Vector Algebra

Scalars are mathematics quantities that can be fully defined by specifying their magnitude in suitable units of measure.

The mass is a scalar and can be expressed in kilograms, the time is a scalar and can be expressed seconds, and the temperature can be expressed in degrees.

Vectors are quantities that require the specification of:
magnitude, orientation, and sense.

The characteristics of a vector are the magnitude, the orientation, and the sense.

The *magnitude* of a vector is specified by a positive number and a unit having appropriate dimensions.

The *orientation* of a vector is specified by the relationship between the vector and given reference lines and/or planes.

The *sense* of a vector is specified by the order of two points on a line parallel to the vector.

Orientation and sense together determine the *direction* of a vector.

The *line of action* of a vector is a hypothetical infinite straight line collinear with the vector.

Displacement, velocity, and force are examples of vectors.

To distinguish vectors from scalars it is customary to denote vectors by boldface letters ($\mathbf{r}$ or $\mathbf{r}_{AB}$).

The symbol $|\mathbf{r}| = r$ represents the magnitude (or module, or absolute value) of the vector $\mathbf{r}$.

In handwritten work a distinguishing mark is used for vectors, such as an arrow over the symbol, $\vec{r}$ or $\vec{AB}$, a line over the symbol, $\bar{r}$, or an underline, $\underline{r}$.

A *bound* (or *fixed*) vector is a vector associated with a particular point P in space.

The point P is the *point of application* of the vector, and the line passing through P and parallel to the vector is the line of action of the vector. The point of application may be represented as the tail, or the head of the vector arrow.

A *free* vector is not associated with a particular point or line in space.

A *transmissible* (or *sliding*) vector is a vector that can be moved along his line of action without change of meaning.

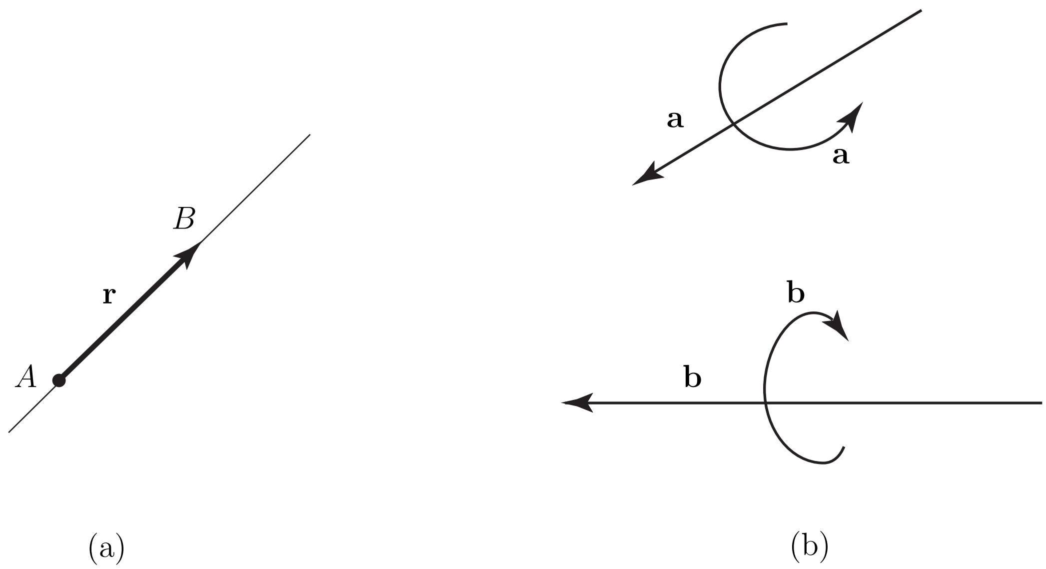


Figure 1

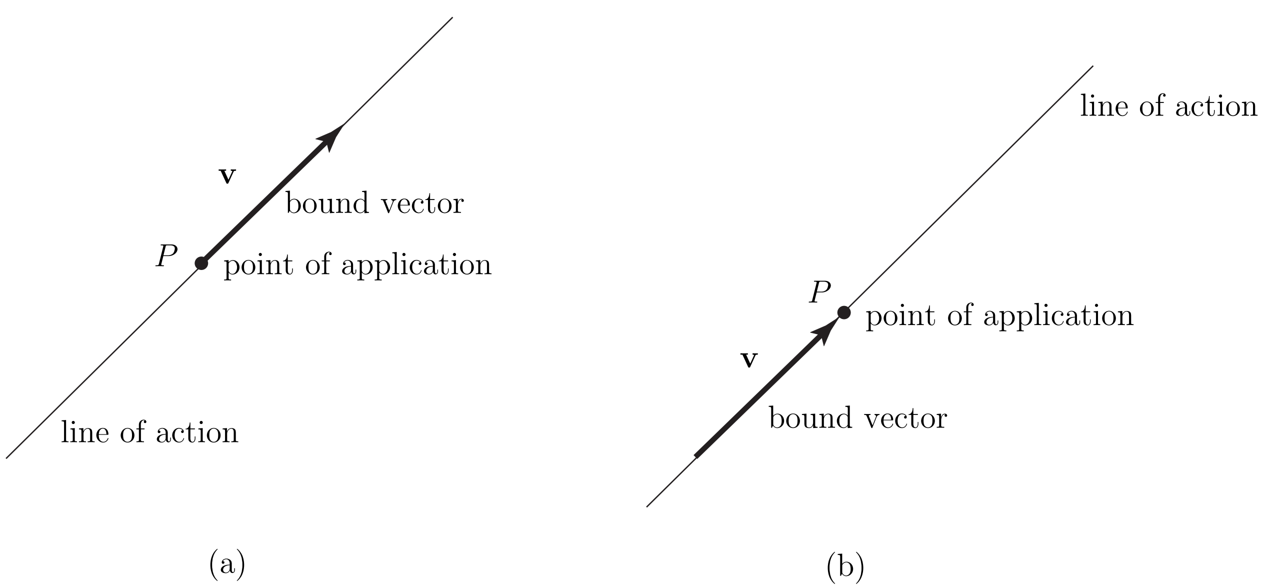


Figure 2

Unit Vectors

Definition. A *unit vector* is a vector with the magnitude equal to 1. Given a vector $\mathbf{v}$, a unit vector $\mathbf{u}$ having the same direction as $\mathbf{v}$ is obtained by forming the quotient of $\mathbf{v}$ and $|\mathbf{v}|$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

A normalized vector is called a versor.

Resolution of Vectors and Components

Let $\boldsymbol{\imath}_1$, $\boldsymbol{\imath}_2$, $\boldsymbol{\imath}_3$ be any three unit vectors not parallel to the same plane (noncollinear vectors)

$$|\boldsymbol{\imath}_1| = |\boldsymbol{\imath}_2| = |\boldsymbol{\imath}_3| = 1.$$

For a given vector $\mathbf{v}$ there exist three unique scalars v_1, v_1, v_3, such that $\mathbf{v}$ can be expressed as

$$\mathbf{v} = v_1\boldsymbol{\imath}_1 + v_2\boldsymbol{\imath}_2 + v_3\boldsymbol{\imath}_3.$$

The opposite action of addition of vectors is the
resolution of vectors.

Thus, for the given vector $\mathbf{v}$ the vectors $v_1\boldsymbol{\imath}_1$, $v_2\boldsymbol{\imath}_2$, and $v_3\boldsymbol{\imath}_3$ sum to the original vector.

The vector $v_k\boldsymbol{\imath}_k$ is called the $\boldsymbol{\imath}_k$ *component* of $\mathbf{v}$ and v_k is called the $\boldsymbol{\imath}_k$ *scalar component* of $\mathbf{v}$, where $k = 1, 2, 3$.

A vector is often replaced by its components since the components are equivalent to the original vector.

Every vector equation $\mathbf{v} = \mathbf{0}$, where
$\mathbf{v} = v_1\boldsymbol{\imath}_1 + v_2\boldsymbol{\imath}_2 + v_3\boldsymbol{\imath}_3$,
is equivalent to three scalar equations

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0.$$

If the unit vectors $\boldsymbol{\imath}_1$, $\boldsymbol{\imath}_2$, $\boldsymbol{\imath}_3$ are mutually perpendicular they form a *cartesian reference frame.*

For a cartesian reference frame the following notation is used:

$$\boldsymbol{\imath}_1 \equiv \boldsymbol{\imath}, \ \boldsymbol{\imath}_2 \equiv \boldsymbol{\jmath}, \ \boldsymbol{\imath}_3 \equiv \mathbf{k},$$

and

$$\imath \perp \jmath, \ \imath \perp \mathbf{k}, \ \jmath \perp \mathbf{k}.$$

The symbol $\perp$ denotes perpendicular.

When a vector $\mathbf{v}$ is expressed in the form
$\mathbf{v} = v_x \imath + v_y \jmath + v_z \mathbf{k}$ where $\imath$, $\jmath$, $\mathbf{k}$ are mutually perpendicular unit vectors (cartesian reference frame or orthogonal reference frame), the magnitude of $\mathbf{v}$ is given by

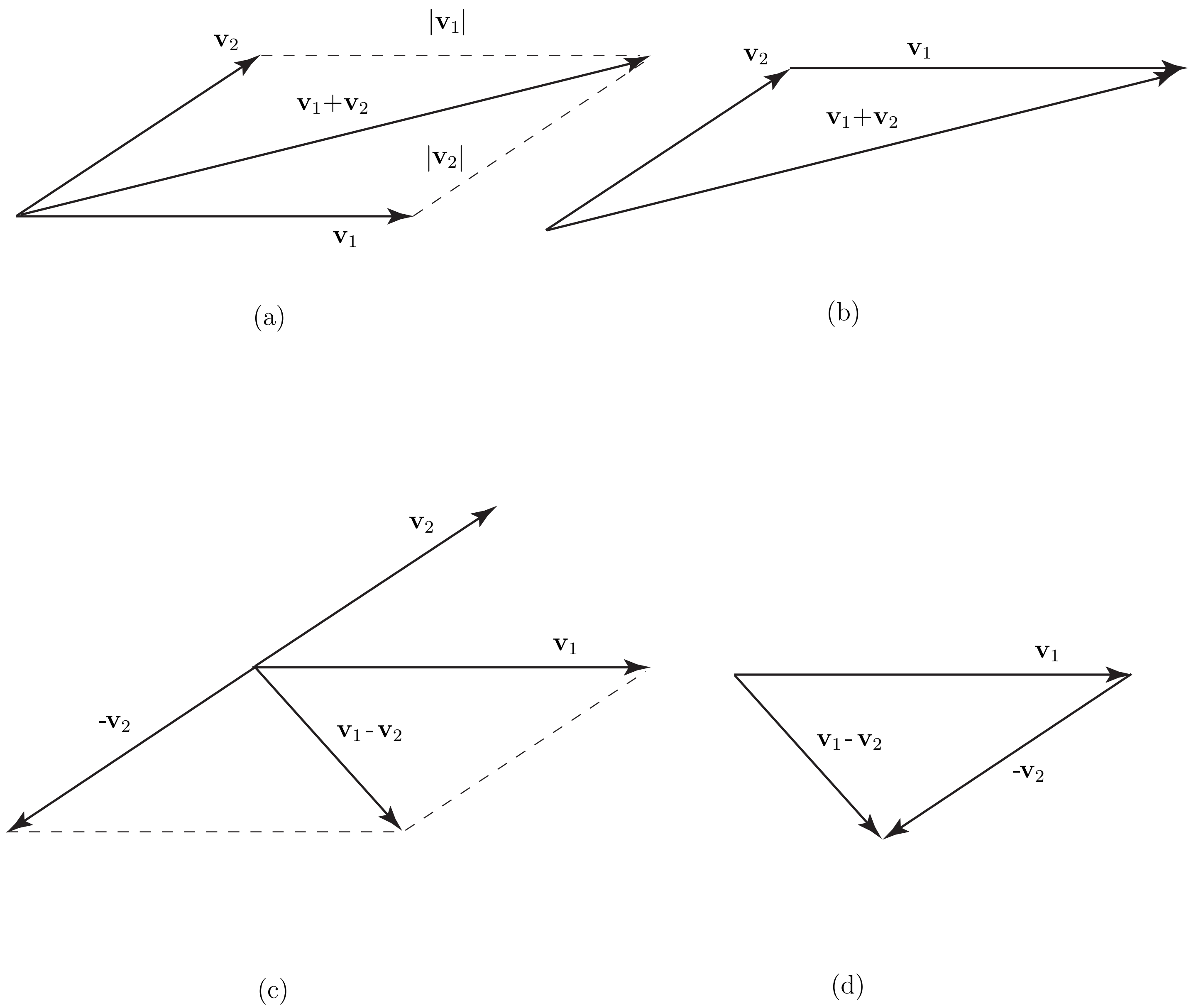
$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

The vectors $\mathbf{v}_x = v_x \imath$, $\mathbf{v}_y = v_y \jmath$, and $\mathbf{v}_z = v_z \mathbf{k}$ are the *orthogonal* or *rectangular component vectors* of the vector $\mathbf{v}$.

The measures v_x, v_y, v_z are the *orthogonal* or *rectangular scalar components* of the vector $\mathbf{v}$.

If $\mathbf{v}_1 = v_{1x} \imath + v_{1y} \jmath + v_{1z} \mathbf{k}$ and $\mathbf{v}_2 = v_{2x} \imath + v_{2y} \jmath + v_{2z} \mathbf{k}$, then the sum of the vectors is

$$\mathbf{v}_1 + \mathbf{v}_2 = (v_{1x} + v_{2x}) \, \imath + (v_{1y} + v_{2y}) \, \jmath + (v_{1z} + v_{2z}) \, v_{1z} \mathbf{k}.$$



$|\mathbf{v}_1|$
$\mathbf{v}_2$
$\mathbf{v}_1+\mathbf{v}_2$
$|\mathbf{v}_2|$
$\mathbf{v}_1$
(a)
$\mathbf{v}_2$
$\mathbf{v}_1$
$\mathbf{v}_1+\mathbf{v}_2$
(b)
$\mathbf{v}_2$
$\mathbf{v}_1$
-$\mathbf{v}_2$
$\mathbf{v}_1$-$\mathbf{v}_2$
(c)
$\mathbf{v}_1$
$\mathbf{v}_1$-$\mathbf{v}_2$
-$\mathbf{v}_2$
(d)

Figure 3

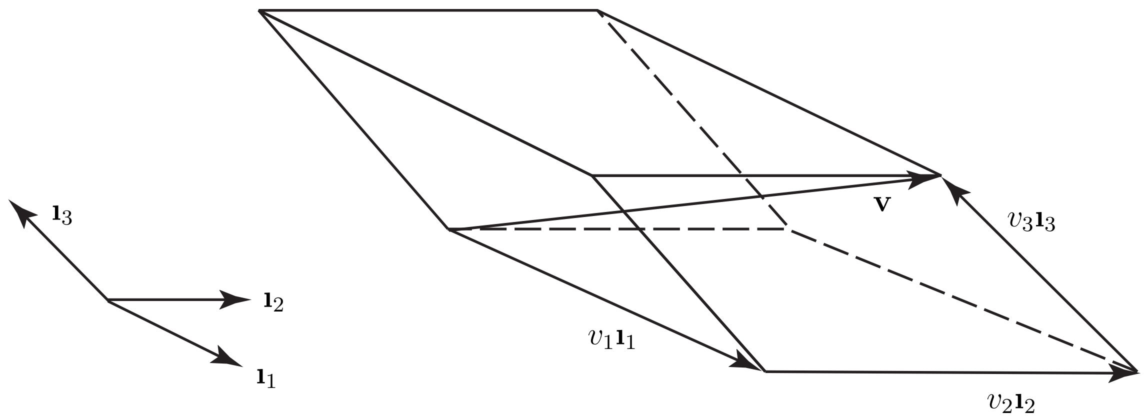


Figure 4

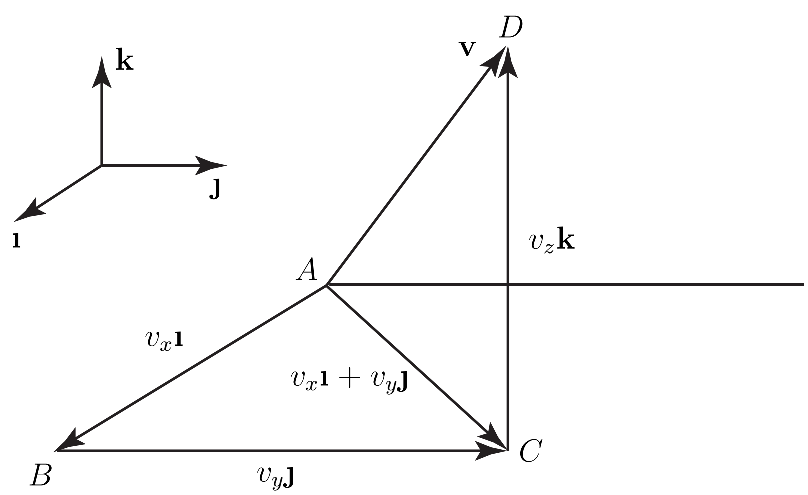
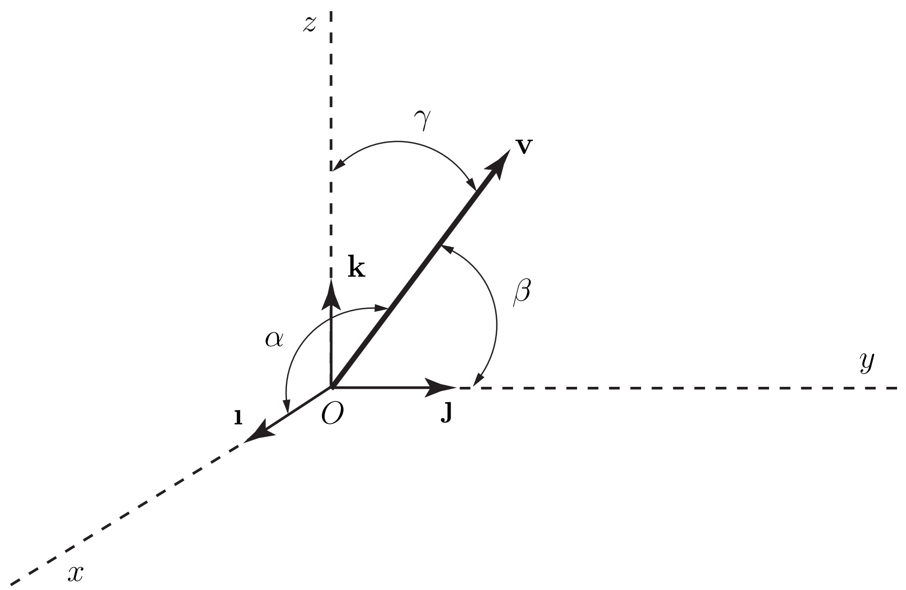


Figure 5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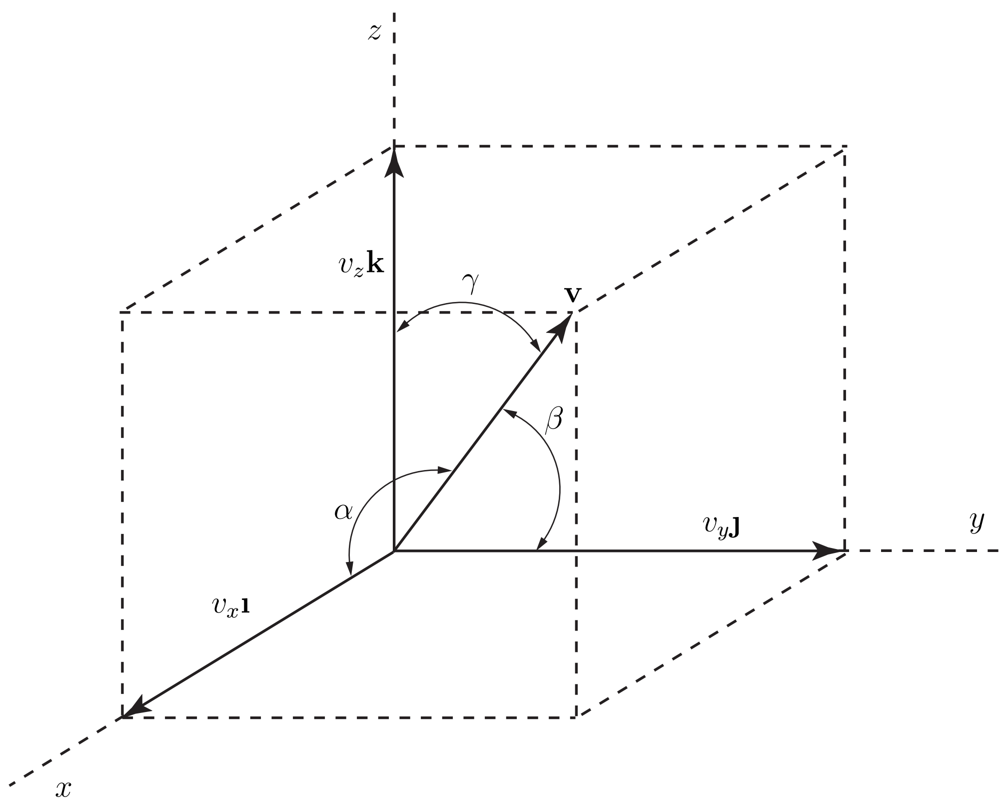

z
$v_z\mathbf{k}$
γ
v
β
α
$v_y\mathbf{\jmath}$
y
$v_x\mathbf{\imath}$
x

Figure 6

Scalar (Dot) Product of Vectors

Definition. The scalar (dot) product of a vector **a** and a vector **b** is

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = |\mathbf{a}|\,|\mathbf{b}| \cos(\mathbf{a}, \mathbf{b}).$$

For any two vectors **a** and **b** and any scalar s

$$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (s\mathbf{b}) = s\mathbf{a} \cdot \mathbf{b}.$$

If

$$\mathbf{a} = a_x \boldsymbol{\imath} + a_y \boldsymbol{\jmath} + a_z \mathbf{k},$$

and

$$\mathbf{b} = b_x \boldsymbol{\imath} + b_y \boldsymbol{\jmath} + b_z \mathbf{k},$$

where $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ are mutually perpendicular unit vectors, then

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \,.$$

The following relationships exist

$$\boldsymbol{\imath} \cdot \boldsymbol{\imath} = \boldsymbol{\jmath} \cdot \boldsymbol{\jmath} = \mathbf{k} \cdot \mathbf{k} = 1,$$
$$\boldsymbol{\imath} \cdot \boldsymbol{\jmath} = \boldsymbol{\jmath} \cdot \mathbf{k} = \mathbf{k} \cdot \boldsymbol{\imath} = 0.$$

Every vector **v** can be expressed in the form

$$\mathbf{v} = \boldsymbol{\imath} \cdot \mathbf{v}\, \boldsymbol{\imath} + \boldsymbol{\jmath} \cdot \mathbf{v}\, \boldsymbol{\jmath} + \mathbf{k} \cdot \mathbf{v}\, \mathbf{k}.$$

The vector **v** can always be expressed as

$$\mathbf{v} = v_x \boldsymbol{\imath} + v_y \boldsymbol{\jmath} + v_z \mathbf{k}.$$

Dot multiply both sides by $\boldsymbol{\imath}$

$$\boldsymbol{\imath} \cdot \mathbf{v} = v_x \boldsymbol{\imath} \cdot \boldsymbol{\imath} + v_y \boldsymbol{\imath} \cdot \boldsymbol{\jmath} + v_z \boldsymbol{\imath} \cdot \mathbf{k}.$$

But,

$$\boldsymbol{\imath} \cdot \boldsymbol{\imath} = 1, \text{ and } \boldsymbol{\imath} \cdot \boldsymbol{\jmath} = \boldsymbol{\imath} \cdot \mathbf{k} = 0.$$

Hence,

$$\boldsymbol{\imath} \cdot \mathbf{v} = v_x.$$

Similarly,

$$\boldsymbol{\jmath} \cdot \mathbf{v} = v_y \quad \text{and} \quad \mathbf{k} \cdot \mathbf{v} = v_z.$$

Vector (Cross) Product of Vectors

Definition. The vector (cross) product of a vector $\mathbf{a}$ and a vector $\mathbf{b}$ is the vector (Fig. 7)

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})\mathbf{n}$$

where $\mathbf{n}$ is a unit vector whose direction is the same as the direction of advance of a right-handed screw rotated from $\mathbf{a}$ toward $\mathbf{b}$, through the angle $(\mathbf{a}, \mathbf{b})$, when the axis of the screw is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
The magnitude of $\mathbf{a} \times \mathbf{b}$ is given by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}).$$

If $\mathbf{a}$ is parallel to $\mathbf{b}$, $\mathbf{a}||\mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = 0$.
The symbol $||$ denotes parallel.

The relation $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ implies only that the product $|\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})$ is equal to zero, and this is the case whenever $|\mathbf{a}| = 0$, or $|\mathbf{b}| = 0$, or $\sin(\mathbf{a}, \mathbf{b}) = 0$.

For any two vectors $\mathbf{a}$ and $\mathbf{b}$ and any real scalar s,

$$(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (s\mathbf{b}) = s\mathbf{a} \times \mathbf{b}.$$

The sense of the unit vector $\mathbf{n}$ which appears in the definition of $\mathbf{a}\times\mathbf{b}$ depends on the order of the factors $\mathbf{a}$ and $\mathbf{b}$ in such a way that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

Vector multiplication obeys the following law of distributivity (Varignon theorem)

$$\mathbf{a} \times \sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} (\mathbf{a} \times \mathbf{v}_i).$$

The cross product is not commutative, but the associative law and the distributive law are valid for cross products.

A set of mutually perpendicular unit vectors $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ is called *right-handed* if $\boldsymbol{\imath} \times \boldsymbol{\jmath} = \mathbf{k}$.

A set of mutually perpendicular unit vectors $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ is called *left-handed* if $\boldsymbol{\imath} \times \boldsymbol{\jmath} = -\mathbf{k}$.

If

$$\mathbf{a} = a_x \boldsymbol{\imath} + a_y \boldsymbol{\jmath} + a_z \mathbf{k},$$

and

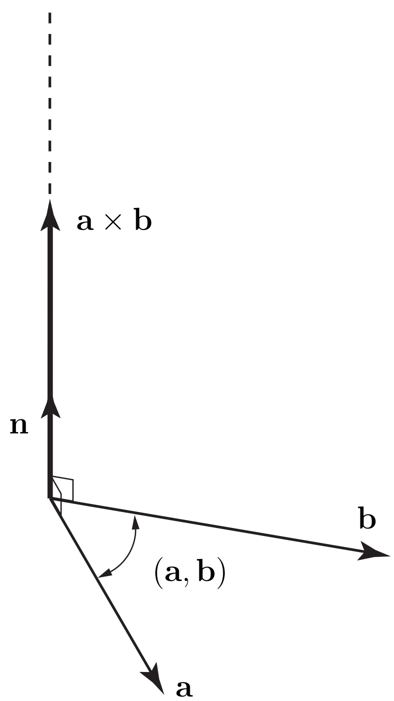
$$\mathbf{b} = b_x \boldsymbol{\imath} + b_y \boldsymbol{\jmath} + b_z \mathbf{k},$$

where $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ are mutually perpendicular unit vectors, then $\mathbf{a} \times \mathbf{b}$ can be expressed in the following determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

The determinant can be expanded by minors of the elements of the first row:

$$\begin{aligned} \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= \boldsymbol{\imath} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \boldsymbol{\jmath} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\ &= \boldsymbol{\imath}(a_y b_z - a_z b_y) - \boldsymbol{\jmath}(a_x b_z - a_z b_x) + \mathbf{k}(a_x b_y - a_y b_x) \\ &= (a_y b_z - a_z b_y)\boldsymbol{\imath} + (a_z b_x - a_x b_z)\boldsymbol{\jmath} + (a_x b_y - a_y b_x)\mathbf{k}. \end{aligned}$$



$$\mathbf{a}\times\mathbf{b} \perp \mathbf{a}$$
$$\mathbf{a}\times\mathbf{b} \perp \mathbf{b}$$

Figure 7

Derivative of a Vector

The derivative of a vector is defined in exactly the same way as is the derivative of a scalar function.

The derivative of a vector has some of the properties of the derivative of a scalar function.

The derivative of the sum of two vector functions **a** and **b** is

$$\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}.$$

The time derivative of the product of a scalar function f and a vector function **a** is

$$\frac{d(f\mathbf{a})}{dt} = \frac{df}{dt}\mathbf{a} + f\frac{d\mathbf{a}}{dt}.$$

Position Vector

The position vector of a point P relative to a point M is a vector $\mathbf{r}_{MP}$ having the following characteristics, Fig. 8

- magnitude ($|\mathbf{r}_{MP}| = r_{MP}$) the length of line MP ;
- orientation parallel to line MP;
- sense MP (from point M to point P).

The vector $\mathbf{r}_{MP}$ is shown as an arrow connecting M to P. The position of a point P relative to P is a zero vector.

Let $\mathbf{\imath}$, $\mathbf{\jmath}$, $\mathbf{k}$ be mutually perpendicular unit vectors (cartesian reference frame) with the origin at O, Fig. 8.

The axes of the cartesian reference frame are x, y, z.

The unit vectors $\mathbf{\imath}$, $\mathbf{\jmath}$, $\mathbf{k}$ are parallel to x, y, z, and they have the senses of the positive x, y, z axes.

The coordinates of the origin O are $x = y = z = 0$, i.e., $O(0, 0, 0)$.

The coordinates of a point P are $x = x_P$, $y = y_P$, $z = z_P$, i.e., $P(x_P, y_P, z_P)$.

The position vector of P relative to the origin O is

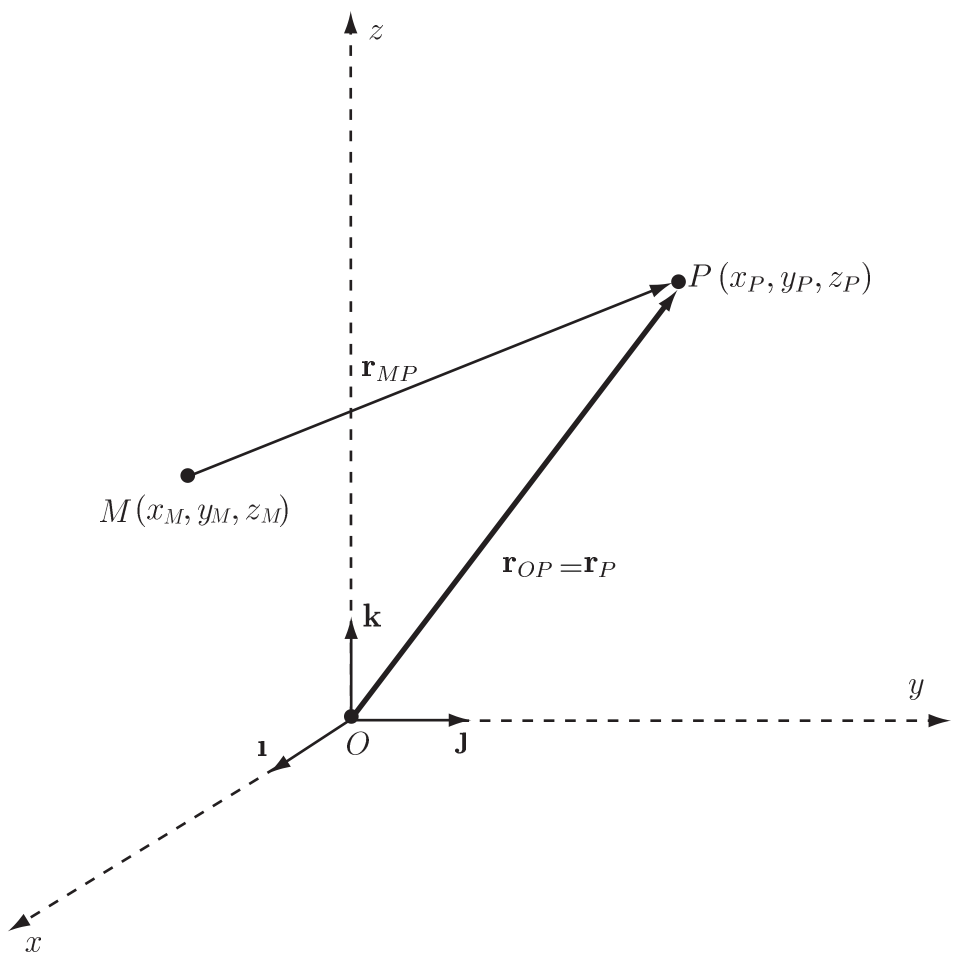
$$\mathbf{r}_{OP} = \mathbf{r}_P = x_P\,\mathbf{\imath} + y_P\,\mathbf{\jmath} + z_P\,\mathbf{k}.$$

The position vector of the point P relative to a point M, $M \neq O$ of coordinates (x_M, y_M, z_M) is

$$\mathbf{r}_{MP} = (x_P - x_M)\,\mathbf{\imath} + (y_P - y_M)\,\mathbf{\jmath} + (z_P - z_M)\,\mathbf{k}.$$

The distance d between P and M is given by

$$\begin{aligned} d &= |\mathbf{r}_P - \mathbf{r}_M| = |\mathbf{r}_{MP}| \\ &= \sqrt{(x_P - x_M)^2 + (y_P - y_M)^2 + (z_P - z_M)^2}. \end{aligned}$$



z
$P\,(x_P, y_P, z_P)$
$\mathbf{r}_{MP}$
$M\,(x_M, y_M, z_M)$
$\mathbf{r}_{OP} = \mathbf{r}_P$
$\mathbf{k}$
y
$\imath$
O
$\jmath$
x

Figure 8

Centroid of a Curve, Surface, or Solid

The position vector of the centroid C of a curve, surface, or solid relative to a point O is

$$\mathbf{r}_C = \frac{\int_D \mathbf{r}\, d\tau}{\int_D d\tau},$$

where D is a curve, surface, or solid, $\mathbf{r}$ denotes the position vector of a typical point of D, relative to O, and $d\tau$ is the length, area, or volume of a differential element of D. Each of the two limits in this expression is called an "integral over the domain D (curve, surface, or solid)."

The integral $\int_D d\tau$ gives the total length, area, or volume of D, that is

$$\int_D d\tau = \tau.$$

The position vector of the centroid is

$$\mathbf{r}_C = \frac{1}{\tau}\int_D \mathbf{r}\, d\tau.$$

Let $\imath$, $\jmath$, $\mathbf{k}$ be mutually perpendicular unit vectors (cartesian reference frame) with the origin at O. The coordinates of C are x_C, y_C, z_C and

$$\mathbf{r}_C = x_C\imath + y_C\jmath + z_C\mathbf{k}.$$

It results that

$$x_C = \frac{1}{\tau}\int_D x\, d\tau, \quad y_C = \frac{1}{\tau}\int_D y\, d\tau, \quad z_C = \frac{1}{\tau}\int_D z\, d\tau.$$

Mass Center of a Curve, Surface, or Solid

The position vector of the mass center C of a continuous body D, curve, surface, or solid, relative to a point O is

$$\mathbf{r}_C = \frac{1}{m}\int_D \mathbf{r}\rho \, d\tau,$$

or using the orthogonal cartesian coordinates

$$x_C = \frac{1}{m}\int_D x\rho \, d\tau, \quad y_C = \frac{1}{m}\int_D y\rho \, d\tau, \quad z_C = \frac{1}{m}\int_D z\rho \, d\tau,$$

where ρ is the mass density of the body: mass per unit of length if D is a curve, mass per unit area if D is a surface, and mass per unit of volume if D is a solid, $\mathbf{r}$ is the position vector of a typical point of D, relative to O, $d\tau$ is the length, area, or volume of a differential element of D, $m = \int_D \rho \, d\tau$ is the total mass of the body, and x_C, y_C, z_C are the coordinates of C.

If the mass density ρ of a body is the same at all points of the body, ρ=constant, the density, as well as the body, are said to be *uniform*. The mass center of a uniform body coincides with the centroid of the figure occupied by the body.

Moments and Couples

Moment of a Bound Vector About a Point

Definition. The moment of a bound vector $\mathbf{v}$ about a point A is the vector

$$\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB} \times \mathbf{v}, \tag{1}$$

$\mathbf{r}_{AB}$ is the position vector of B relative to A, and

B is any point of line of action, Δ, of the vector $\mathbf{v}$, Fig. 9.

The vector $\mathbf{M}_A^{\mathbf{v}} = \mathbf{0}$ if and only the line of action of $\mathbf{v}$ passes through A or $\mathbf{v} = \mathbf{0}$.

The magnitude of $\mathbf{M}_A^{\mathbf{v}}$ is

$$|\mathbf{M}_A^{\mathbf{v}}| = M_A^{\mathbf{v}} = |\mathbf{r}_{AB}|\,|\mathbf{v}|\,\sin\theta,$$

θ is the angle between $\mathbf{r}_{AB}$ and $\mathbf{v}$ when they are placed tail to tail.

The perpendicular distance from A to the line of action of $\mathbf{v}$ is

$$d = |\mathbf{r}_{AB}|\,\sin\theta,$$

and the magnitude of $\mathbf{M}_A^{\mathbf{v}}$ is

$$|\mathbf{M}_A^{\mathbf{v}}| = M_A^{\mathbf{v}} = d\,|\mathbf{v}|.$$

The vector $\mathbf{M}_A^{\mathbf{v}}$ is perpendicular to both $\mathbf{r}_{AB}$ and $\mathbf{v}$:

$$\mathbf{M}_A^{\mathbf{v}} \perp \mathbf{r}_{AB} \quad \text{and} \quad \mathbf{M}_A^{\mathbf{v}} \perp \mathbf{v}$$

The vector $\mathbf{M}_A^{\mathbf{v}}$ being perpendicular to $\mathbf{r}_{AB}$ and $\mathbf{v}$ is perpendicular to the plane containing $\mathbf{r}_{AB}$ and $\mathbf{v}$.

The moment given by Eq. (1) does not depend on the point B of the line of action of $\mathbf{v}$, Δ, where $\mathbf{r}_{AB}$ intersects Δ.

Instead of using the point B one could use the point B', Fig. 9.

The vector $\mathbf{r}_{AB} = \mathbf{r}_{AB'} + \mathbf{r}_{B'B}$ where the vector $\mathbf{r}_{B'B}$ is parallel to $\mathbf{v}$, $\mathbf{r}_{B'B}||\mathbf{v}$.

$$\begin{aligned}\mathbf{M}_A^{\mathbf{v}} &= \mathbf{r}_{AB} \times \mathbf{v} = (\mathbf{r}_{AB'} + \mathbf{r}_{B'B}) \times \mathbf{v} \\ &= \mathbf{r}_{AB'} \times \mathbf{v} + \mathbf{r}_{B'B} \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v},\end{aligned}$$

because $\mathbf{r}_{B'B} \times \mathbf{v} = \mathbf{0}$.

Couples

Definition. A *couple* is a system of bound vectors whose resultant is equal to zero and whose moment about some point is not equal to zero.

A system of vectors is not a vector, therefore couples are not vectors.

A couple consisting of only two vectors is called a *simple couple*.

The vectors of a simple couple have equal magnitudes, parallel lines of action, and opposite senses.

Writers use the word "couple" to denote the simple couple.

The moment of a couple about a point is called the *torque* of the couple, $\mathbf{M}$ or $\mathbf{T}$.

The moment of a couple about one point is equal to the moment of the couple about any other point i.e., it is unnecessary to refer to a specific point.

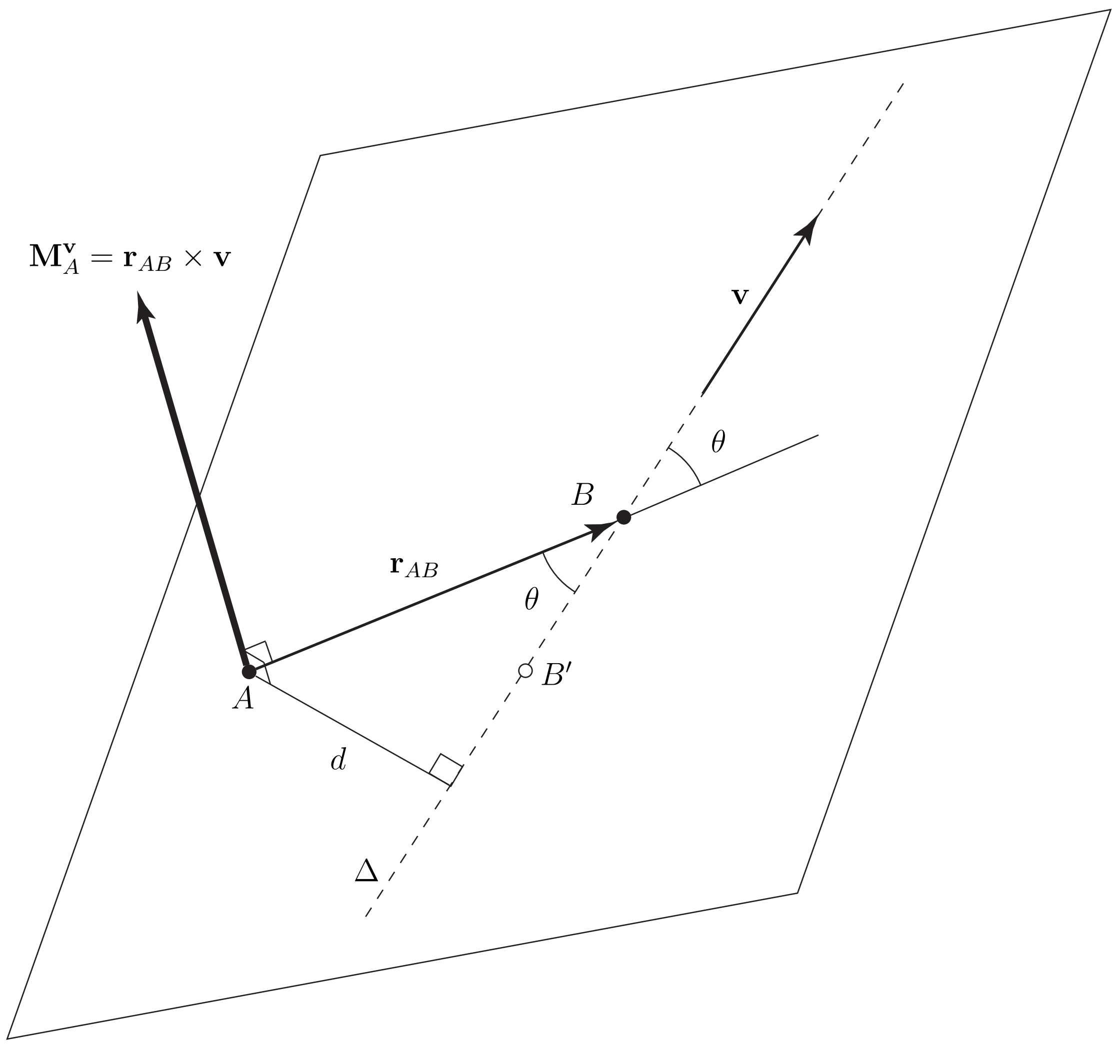
The moment of a couple is a free vector.

The torques are vectors and the magnitude of a torque of a simple couple is given by

$$|\mathbf{M}| = d|\mathbf{v}|,$$

where d is the distance between the lines of action of the two vectors comprising the couple, and $\mathbf{v}$ is one of these vectors.

The direction of the torque of a simple couple can be determined by inspection: $\mathbf{M}$ is perpendicular to the plane determined by the lines of action of the two vectors comprising the couple, and the sense of $\mathbf{M}$ is the same as that of $\mathbf{r} \times \mathbf{v}$.



$\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB} \times \mathbf{v}$
$\mathbf{v}$
θ
B
$\mathbf{r}_{AB}$
θ
B'
A
d
Δ

Figure 9

Force Vector and Moment of a Force

Force is a vector quantity, having both magnitude and direction.

Force is commonly explained in terms of Newton's three laws of motion set forth in his Principia Mathematica (1687).

Newton's first principle: a body that is at rest or moving at a uniform rate in a straight line will remain in that state until some force is applied to it.

Newton's second law of motion states that a particle acted on by forces whose resultant is not zero will move in such a way that the time rate of change of its momentum will at any instant be proportional to the resultant force.

Newton's third law states that when one body exerts a force on another body, the second body exerts an equal force on the first body.

This is the principle of action and reaction.

Because force is a vector quantity can be represented graphically as a directed line segment.

The representation of forces by vectors implies that they are concentrated either at a single point or along a single line.

The force of gravity is invariably distributed throughout the volume of a body.

Nonetheless, when the equilibrium of a body is the primary consideration, it is generally valid as well as convenient to assume that the forces are concentrated at a single point.

In the case of gravitational force, the total weight of a body may be assumed to be concentrated at its centre of gravity.

Force is measured in newtons (N); a force of 1 N will accelerate a mass of one kilogram at a rate of one meter per second per second. The newton is a unit of the International System (SI), for measuring force.

Using the English system, the force is measured in pounds. One pound of force imparts to a one-pound object an acceleration of 32.17 feet per second squared.

The force vector $\mathbf{F}$ can be expressed in terms of a cartesian reference frame, with the unit vectors $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, and $\mathbf{k}$, Fig. 10(a)

$$\mathbf{F} = F_x\boldsymbol{\imath} + F_y\boldsymbol{\jmath} + F_z\mathbf{k}. \tag{2}$$

The components of the force in the x, y, and z directions are F_x, F_y, and F_z.

The resultant of two forces $\mathbf{F}_1 = F_{1x}\mathbf{\imath} + F_{1y}\mathbf{\jmath} + F_{1z}\mathbf{k}$ and $\mathbf{F}_2 = F_{2x}\mathbf{\imath} + F_{2y}\mathbf{\jmath} + F_{2z}\mathbf{k}$ is the vector sum of those forces

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (F_{1x} + F_{2x})\mathbf{\imath} + (F_{1y} + F_{2y})\mathbf{\jmath} + (F_{1z} + F_{2z})\mathbf{k}. \tag{3}$$

A moment is defined as the moment of a force about (with respect to) a point.

The moment of the force $\mathbf{F}$ about the point O is the cross product vector

$$\begin{aligned} \mathbf{M}_O^{\mathbf{F}} &= \mathbf{r} \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \\ &= (r_y\, F_z - r_z\, F_y)\mathbf{\imath} + (r_z\, F_x - r_x\, F_z)\mathbf{\jmath} + (r_x\, F_y - r_y\, F_x)\mathbf{k}. \end{aligned}$$

where $\mathbf{r} = r_x\mathbf{\imath} + r_y\mathbf{\jmath} + r_z\mathbf{k}$ is a position vector directed from the point about which the moment is taken (O in this case) to any point A on the line of action of the force, Fig. 10(a).

If the coordinates of O are x_O, y_O, z_O and the coordinates of A are x_A, y_A, z_A then

$\mathbf{r} = \mathbf{r}_{OA} = (x_A - x_O)\mathbf{\imath} + (y_A - y_O)\mathbf{\jmath} + (z_A - z_O)\mathbf{k}$ and the the moment of the force $\mathbf{F}$ about the point O is

$$\mathbf{M}_O^{\mathbf{F}} = \mathbf{r}_{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_A - x_O & y_A - y_O & z_A - z_O \\ F_x & F_y & F_z \end{vmatrix}.$$

The magnitude of $\mathbf{M}_O^{\mathbf{F}}$ is

$$|\mathbf{M}_O^{\mathbf{F}}| = M_O^{\mathbf{F}} = r\, F\, |\sin\theta|,$$

where $\theta = \angle(\mathbf{r}, \mathbf{F})$ is the angle between vectors $\mathbf{r}$ and $\mathbf{F}$, and $r = |\mathbf{r}|$ and $F = |\mathbf{F}|$ are the magnitudes of the vectors.

The line of action of $\mathbf{M}_O^{\mathbf{F}}$ is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{F}$ ($\mathbf{M}_O^{\mathbf{F}} \perp \mathbf{r} \,\&\, \mathbf{M}_O^{\mathbf{F}} \perp \mathbf{F}$) and the sense is given by the right-hand rule.

The moment of the force $\mathbf{F}$ about another point P is

$$\mathbf{M}_P^{\mathbf{F}} = \mathbf{r}_{PA} \times \mathbf{F} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_A - x_P & y_A - y_P & z_A - z_P \\ F_x & F_y & F_z \end{vmatrix},$$

where x_P, y_P, z_P are the coordinates of the point P.

The system of two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, which have equal magnitudes $|\mathbf{F}_1| = |\mathbf{F}_2|$, opposite senses $\mathbf{F}_1 = -\mathbf{F}_2$, and parallel directions ($\mathbf{F}_1 || \mathbf{F}_2$) is a couple.

The resultant force of a couple is zero $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$.

The resultant moment $\mathbf{M} \neq \mathbf{0}$ about an arbitrary point is

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2,$$

or

$$\mathbf{M} = \mathbf{r}_1 \times (-\mathbf{F}_2) + \mathbf{r}_2 \times \mathbf{F}_2 = (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_2 = \mathbf{r} \times \mathbf{F}_2,$$

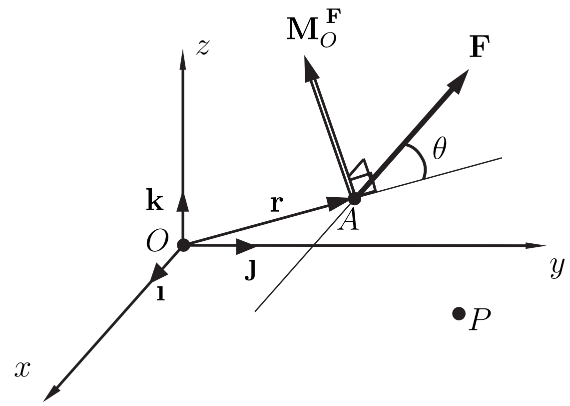
where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is a vector from any point on the line of action of $\mathbf{F}_1$ to any point of the line of action of $\mathbf{F}_2$.

The direction of the torque of the couple is perpendicular to the plane of the couple and the magnitude is given by, Fig. 10(b)

$$|\mathbf{M}| = M = r\, F_2\, |\sin\theta| = h\, F_2, \tag{4}$$

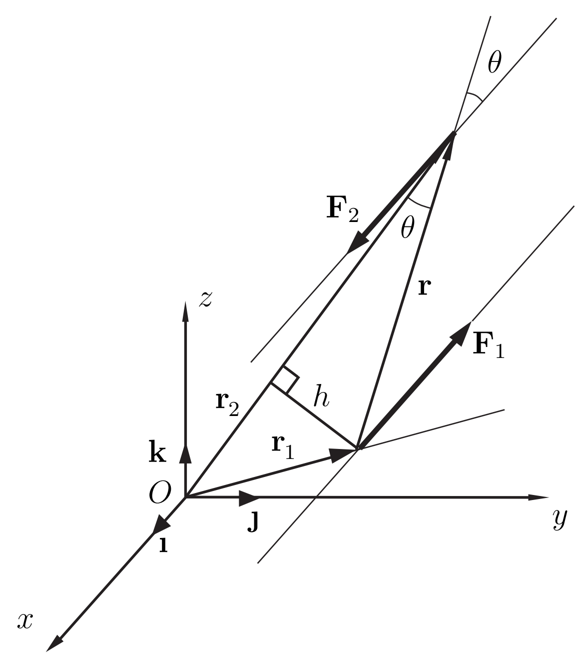
where $h = r\, |\sin\theta|$ is the perpendicular distance between the lines of action.

The resultant moment of a couple is independent of the point with respect to which moments are taken.



$\mathbf{M}_O^{\mathbf{F}}$
z
$\mathbf{F}$
θ
$\mathbf{k}$
$\mathbf{r}$
A
O
$\mathbf{j}$
y
$\mathbf{i}$
P
x

(a)



θ
$\mathbf{F}_2$
θ
$\mathbf{r}$
z
$\mathbf{F}_1$
$\mathbf{r}_2$
h
$\mathbf{k}$
$\mathbf{r}_1$
O
$\mathbf{j}$
y
$\mathbf{i}$
x

(b)

Figure 10

Equilibrium

Equilibrium Equations

A body is in *equilibrium* when it is stationary or in steady translation relative to an inertial reference frame.

The following conditions are satisfied when a body, acted upon by a system of forces and moments, is in equilibrium:

1. the sum of the forces is zero

$$\sum \mathbf{F} = 0. \tag{5}$$

2. the sum of the moments about any point is zero

$$\sum \mathbf{M}_P = 0, \ \ \forall P. \tag{6}$$

If the sum of the forces acting on a body is zero and the sum of the moments about one point is zero, then the sum of the moments about every point is zero.

A body subjected to concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$, …, $\mathbf{F}_n$ and no couples.

If the sum of the concurrent forces is zero,

$$\mathbf{F}_1 + \mathbf{F}_2 + ... + \mathbf{F}_n = \mathbf{0},$$

the sum of the moments of the forces about the concurrent point is zero, so the sum of the moments about every point is zero.

The only condition imposed by equilibrium on a set of concurrent forces is that their sum is zero.

Free-Body Diagrams

Free-body diagrams are used to determine forces and moments acting on simple bodies in equilibrium.

The beam in Figure 11(a) has a pin support at the left end A and a roller support at the right end B.
The beam is loaded by a force F and a moment M at C.
To obtain the free-body diagram first the beam is isolated from its supports.
Next, the reactions exerted on the beam by the supports are shown on the the free-body diagram, Figure 11(b).

Once the free-body diagram is obtained one can apply the equilibrium equations.

The steps required to determine the reactions on bodies are

1. draw the free-body diagram, isolating the body from its supports and showing the forces and the reactions;

2. apply the equilibrium equations to determine the reactions.

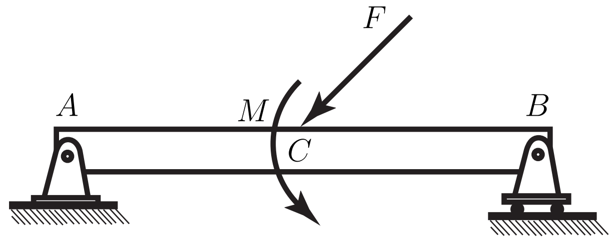
For two-dimensional systems, the forces and moments are related by three scalar equilibrium equations

$$\sum F_x = 0, \tag{7}$$

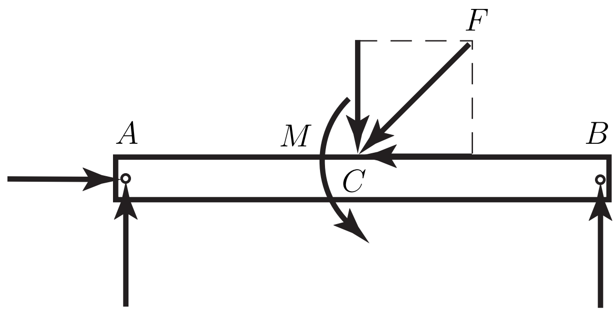
$$\sum F_y = 0, \tag{8}$$

$$\sum M_P = 0, \ \ \forall P. \tag{9}$$

One can obtain more than one equation from Eq. (9) by evaluating the sum of the moments about more than one point. The additional equations will not be independent of Eqs. (7)-(9). One cannot obtain more than three independent equilibrium equations from a two-dimensional free-body diagram, which means one can solve for at most three unknown forces or moments.



(a)



(b)

Figure 11