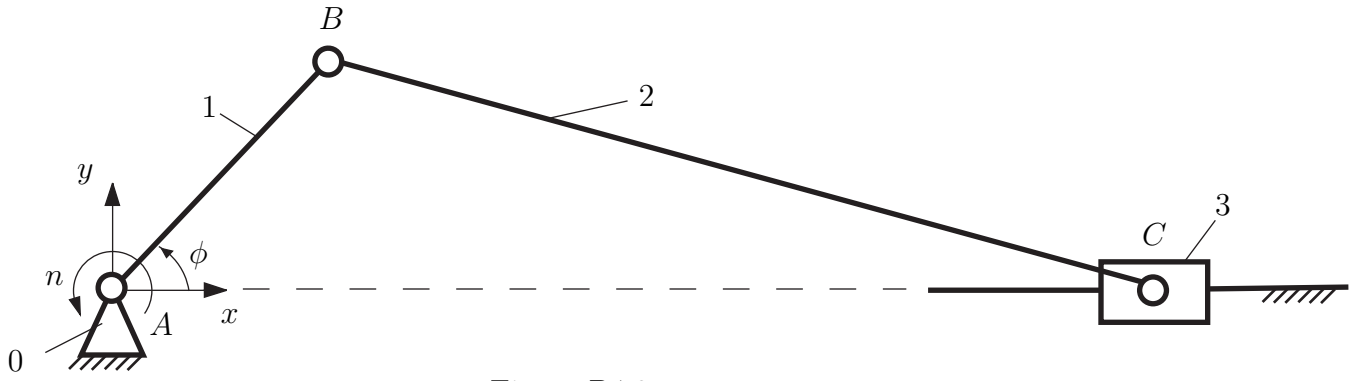


Figure P4.2

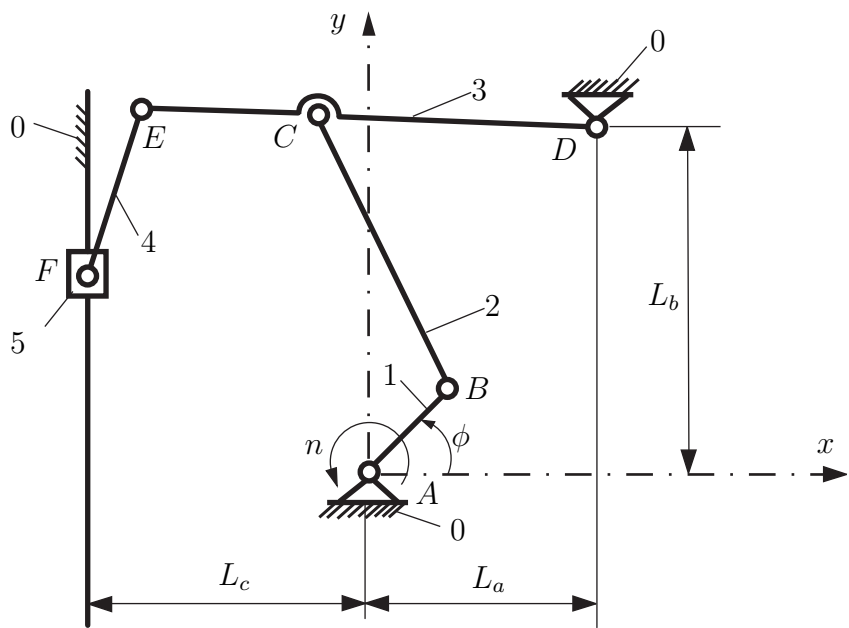


Figure P4.3

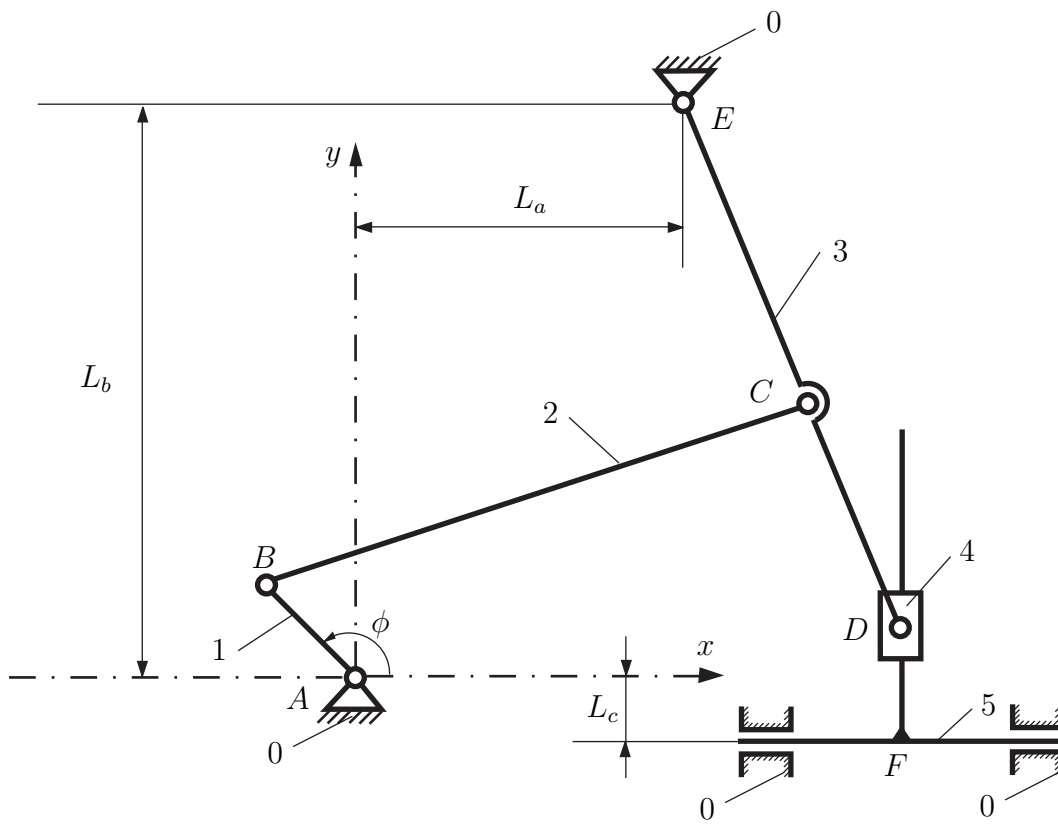


Figure P4.4

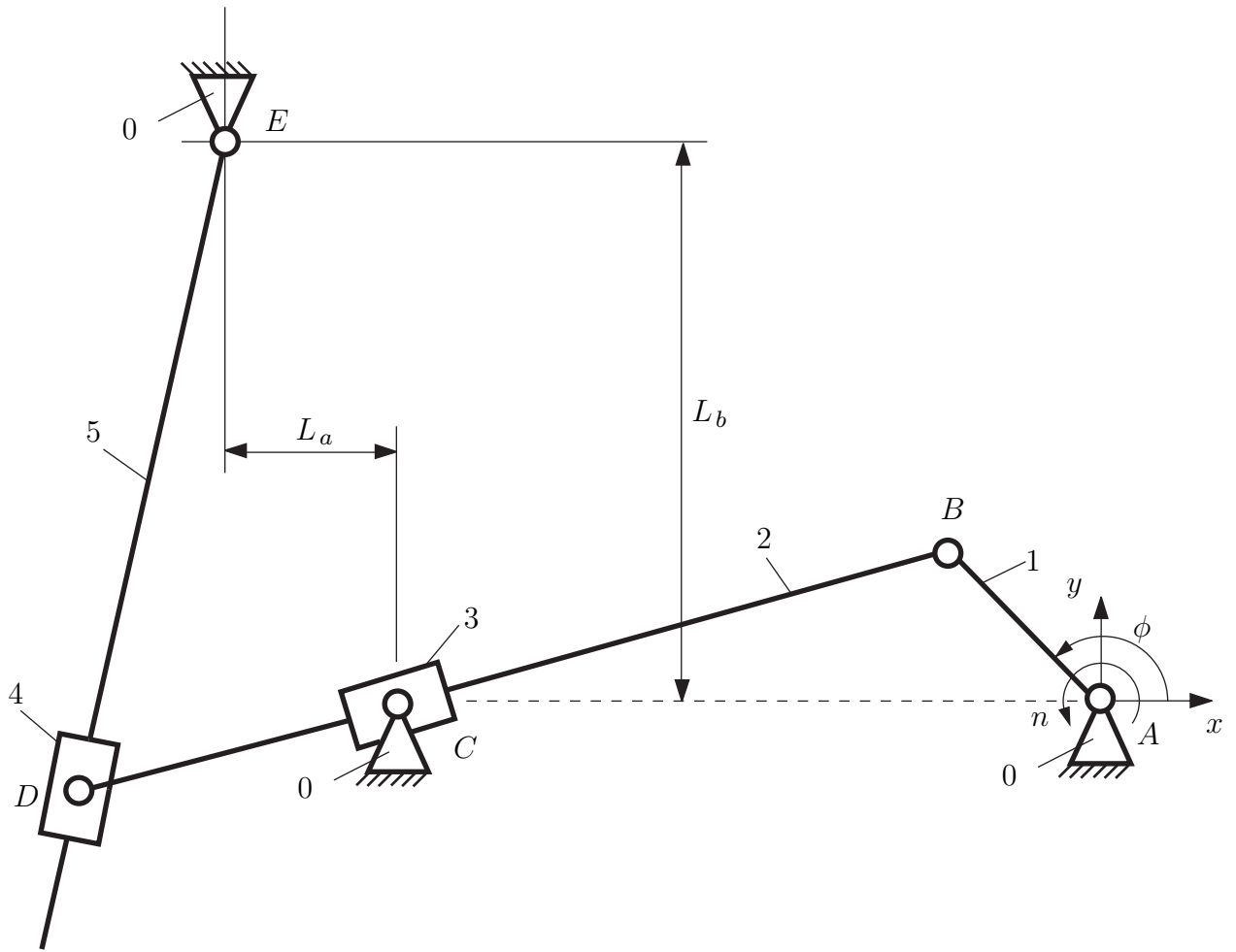


Figure P4.5

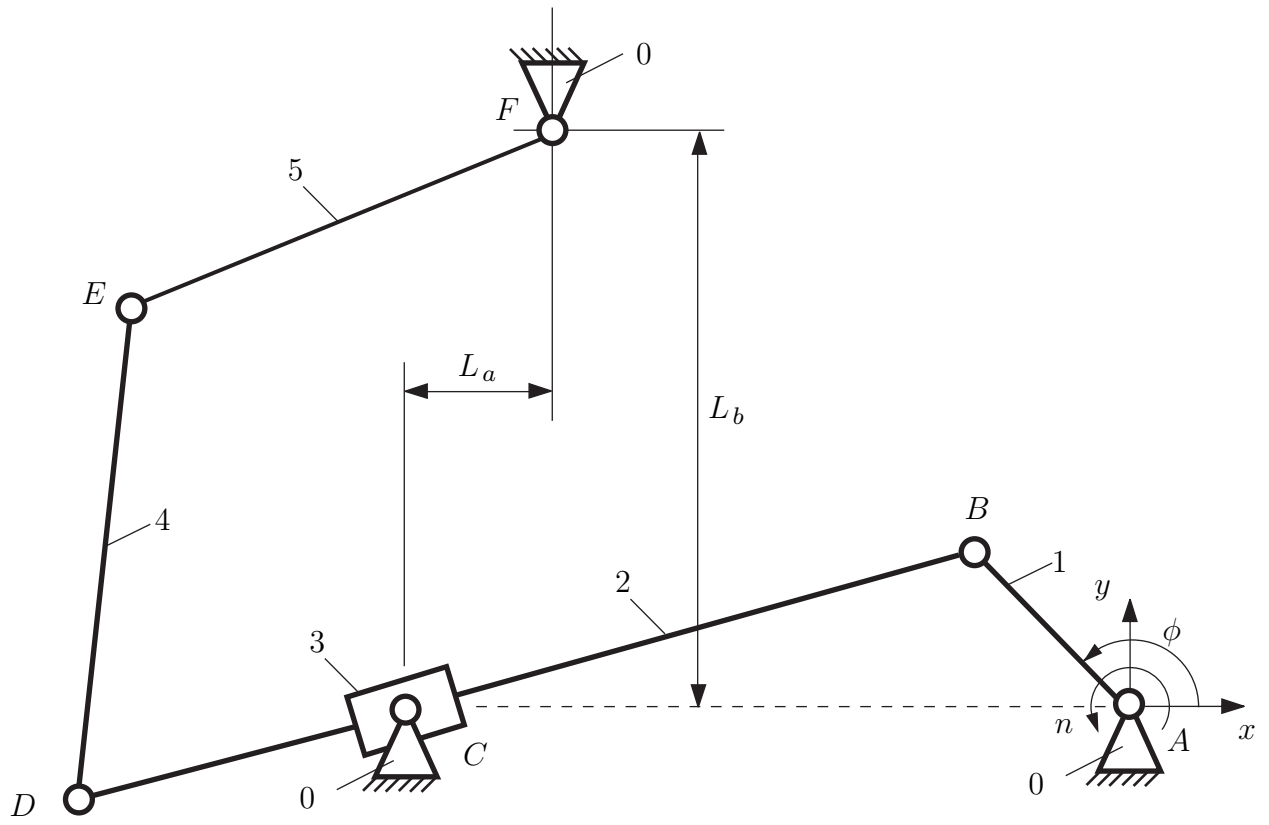


Figure P4.6

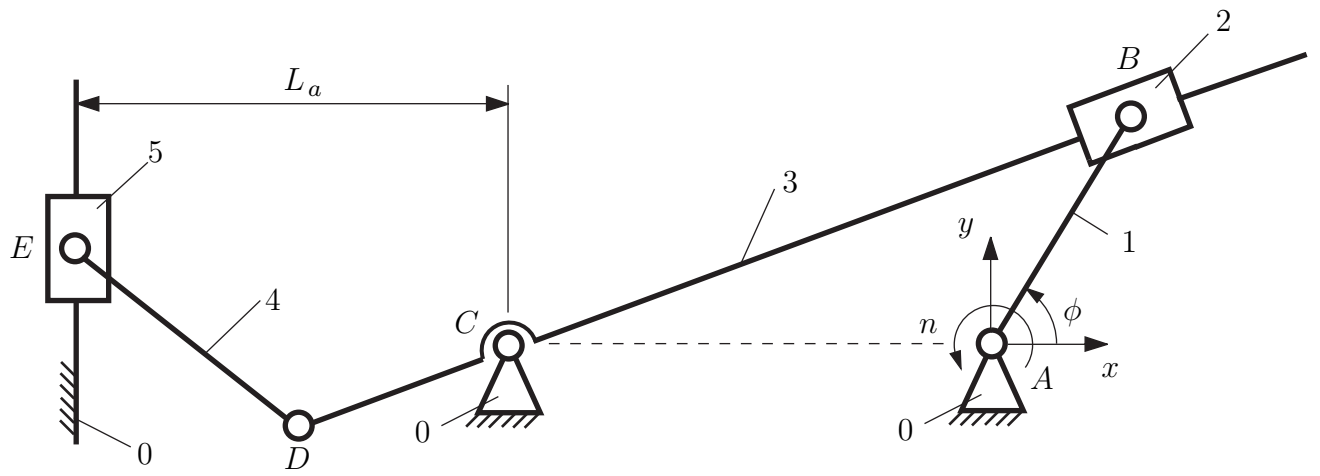


Figure P4.7

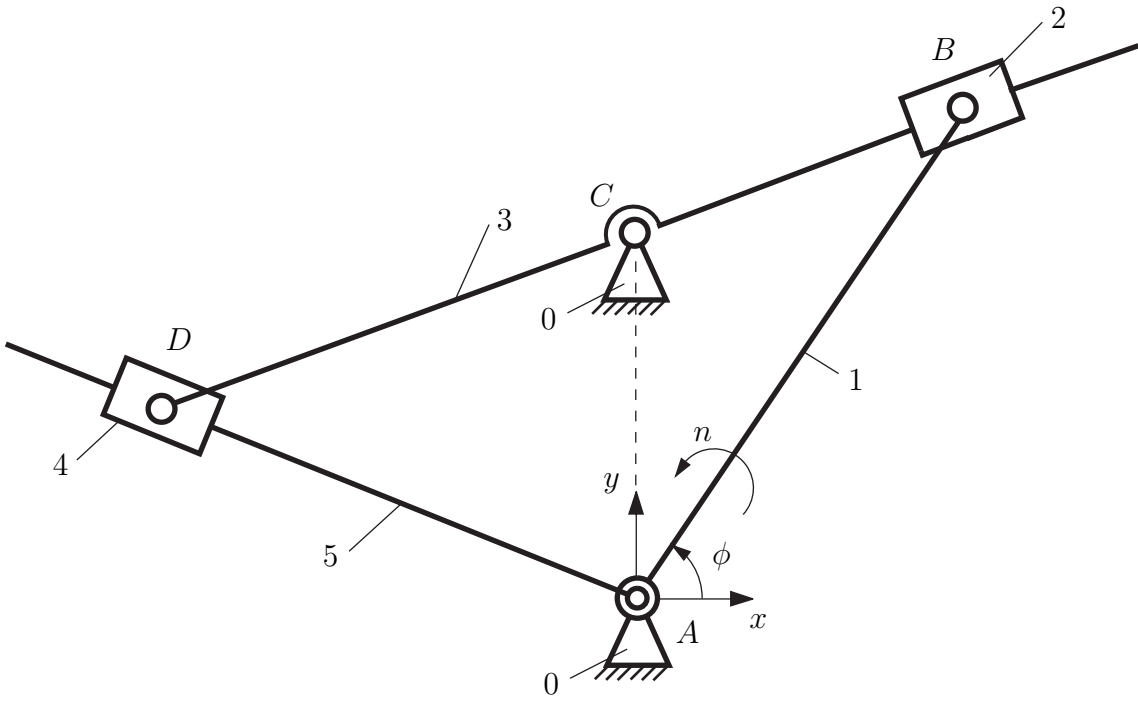


Figure P4.8