1 Direct Dynamics Newton Euler Equations of Motion

1.1 Dynamics of a Compound Pendulum

Figure 1.1(a) depicts a uniform rod of mass m and length L. The rod is connnected to the ground by a pin joint and is free to swing in a vertical plane. The rod is moving and makes an instant angle $\theta(t)$ with the horizontal. The local acceleration of gravity is g. Numerical application: $L = 3$ ft, g=32.2 ft/s^2, $G = m\,g$=12 lb. Find the Newton Euler equations of motion.

Solution.

The system of interest is the rod during the interval of its motion. The rod is constrained to move in a vertical plane. First a reference frame will be introduced. The plane of motion will be designated the xy plane. The y-axis is vertical, with the positive sense directed vertically downward. The x-axis is horizontal and is contained in the plane of motion. The z-axis is also horizontal and is perpendicular to the plane of motion. These axes define an inertial reference frame. The unit vectors for the inertial reference frame are $\mathbf{\imath}$, $\mathbf{\jmath}$, and $\mathbf{k}$. The angle between the x and the rod axis is denoted by θ. The rod is moving and hence the angle is changing with time at the instant of interest. In the static equilibrium position of the rod, the angle, θ, is equal to $\pi/2$. The system has one degree of freedom. The angle, θ, is an appropriate generalized coordinate describing this degree of freedom. The system has a single moving body. The only motion permitted that body is rotation about a fixed horizontal axis (z-axis). The body is connected to the ground with the rotating pin joint (R) at O. The rod is referred to as link 1 and the ground is referred to as link 0. The mass center of the rod is at the point C. As the rod is uniform, its mass center is coincident with its geometric center.

Kinematics

The mass center, C, is at a distance $L/2$ from the pivot point O and the position vector is

$$\mathbf{r}_{OC} = \mathbf{r}_C = x_C\mathbf{\imath} + y_C\mathbf{\jmath}, \tag{1.1}$$

where x_C and y_C are the coordinates of C

$$x_C = \frac{L}{2}\cos\theta,\ \ y_C = \frac{L}{2}\sin\theta. \tag{1.2}$$

The rod is constrained to move in a vertical plane, with its pinned location, O, serving as a pivot point. The motion of the rod is planar, consisting of pure rotation about the pivot point. The directions of the angular velocity and angular acceleration vectors will be perpendicular to this plane, in the z direction.The angular velocity of the rod can be expressed as:

$$\boldsymbol{\omega} = \omega \mathbf{k} = \frac{d\theta}{dt}\mathbf{k} = \dot{\theta}\mathbf{k}, \tag{1.3}$$

ω is the rate of rotation of the rod. The positive sense is clockwise (consistent with the x and y directions defined above). This problem involves only a single moving rigid body and the angular velocity vector refers to that body. For this reason, no explicit indication of the body, 1, is included in the specification of the angular velocity vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1$. The angular acceleration of the rod can be expressed as

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \alpha \mathbf{k} = \frac{d^2\theta}{dt^2}\mathbf{k} = \ddot{\theta}\mathbf{k}, \tag{1.4}$$

α is the angular acceleration of the rod. The positive sense is clockwise.

The velocity of the mass center can be related to the velocity of the pivot point using the relationship between the velocities of two points attached to the same rigid body

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OC} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega \\ x_C & y_C & 0 \end{vmatrix} = \omega(-y_C \boldsymbol{\imath} + x_C \boldsymbol{\jmath}) = \\ & \frac{L\omega}{2}(-\sin\theta \boldsymbol{\imath} + \cos\theta \boldsymbol{\jmath}) = \frac{L\dot{\theta}}{2}(-\sin\theta \boldsymbol{\imath} + \cos\theta \boldsymbol{\jmath}).\end{aligned} \tag{1.5}$$

The velocity of the pivot point, O, is zero.
The acceleration of the mass center can be related to the acceleration of the pivot point ($\mathbf{a}_O = \mathbf{0}$) using the relationship between the accelerations of two points attached to the same rigid body

$$\begin{aligned}&\mathbf{a}_C = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OC} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{OC}) = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OC} - \omega^2\, \mathbf{r}_{OC} = \\ &\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha \\ x_C & y_C & 0 \end{vmatrix} - \omega^2(x_C \boldsymbol{\imath} + y_C \boldsymbol{\jmath}) = \alpha(-y_C \boldsymbol{\imath} + x_C \boldsymbol{\jmath}) - \omega^2(x_C \boldsymbol{\imath} + y_C \boldsymbol{\jmath}) = \\ &-(\alpha y_C + \omega^2 x_C)\boldsymbol{\imath} + (\alpha x_C - \omega^2 y_C)\boldsymbol{\jmath} = \\ &-\frac{L}{2}(\alpha \sin\theta + \omega^2 \cos\theta)\boldsymbol{\imath} + \frac{L}{2}(\alpha \cos\theta - \omega^2 \sin\theta)\boldsymbol{\jmath} = \\ &-\frac{L}{2}(\ddot{\theta} \sin\theta + \dot{\theta}^2 \cos\theta)\boldsymbol{\imath} + \frac{L}{2}(\ddot{\theta} \cos\theta - \dot{\theta}^2 \sin\theta)\boldsymbol{\jmath}.\end{aligned} \tag{1.6}$$

It is also useful to define a set of body-fixed coordinate axes. These are axes that move with the rod (body-fixed axe). The n-axis is along the length of the rod, the positive direction running from the origin O toward the mass center C. The unit vector of the n-axis is $\mathbf{n}$. The t-axis will be perpendicular to the rod and be contained in the plane of motion as shown in Fig. 1.1(a).The unit vector of the t-axis is $\mathbf{t}$ and $\mathbf{n} \times \mathbf{t} = \mathbf{k}$. The velocity of the mass center C in body-fixed reference frame is

$$\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OC} = \begin{vmatrix} \mathbf{n} & \mathbf{t} & \mathbf{k} \\ 0 & 0 & \omega \\ \frac{L}{2} & 0 & 0 \end{vmatrix} = \frac{L\omega}{2}\mathbf{t} = \frac{L\dot{\theta}}{2}\mathbf{t}, \tag{1.7}$$

where $\mathbf{r}_{OC} = (L/2)\mathbf{n}$. The acceleration of the mass center C in body-fixed reference frame is

$$\mathbf{a}_C = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OC} - \omega^2\,\mathbf{r}_{OC} = \frac{L\alpha}{2}\mathbf{t} - \omega^2\frac{L}{2}\mathbf{n} = \frac{L\ddot{\theta}}{2}\mathbf{t} - \dot{\theta}^2\frac{L}{2}\mathbf{n}, \tag{1.8}$$

or

$$\mathbf{a}_C = \mathbf{a}_C^t + \mathbf{a}_C^n,$$

with the components

$$\mathbf{a}_C^t = \frac{L\ddot{\theta}}{2}\mathbf{t} \quad \text{and} \quad \mathbf{a}_C^n = -\frac{L\dot{\theta}^2}{2}\mathbf{n}.$$

Newton-Euler equation of motion

The rod is rotating about a fixed axis. The mass moment of inertia of the rod about the fixed pivot point O can be evaluated from the mass moment of inertia about the mass center C using the transfer theorem. Thus

$$I_O = I_C + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{12} + \frac{mL^2}{4} = \frac{mL^2}{3}. \tag{1.9}$$

The pin is frictionless and is capable of exerting horizontal and vertical forces on the rod at O

$$\mathbf{F}_{01} = F_{01x}\mathbf{\imath} + F_{01y}\mathbf{\jmath}, \tag{1.10}$$

where F_{01x} and F_{01y} are the components of the pin force on the rod in the fixed axis system.

The force driving the motion of the rod is gravity. The weight of the rod is acting through its mass center will cause a moment about the pivot point. This moment will give the rod a tendency to rotate about the pivot point. This moment will be given by the cross product of the vector from the pivot point, O, to the mass center, C, crossed into the weight force $\mathbf{G} = mg\mathbf{j}$.

As the pivot point, O, of the rod is fixed, the appropriate moment summation point will be about that pivot point. The sum of the moments about this point will be equal to the mass moment of inertia about the pivot point multiplied by the angular acceleration of the rod. The only contributor to the moment is the weight of the rod. Thus we should be able to directly determine the angular acceleration from the moment equation. The sum of the forces acting on the rod should be equal to the product of the rod mass and the acceleration of its mass center. This should be useful in determining the forces exerted by the pin on the rod.

The free body diagram shows the rod at the instant of interest, Fig. 1.1(b). The rod is acted upon by its weight acting vertically downward through the mass center of the rod. The rod is acted upon by the pin force at its pivot point. The components of the pin force parallel (axial) and perpendicular (shear) to the rod are shown.

The motion diagram shows the rod at the instant of interest, Fig. 1.1(c). The motion diagram shows the relevant velocity and acceleration information. The angular velocity and angular acceleration vectors are depicted as arcs. The resulting mass center velocity and acceleration vectors are shown as arrows.The motion of the mass center is circular, the velocity vector has been shown as tangent to that circle. The acceleration vector has been broken into its tangent and normal components. Expressions relating these kinematic quantities to the angle θ and its time derivatives were developed above. Newton Euler equations of motion for the rod are

$$m\,\mathbf{a}_C = \Sigma\mathbf{F} = \mathbf{G} + \mathbf{F}_{01}, \tag{1.11}$$

$$I_C\,\boldsymbol{\alpha} = \Sigma\mathbf{M}_C = \mathbf{r}_{CO} \times \mathbf{F}_{01}. \tag{1.12}$$

Since the rigid body has a fixed point at O the equations of motion state that the moment sum about the fixed point must be equal to to the product of the rod mass moment of inertia about that point and the rod angular acceleration. Thus

$$I_O\,\boldsymbol{\alpha} = \Sigma\mathbf{M}_O = \mathbf{r}_{OC} \times \mathbf{G} \tag{1.13}$$

Using Eqs. (1.4),(1.10), and (1.13) the equation of motion is

$$\frac{mL^2}{3}\ddot{\theta}\mathbf{k} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ \frac{L}{2}\cos\theta & \frac{L}{2}\sin\theta & 0 \\ 0 & mg & 0 \end{vmatrix}, \tag{1.14}$$

or

$$\ddot{\theta} = \frac{3g}{2L}\cos\theta. \tag{1.15}$$

The equation of motion, Eq. (1.15) is a nonlinear, second order, differential equation relating the second time derivative of the angle, θ, to the value of that angle and various problem parameters g and L. The equation is nonlinear due to the presence of the $\cos\theta$, where $\theta(t)$ is the unknown function of interest.

The force exerted by the pin on the rod have are obtained from Eq. (1.12)

$$\mathbf{F}_{01} = m\,\mathbf{a}_C - \mathbf{G},$$

and the components of the force are

$$F_{01x} = m\,\ddot{x}_C = -\frac{mL}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta),$$
$$F_{01y} = m\,\ddot{y}_C + mg = \frac{mL}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) - mg. \tag{1.16}$$

Using the moving reference frame (body-fixed) the components of the reaction force on n and t axes are

$$F_{01n} = m\,a_C^n - mg\sin\theta = -\frac{mL\dot{\theta}^2}{2} - mg\sin\theta,$$
$$F_{01t} = m\,a_C^t - mg\cos\theta = \frac{mL\ddot{\theta}}{2} - mg\cos\theta. \tag{1.17}$$

If the rod is released from rest, then the initial value of the angular velocity is zero $\omega(t=0) = \omega(0) = \dot{\theta}(0) = 0$ rad/s. If the initial angle is $\theta(0) = 0$ radians, then the cosine of that initial angle is unity and the sine is zero. The initial angular acceleration can be determined from Eq. (1.15)

$$\ddot{\theta}(0) = \alpha(0) = \frac{3g}{2L}\cos\theta(0) = \frac{3g}{2L} = 16.1 \text{ rad/s}^2. \tag{1.18}$$

The positive sign indicates that the initial angular acceleration of the rod is clockwise, as one would expect.

The initial reaction force components can be evaluated from Eq. (1.19)

$$F_{01x}(0) = 0 \text{ lb},$$
$$F_{01y}(0) = \frac{mL}{2}\ddot{\theta}(0) - mg = -\frac{gm}{4} = -3 \text{ lb}. \tag{1.19}$$

An analytical solution to the differential equation is difficult to obtain. Numerical approaches have the advantage of being simple to apply even for complex mechanical systems. A *Mathematica*^{TM} computer program to solve the governing differential equation is given in the Appendix 1.1. The *Mathematica*^{TM} function `NDSolve` is used to solvethe differential equation. The function `NDSolve[eqns, y, {x, xmin, xmax}]` finds a numerical solution to the ordinary differential equations `eqns` for the function `y` with the independent variable `x` in the range `xmin` to `xmax`.

1.2 Dynamics of a Double Pendulum

A two link kinematic chain (double pendulum) is considered, Fig. 1.2(a). The links 1 and 2 have the masses m_1 and m_2 and the lengths $AB = L_1$ and $BD = L_2$. The system is free to move in a vertical plane. The local acceleration of gravity is g. Numerical application: $m_1 = m_2 = m$=1 kg, $L_1 = L_2 = L = 1$ m, and g=10 m/s^2. Find the equations of motion.

Solution.

The plane of motion is xy plane with the y-axis vertical, with the positive sense directed upward. The origin of the reference frame is at A. The mass centers of the links are designated by $C_1(x_{C1}, y_{C1}, 0)$ and $C_2(x_{C2}, y_{C2}, 0)$. The number of degrees of freedom are computed using the relation

$$M = 3n - 2c_5 - c_4,$$

where n is the number of moving links, c_5 is the number of one degree of freedom joints, and c_4 is the number of two degrees of freedom joints. For the double pendulum $n = 2,\ c_5 = 2,\ c_4 = 0$, and the system has two degrees of freedom, $M = 2$, and two generalized coordinates. The angles $q_1(t)$ and $q_2(t)$ are selected as the generalized coordinates as shown in Fig. 1.2(a).

Kinematics

The position vector of the center of the mass C_1 of the link 1 is

$$\mathbf{r}_{C_1} = x_{C_1}\mathbf{\imath} + y_{C_1}\mathbf{\jmath}, \tag{1.20}$$

where x_{C_1} and y_{C_1} are the coordinates of C_1

$$x_{C_1} = \frac{L_1}{2}\cos q_1,\ \ y_{C_1} = \frac{L_1}{2}\sin q_1. \tag{1.21}$$

The position vector of the center of the mass C_2 of the link 2 is

$$\mathbf{r}_{C_2} = x_{C_2}\mathbf{\imath} + y_{C_2}\mathbf{\jmath}, \tag{1.22}$$

where x_{C_2} and y_{C_2} are the coordinates of C_2

$$x_{C_2} = L_1\cos q_1 + \frac{L_2}{2}\cos q_2,\ \ y_{C_2} = L_1\sin q_1 + \frac{L_2}{2}\sin q_2. \tag{1.23}$$

The velocity vector of C_1 is the derivative with respect to time of the position vector of C_1

$$\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1}\mathbf{\imath} + \dot{y}_{C_1}\mathbf{\jmath}, \tag{1.24}$$

where

$$\dot{x}_{C_1} = -\frac{L_1}{2}\dot{q}_1 \sin q_1, \;\; \dot{y}_{C_1} = \frac{L_1}{2}\dot{q}_1 \cos q_1. \tag{1.25}$$

The velocity vector of C_2 is the derivative with respect to time of the position vector of C_2

$$\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2}\mathbf{ı} + \dot{y}_{C_2}\mathbf{ȷ}, \tag{1.26}$$

where

$$\begin{aligned} \dot{x}_{C_2} &= -L_1\dot{q}_1 \sin q_1 - \frac{L_2}{2}\dot{q}_2 \sin q_2, \\ \dot{y}_{C_2} &= L_1\dot{q}_1 \cos q_1 + \frac{L_2}{2}\dot{q}_2 \cos q_2. \end{aligned} \tag{1.27}$$

The acceleration vector of C_1 is the double derivative with respect to time of the position vector of C_1

$$\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1}\mathbf{ı} + \ddot{y}_{C_1}\mathbf{ȷ}, \tag{1.28}$$

where

$$\begin{aligned} \ddot{x}_{C_1} &= -\frac{L_1}{2}\ddot{q}_1 \sin q_1 - \frac{L_1}{2}\dot{q}_1^2 \cos q_1, \\ \ddot{y}_{C_1} &= \frac{L_1}{2}\ddot{q}_1 \cos q_1 - \frac{L_1}{2}\dot{q}_1^2 \sin q_1. \end{aligned} \tag{1.29}$$

The acceleration vector of C_2 is the double derivative with respect to time of the position vector of C_2

$$\mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2}\mathbf{ı} + \ddot{y}_{C_2}\mathbf{ȷ}, \tag{1.30}$$

where

$$\begin{aligned} \ddot{x}_{C_2} &= -L_1\ddot{q}_1 \sin q_1 - L_1\dot{q}_1^2 \cos q_1 - \frac{L_2}{2}\ddot{q}_2 \sin q_2 - \frac{L_2}{2}\dot{q}_2^2 \cos q_2, \\ \ddot{y}_{C_2} &= L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2. \end{aligned} \tag{1.31}$$

The angular velocity vectors of the links 1 and 2 are

$$\boldsymbol{\omega}_1 = \dot{q}_1\mathbf{k}, \;\; \boldsymbol{\omega}_2 = \dot{q}_2\mathbf{k}. \tag{1.32}$$

The angular acceleration vectors of the links 1 and 2 are

$$\boldsymbol{\alpha}_1 = \ddot{q}_1\mathbf{k}, \;\; \boldsymbol{\alpha}_2 = \ddot{q}_2\mathbf{k}. \tag{1.33}$$

Newton-Euler equations of motion

The gravitational forces of the links 1 and 2 are

$$\mathbf{G}_1 = -m_1 g \jmath, \ \mathbf{G}_2 = -m_2 g \jmath. \tag{1.34}$$

The mass moments of inertia of the link 1 with respect to the center of mass C_1 is

$$I_{C_1} = \frac{m_1 L_1^2}{12}.$$

The mass moments of inertia of the link 1 with respect to the fixed point of rotation A is

$$I_A = I_{C_1} + m_1 \left(\frac{L_1}{2}\right)^2 = \frac{m_1 L_1^2}{3}.$$

The mass moment of inertia of the link 2 with respect to the center of mass C_2 is

$$I_{C_2} = \frac{m_2 L_2^2}{12}.$$

The equations of motion of the double pendulum are inferred using the Newton-Euler method. There are two rigid bodies in the system and the Newton-Euler equations are written for each link using the free body diagrams shown in Fig. 1.2(b).

Link 1.

The Newton-Euler equations for the link 1 are

$$m_1 \mathbf{a}_{C_1} = \mathbf{F}_{01} + \mathbf{F}_{21} + \mathbf{G}_1, \tag{1.35}$$

$$I_{C_1} \boldsymbol{\alpha}_1 = \mathbf{r}_{C_1 A} \times \mathbf{F}_{01} + \mathbf{r}_{C_1 B} \times \mathbf{F}_{21}, \tag{1.36}$$

where $\mathbf{F}_{01}$ is the joint reaction of the ground 0 on the link 1 at point A, and $\mathbf{F}_{21}$ is the joint reaction of the link 2 on the link 1 at point B

$$\mathbf{F}_{01} = F_{01x} \imath + F_{01y} \jmath, \quad \mathbf{F}_{21} = F_{21x} \imath + F_{21y} \jmath.$$

Since the link 1 has a fixed point of rotation at A the moment sum about the fixed point must be equal to to the product of the rod mass moment of inertia about that point and the rod angular acceleration. Thus

$$I_A \boldsymbol{\alpha}_1 = \mathbf{r}_{AC_1} \times \mathbf{G}_1 + \mathbf{r}_{AB} \times \mathbf{F}_{21}, \quad \text{or} \tag{1.37}$$

$$\frac{m_1 L_1^2}{3} \ddot{q}_1 \mathbf{k} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_{C_1} & y_{C_1} & 0 \\ 0 & -m_1 g & 0 \end{vmatrix} + \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_B & y_B & 0 \\ F_{21x} & F_{21y} & 0 \end{vmatrix}, \quad \text{or}$$

$$\frac{m_1 L_1^2}{3} \ddot{q}_1 \mathbf{k} = (-m_1 g x_{C_1} + F_{21y} x_B - F_{21x} y_B) \mathbf{k}.$$

The equation of motion for link 1 is

$$\frac{m_1 L_1^2}{3}\,\ddot{q}_1 = \left(-m_1 g\frac{L_1}{2}\cos q_1 + F_{21y}L_1\cos q_1 - F_{21x}L_1\sin q_1\right). \tag{1.38}$$

Link 2.

The Newton-Euler equations for the link 2 are

$$\begin{aligned} m_2\mathbf{a}_{C_2} &= \mathbf{F}_{12} + \mathbf{G}_2, \\ I_{C_2}\boldsymbol{\alpha}_2 &= \mathbf{r}_{C_2B}\times\mathbf{F}_{12}, \end{aligned} \tag{1.39}$$

where $\mathbf{F}_{12} = -\mathbf{F}_{21}$ is the joint reaction of the link 1 on the link 2 at B. Equation (1.39) becomes

$$\begin{aligned} m_2\ddot{x}_{C_2} &= -F_{21x}, \\ m_2\ddot{y}_{C_2} &= -F_{21y} - m_2\,g, \\ \frac{mL_2^2}{12}\,\ddot{q}_2\mathbf{k} &= \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_B - x_{C_2} & y_B - y_{C_2} & 0 \\ -F_{21x} & -F_{21y} & 0 \end{vmatrix}, \end{aligned} \tag{1.40}$$

or

$$m_2\left(-L_1\ddot{q}_1\sin q_1 - L_1\dot{q}_1^2\cos q_1 - \frac{L_2}{2}\ddot{q}_2\sin q_2 - \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)$$
$$= -F_{21x}, \tag{1.41}$$
$$m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2\right)$$
$$= -F_{21y} - m_2\,g, \tag{1.42}$$
$$\frac{m_2 L_2^2}{12}\,\ddot{q}_2 = \frac{L_2}{2}\left(-F_{21y}\cos q_2 + F_{21x}\sin q_2\right). \tag{1.43}$$

The reaction components F_{21x} and F_{21y} are obtained from Eqs. (1.41)(1.42)

$$F_{21x} = m_2\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right), \tag{1.44}$$
$$F_{21y} = -m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2\right) + m_2\,g. \tag{1.45}$$

The equations of motion are obtained substituting F_{21x} and F_{21y} in Eq. (1.38) and Eq. (1.43)

$$\frac{m_2 L_1^2}{3}\,\ddot{q}_1 = -m_1 g \frac{L_1}{2}\cos q_1 - m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2 - g\right)L_1\cos q_1 - m_2\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)L_1\sin q_1, \quad (1.46)$$

$$\frac{m_2 L_2^2}{12}\,\ddot{q}_2 = \frac{m_2\,L_2}{2}\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2 - g\right)\cos q_2 + \frac{m_2\,L_2}{2}\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)\sin q_2. \quad (1.47)$$

The eqautions of motion represent two nonlinear differential equations. The initial conditions (Cauchy problem) are necessary to solve the equations. At $t = 0$ the initial conditions are

$$q_1(0) = q_{10},\ q_2(0) = q_{20},\ \dot{q}_1(0) = \omega_{10},\ \dot{q}_2(0) = \omega_{20}$$

The numerical solutions for $q_1(0) = \pi/6$, $q_2(0) = \pi/3$, $\dot{q}_1(0) = 0$, $\dot{q}_2(0) = 0$ is given in Appendix 1.2.

1.3 Dynamics of a Disk on an Inclined Plane

A homogeneous circular disk in motion on a rough inclined plane is shown in Fig. 1.3. The fixed cartesian reference frame $xOyz$ is chosen with the origin at O. The angle between the axis Ox and the horizontal is α. The contact point between the disk and the plane is B. The disk has the mass m, the radius r, and the center of mass at C. The gravitational acceleration is g. Find the equations of motion for the disk.

Solution.

Pure rolling (no sliding)

The forces that act on the disk are the gravitational force $\mathbf{G} = -mg\boldsymbol{\jmath}$ at the point C, the normal reaction force $\mathbf{N}$ of the plane, and the friction force $\mathbf{F}_f$ at the contact point B. The rolling friction is considered negligible. The position vector $\mathbf{r}_C$ of the center of mass C of the disk is

$$\mathbf{r}_C = x_C\boldsymbol{\imath} + r\boldsymbol{\jmath}. \tag{1.48}$$

The velocity vector $\mathbf{v}_C$ of the center of mass C of the disk is

$$\mathbf{v}_C = \dot{\mathbf{r}}_C = \dot{x}_C\boldsymbol{\imath} + \dot{r}\boldsymbol{\jmath} = \dot{x}_C\boldsymbol{\imath}. \tag{1.49}$$

Denoting $\dot{x}_C = v$, the velocity of the center of mass C becomes

$$\mathbf{v}_C = v\boldsymbol{\imath}. \tag{1.50}$$

Thus, the acceleration vector of $\mathbf{a}_C$ of the center of mass C of the disk is

$$\mathbf{a}_C = \dot{\mathbf{v}}_C = \dot{v}\boldsymbol{\imath}. \tag{1.51}$$

One can express the velocity $\mathbf{v}_B$ of the contact point B as

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{CB} = v\,\boldsymbol{\imath} + (-\omega\mathbf{k}) \times (-r\boldsymbol{\jmath}) = (v - r\omega)\boldsymbol{\imath}. \tag{1.52}$$

In order to find the equation of motion for the disk, one can write the Newton's equation

$$m\mathbf{a}_C = \sum \mathbf{F}. \tag{1.53}$$

The sum of the external forces can be written as

$$\sum \mathbf{F} = m\mathbf{g} + \mathbf{F}_f + \mathbf{N} = (mg\sin\alpha - F_f)\boldsymbol{\imath} + (N - mg\cos\alpha)\boldsymbol{\jmath}, \tag{1.54}$$

where $\mathbf{F}_f = -F_f \boldsymbol{\imath}$ is the friction force, and $\mathbf{N} = N\boldsymbol{\jmath}$ is the reaction force of the plane on the disk. From Eqs. (1.51), (1.53), and (1.54) one can write the following equations

$$m\dot{v} = mg\sin\alpha - F_f, \tag{1.55}$$

$$N = mg\cos\alpha. \tag{1.56}$$

The following moment equation can be written for the disk with respect to its center of mass C

$$I_C \boldsymbol{\alpha} = \sum \mathbf{M}_C, \tag{1.57}$$

where $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = -\dot{\omega}\mathbf{k}$ is the angular acceleration of the disk ($\boldsymbol{\omega} = -\omega\mathbf{k}$ is the angular velocity) and I_C is the mass moment of inertia with respect to the point C. The sum of the external moments can be written as

$$\sum \mathbf{M}_C = \mathbf{r}_{CB} \times \mathbf{F} = (-r\boldsymbol{\jmath}) \times (-F\boldsymbol{\imath}) = -rF_f\mathbf{k}. \tag{1.58}$$

From Eqs. (1.57) and (1.58) one can write the following equation

$$I_C\dot{\omega} = rF_f. \tag{1.59}$$

For no sliding, the velocity $\mathbf{v}_B$ is zero ($\mathbf{v}_B = \mathbf{0}$). Thus, from Eq. (1.52) one can write $\omega = v/r$ and Eq. (1.59) becomes

$$I_C\frac{\dot{v}}{r^2} = F_f. \tag{1.60}$$

From Eqs. (1.55) and (1.60) the following equation of motion can be derived

$$\left(m + \frac{I_C}{r^2}\right)\dot{v} = mg\sin\alpha. \tag{1.61}$$

A homogeneous disk is considered in our case and the mass moment of inertia with respect to its center of mass is $I_C = m\dfrac{r^2}{2}$. Thus, Eq. (1.61) can be written as

$$\dot{v} = \frac{2}{3}g\sin\alpha. \tag{1.62}$$

Condition for pure rolling

From Eq. (1.55) and Eq. (1.62) one can compute the friction force F_f as

$$F_f = \frac{m}{3}g\sin\alpha. \tag{1.63}$$

The condition for the disk of rolling without sliding on the plane is

$$F_f \leq \mu_k N, \tag{1.64}$$

where μ_k is the coefficient of kinetic friction. From Eq. (1.56), Eq. (1.63), and Eq. (1.64) one can obtain

$$\tan\alpha \leq 3\mu_k, \tag{1.65}$$

or

$$\alpha \leq \Phi, \tag{1.66}$$

where the sliding friction angle Φ can be determined from the equation

$$\tan\Phi = 3\mu_k, \text{ where } \mu_k = \tan\phi. \tag{1.67}$$

Eq. (1.66) represents the condition for rolling without sliding of the disk on the plane. If the angle α of the plane is smaller than the sliding friction angle Φ, the disk rolls on the plane without sliding. If the angle α of the plane is greater than the sliding friction angle Φ, the disk rolls and slides on the plane simultaneously.

Moment of rolling friction

Experimentally one can observe that if the angle α of the plane is small enough, the disk does not move. The equilibrium conditions for the disk are $v = 0$, and $\omega = 0$. The rolling is stopped by a rolling resistant moment M_f that balances the active moment rF_f

$$M_f = rF_f. \tag{1.68}$$

The acceleration $\dot{v}$ is zero and from Eq. (1.55) one can express the friction force as $F_f = G\sin\alpha$. Thus, one can write

$$M_f = rmg\sin\alpha = rN\tan\alpha. \tag{1.69}$$

If α_0 is the value of the angle α when the rolling starts, the moment M_f is called the *rolling friction moment* and has the value

$$M_f = rN\tan\alpha_0. \tag{1.70}$$

The constant $r\tan\alpha_0$ is denoted by s and represents the coefficient of rolling friction

$$s = r\tan\alpha_0. \tag{1.71}$$

The rolling friction moment M_f become

$$M_f = sN. \tag{1.72}$$

The rolling friction moment M_f is proportional to the normal reaction N and has the expression

$$\mathbf{M}_f = -sN\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}. \tag{1.73}$$

Rolling with moment of friction

In this case, Eq. (1.57) becomes

$$I\dot{\omega} = rF_f - sN. \tag{1.74}$$

From Eq. (1.55) and Eq. (1.74) one can write

$$\dot{v} = \frac{2}{3}(\sin\alpha - \cos\alpha\tan\alpha_0)g = \frac{2\sin(\alpha-\alpha_0)}{3\cos\alpha_0}g. \tag{1.75}$$

In this case, the rolling condition is

$$\frac{sin(\alpha-\alpha_0)}{\cos\alpha_0} \le 3\frac{sin(\phi-\alpha_0)}{\cos\phi}, \tag{1.76}$$

or

$$\tan\alpha \le \tan\phi + 2(\tan\phi - \tan\alpha_0). \tag{1.77}$$

Eq. (1.77) can also be written as

$$\alpha \le \Phi, \tag{1.78}$$

where the angle Φ can be obtained from

$$\tan\Phi = \tan\phi + 2(\tan\phi - \tan\alpha_0). \tag{1.79}$$

For the motion of the homogeneous disk on the plane of slope α, the following three cases are possible:

Case 1. $\alpha < \alpha_0$ (Eq. (1.75)). The disk has no motion.

Case 2. $\alpha_0 \le \alpha < \Phi$ (Eq. (1.79)). The disk has pure rolling (no sliding) motion.

Case 3. $\alpha \ge \Phi$. The disk has rolling and sliding motion simultaneously .

Problems

1. Outside the long slender pendulum 1, of mass m_1, a translational joint 2, of mass m_2, is sliding without friction (Fig. 1.4). The length of the pendulum is L. The mass moment of inertia of the slider 2 with respect to its mass center point A is I_A. The acceleration due to gravity is g. Find the equations of motion, using Newton-Euler method, for the system with $m_1 = m_2 = m$.

   Numerical application: $m = 1$ kg, $L = 1$ m, $I_A = 1$ kg m^2, and $g = 10$ m/s^2.

2. A slender rod $AB = L$ (link 2) is moving without friction along the slider 1(Fig. 1.5). The slider is connnected to the ground by a pin joint at O and is free to swing in a vertical plane. The mass of the rod is m_1 the mass center is located at C. The mass of the slider is m_2 and the mass moment of inertia of the slider with respect to its mass center point O is I_O. The acceleration due to gravity is g. Find the Newton Euler equations of motion.

   Numerical application: $m_1 = m_2 = 1$ kg, $L = 1$ m, $I_O = 1$ kg m^2, and $g = 10$ m/s^2.

3. Two uniform hinged rods 1 and 2 of mass $m_1 = m_2 = m$ and length $AB = BC = L$ are shown in Fig. 1.6. The rod 1 is connnected to the ground by a pin joint at A and to the rod 2 by a pin joint at B. The end B is moving with friction along the horizontal surface. The coefficient of friction between rod 2 and the horizontal surface is μ. The acceleration due to gravity is g. Find the equation of motion of the system.

   Numerical application: $m = 1$ kg, $L = 1$ m, $\mu = 0.1$, and $g = 10$ m/s^2.

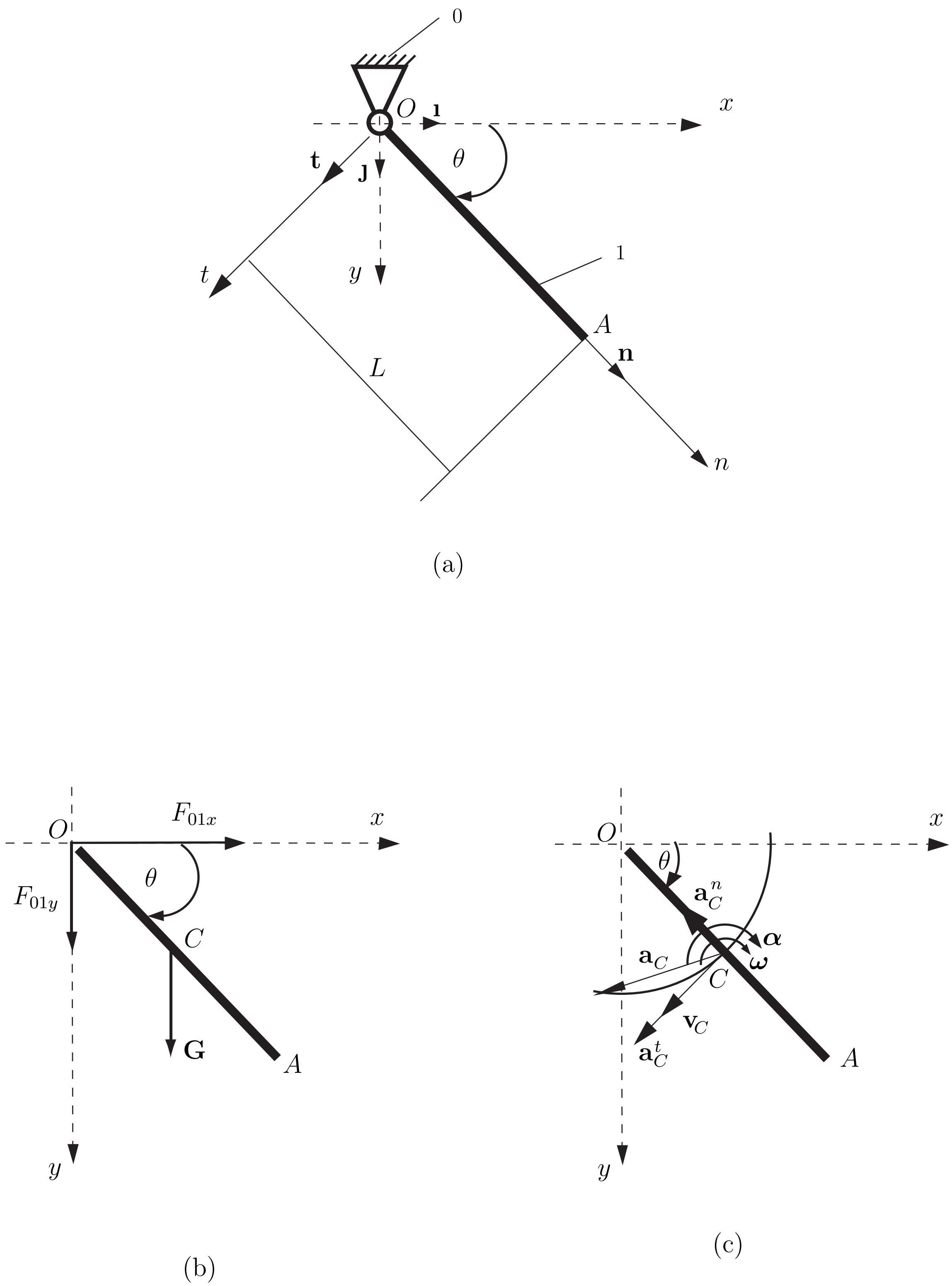
4. Figure 1.7 depicts two uniform rods 1 and 2 of mass $m_1 = m_2 = m$ and length $OA = AB = 2\,L$. The rod 1 is connnected to the ground by a pin joint at O and to the rod 2 by a pin joint at A. The rods are constrained to move in a vertical plane xOy. The x-axis is vertical, with the positive sense directed vertically downward. The y-axis is horizontal and is contained in the plane of motion. The rod 1 is moving and the instant angle with the vertical axis Ox is $q(t)$. The rod 2 is connnected to the ground by a pin joint at B which is confined to move in a vertical

slot. The local acceleration of gravity is g. Find the equations of motion of the system.

Numerical application: $m = 1$ kg, $L = 1$ m, and $g = 10$ m/s^2.

5. Figure 1.8 shows an open kinematic chain with two uniform rigid rods 1 and 2 of mass $m_1 = m_2 = m$ and length $L_1 = L_2 = L$. The rod 1 is connnected to the ground by a pin joint at A and to the rod 2 by a pin joint at B. The rods are constrained to move in a vertical plane xy. A spring of elastic constant k and a viscous damper with a damping constant c are opposing the relative motion of the link 2 with respect to link 1. The local acceleration of gravity is g. Find the equations of motion of the system.

   Numerical application: $m_1 = m_2 = 1$ kg, $L_1 = L_2 = 1$ m, $g = 10$ m/s^2, $k = 100$ N m/rad, $c = 10$ N m s/rad, $q_1(0) = \pi/3$ rad, $q_2(0) = \pi/6$ rad, $\dot{q}_1(0) = \dot{q}_2(0) = 0$ rad/s.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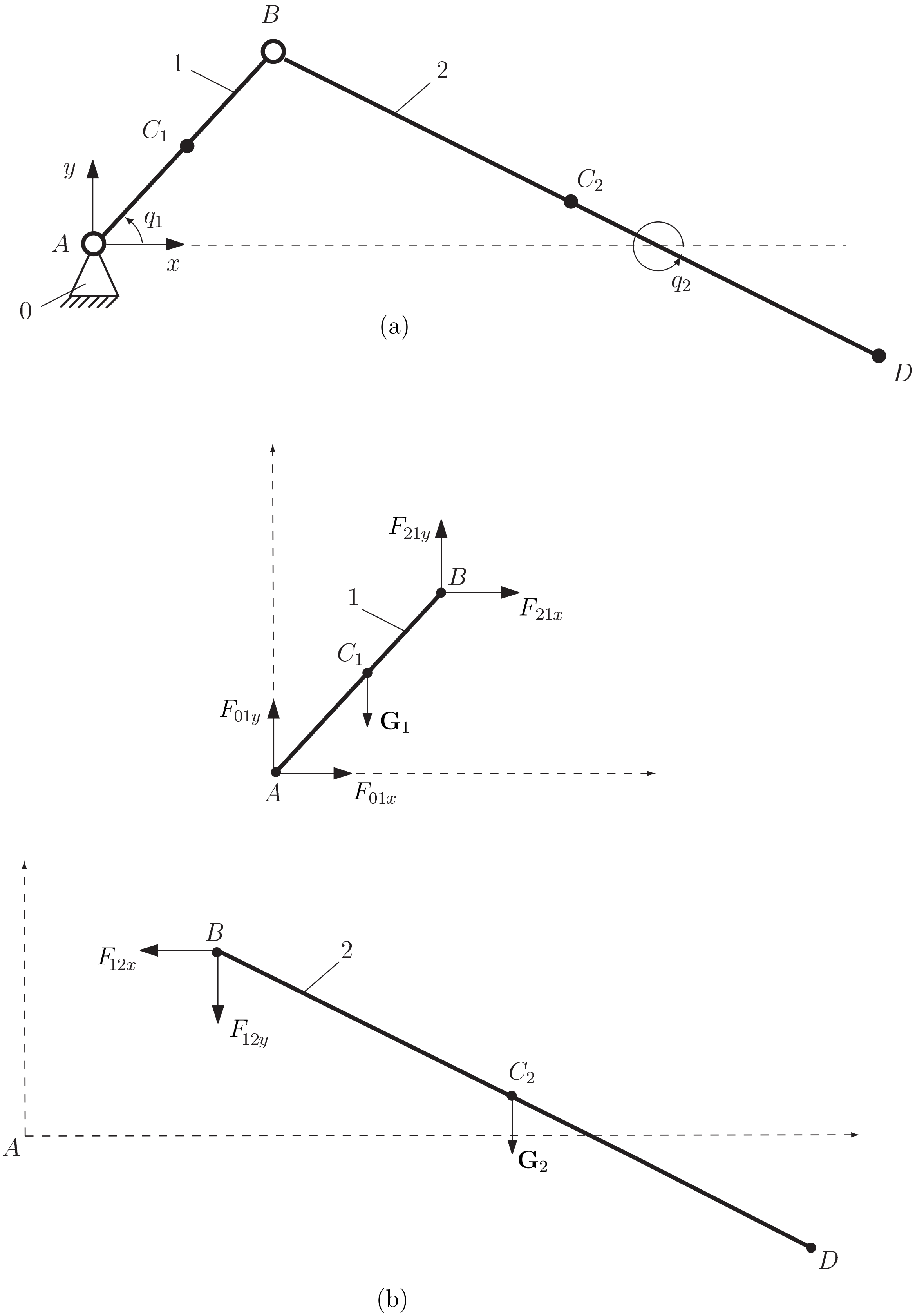
ω
a_C
C
v_C
a_C^t
A
y
(c)

Figure 1



B
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2
C_1
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C_2
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x
q_2
0
(a)
D
F_{21y}
B
1
F_{21x}
C_1
F_{01y}
$\mathbf{G}_1$
A
F_{01x}
B
F_{12x}
2
F_{12y}
C_2
A
$\mathbf{G}_2$
D
(b)

Figure 2

```
Apply[Clear, Names["Global`*"]];
Off[General::spell];
Off[General::spell1];

Print["Kinematics"];
omega[t] = {0, 0, theta'[t]};
Print["angular velocity of RB: omega=", omega[t]];
alpha[t] = {0, 0, theta''[t]};
Print["angular acceleration of RB: alpha=", alpha[t]];
xC = (L / 2) * Cos[theta[t]];
yC = (L / 2) * Sin[theta[t]];
rC = {xC, yC, 0};
Print["position vector of C: rC=", rC];
vC = D[rC, t];
Print["velocity of C: vC=d(rC)/dt=", vC];
aC = D[vC, t];
Print["acceleration of C: aC=d(vC)/dt=", Simplify[aC]];
Print["another way of calculating vC and aC"];
vC = Cross[omega[t], rC];
Print["vC=omega x rC=", vC];
aC = Cross[alpha[t], rC] - omega[t].omega[t] * rC;
Print["aC=alpha x rC - omega.omega rC=", Simplify[aC]];
Print["Forces"];
FO = {FOx, FOy, 0};
Print["reaction force at pin joint O: FO=", FO];
G = {0, m g, 0};
Print["gravitational force at C: G=", G];
rCO = -rC;
IC = m * L^2 / 12;
Print["mass moment of inertia wrt C: ICz=", IC];
IO = IC + m * (L / 2)^2;
Print["mass moment of inertia wrt O: IOz=ICz+m*(L/2)^2=",
  Simplify[IO]];
Print["Method I"];
eqI = Simplify[IO * alpha[t] - Cross[rC, G]];
solutionI = Solve[eqI[[3]] == 0, theta''[t]][[1]];
Print["moment equation: IO alpha = sum M wrt O = rC x G"];
Print[eqI[[3]], "=0"];
Print["Solution: theta''[t]=", theta''[t] /. Simplify[solutionI]];
Print["Method II"];
eqIIF = Simplify[m * aC - (FO + G)];
Print["force equation: m aC = sum F = FO + G"];
Print["projection on x:"];
Print[eqIIF[[1]], "=0", " (1)"];
Print["projection on y:"];
Print[eqIIF[[2]], "=0", " (2)"];
eqIIM = Simplify[IC * alpha[t] - Cross[rCO, FO]];
Print["moment eqation: IC alpha = sum M wrt C = -rC x FO"];
Print["projection on z: "]
Print[eqIIM[[3]], "=0", " (3)"];
solFOx = Solve[eqIIF[[1]] == 0, FOx][[1]];
Fx = FOx /. solFOx;
Print["from Eq.(1) => FOx = ", Fx];
solFOy = Solve[eqIIF[[2]] == 0, FOy][[1]];
Fy = FOy /. solFOy;
Print["from Eq.(2) => FOy = ", Fy];
solutionII = Solve[(eqIIM[[3]] /. solFOx /. solFOy) == 0, theta''[t]][[1]];
```

```
ddtheta = theta''[t] /. Simplify[solutionII];
Print["from Eqs.(1)(2)(3) => theta''[t] = ", ddtheta];
Print["Initial Conditions"];
Print["at t=0: theta[0]=0, theta'[0]=omega[0]=0"];
ic = {theta[0] → 0, theta'[0] → 0};
ddtheta0 = ddtheta /. {t → 0} /. ic;
Print["numerical data: m=12/32.2,L=3,g=32.2"];
data = {m → 12 / 32.2, L → 3, g → 32.2};
Print["theta''[0]=alpha[0]= ", ddtheta0, "=", ddtheta0 /. data, " rad/s^2"];
V = Simplify[Fy /. {t → 0} /. ic];
V0 = Simplify[V /. theta''[0] -> ddtheta0];
Print["V=F0y[0] = ", V, "=", V0, "=", V0 /. data, " lb"];
H = Simplify[Fx /. {t → 0} /. ic];
Print["H=F0x[0] = ", H, " lb"];
(* Numerical solution of differential equation *)
soldif =
  NDSolve[{(eqI[[3]] /. data) == 0, theta[0] == 0, theta'[0] == 0}, theta[t], {t, 0, 2}];
Plot[Evaluate[theta[t] /. soldif] * 180 / Pi, {t, 0, 2}, AxesLabel → {"t[s]", "theta[deg]"}];
Plot[Evaluate[D[theta[t] /. soldif, t]], {t, 0, 2}, AxesLabel → {"t[s]", "omega[rad/s]"}];
Plot[Evaluate[D[theta[t] /. soldif, {t, 2}]],
  {t, 0, 2}, AxesLabel → {"t[s]", "alpha[rad/s]"}];

DSolve[theta''[t] + C Sin[theta[t]] == 0, theta[t], t]
```

Kinematics

angular velocity of RB: omega={0, 0, theta′[t]}

angular acceleration of RB: alpha={0, 0, theta″[t]}

position vector of C: rC= $\left\{\frac{1}{2} L \text{Cos}[\text{theta}[t]], \frac{1}{2} L \text{Sin}[\text{theta}[t]], 0\right\}$

velocity of C: vC=d(rC)/dt= $\left\{-\frac{1}{2} L \text{Sin}[\text{theta}[t]]\, \text{theta}'[t], \frac{1}{2} L \text{Cos}[\text{theta}[t]]\, \text{theta}'[t], 0\right\}$

acceleration of C: aC=d(vC)/dt= $\left\{-\frac{1}{2} L \left(\text{Cos}[\text{theta}[t]]\, \text{theta}'[t]^2 + \text{Sin}[\text{theta}[t]]\, \text{theta}''[t]\right), \frac{1}{2} \left(-L \text{Sin}[\text{theta}[t]]\, \text{theta}'[t]^2 + L \text{Cos}[\text{theta}[t]]\, \text{theta}''[t]\right), 0\right\}$

another way of calculating vC and aC

vC=omega x rC= $\left\{-\frac{1}{2} L \text{Sin}[\text{theta}[t]]\, \text{theta}'[t], \frac{1}{2} L \text{Cos}[\text{theta}[t]]\, \text{theta}'[t], 0\right\}$

aC=alpha x rC - omega.omega rC= $\left\{-\frac{1}{2} L \left(\text{Cos}[\text{theta}[t]]\, \text{theta}'[t]^2 + \text{Sin}[\text{theta}[t]]\, \text{theta}''[t]\right), \frac{1}{2} \left(-L \text{Sin}[\text{theta}[t]]\, \text{theta}'[t]^2 + L \text{Cos}[\text{theta}[t]]\, \text{theta}''[t]\right), 0\right\}$

Forces

reaction force at pin joint O: FO={FOx, FOy, 0}

gravitational force at C: G={0, g m, 0}

mass moment of inertia wrt C: ICz= $\frac{L^2 m}{12}$

mass moment of inertia wrt O: IOz=ICz+m*(L/2)^2= $\frac{L^2 m}{3}$

Method I

moment equation: IO alpha = sum M wrt O = rC x G

$\frac{1}{6}$ L m (-3 g Cos[theta[t]] + 2 L theta''[t]) = 0

Solution: theta''[t] = $\frac{3 \text{ g Cos[theta[t]]}}{2 \text{ L}}$

Method II

force equation: m aC = sum F = FO + G

projection on x:

$\frac{1}{2}$ (-2 FOx - L m Cos[theta[t]] theta'[t]^2 - L m Sin[theta[t]] theta''[t]) = 0 (1)

projection on y:

$\frac{1}{2}$ (-2 (FOy + g m) - L m Sin[theta[t]] theta'[t]^2 + L m Cos[theta[t]] theta''[t]) = 0 (2)

moment eqation: IC alpha = sum M wrt C = -rC x FO

projection on z:

$\frac{1}{12}$ L (6 FOy Cos[theta[t]] - 6 FOx Sin[theta[t]] + L m theta''[t]) = 0 (3)

from Eq.(1) => FOx = $\frac{1}{2}$ (-L m Cos[theta[t]] theta'[t]^2 - L m Sin[theta[t]] theta''[t])

from Eq.(2) => FOy = $\frac{1}{2}$ (-2 g m - L m Sin[theta[t]] theta'[t]^2 + L m Cos[theta[t]] theta''[t])

from Eqs.(1)(2)(3) => theta''[t] = $\frac{3 \text{ g Cos[theta[t]]}}{2 \text{ L}}$

Initial Conditions

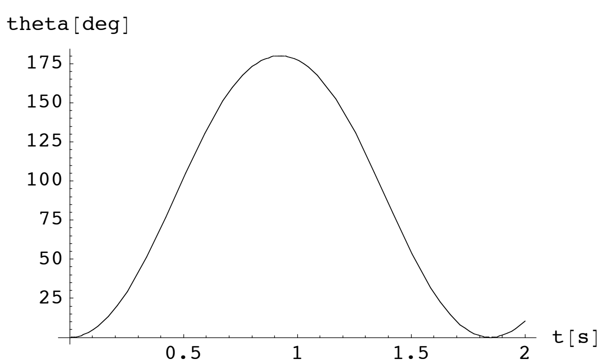
at t=0: theta[0]=0, theta'[0]=omega[0]=0

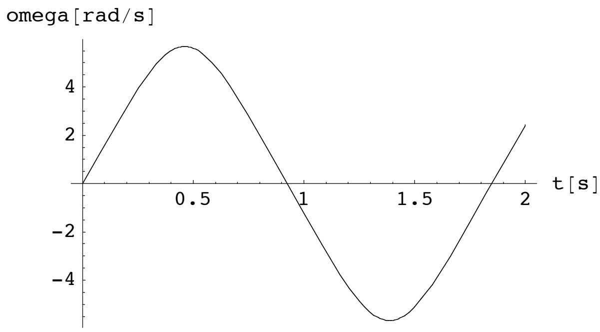
numerical data: m=12/32.2,L=3,g=32.2

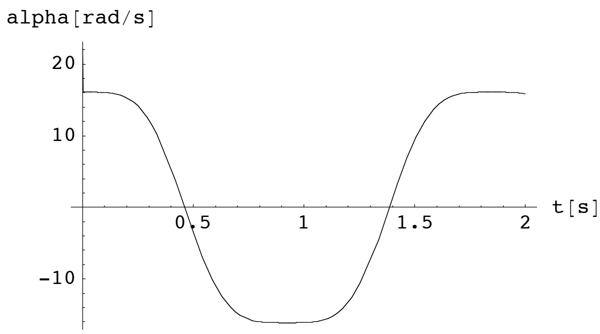
theta''[0]=alpha[0]= $\frac{3 \text{ g}}{2 \text{ L}}$=16.1 rad/s^2

V=F0y[0] = -g m + $\frac{1}{2}$ L m theta''[0]=-$\frac{\text{g m}}{4}$=-3. lb

H=F0x[0] = 0 lb








```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

$$\left\{\left\{\text{theta}[t]\to 2\,\text{JacobiAmplitude}\left[-\frac{1}{2}\sqrt{(-2\,C-C[1])\,(-t^2-2\,t\,C[2]-C[2]^2)},\ \frac{4\,C}{2\,C+C[1]}\right]\right\},\right.$$

$$\left.\left\{\text{theta}[t]\to 2\,\text{JacobiAmplitude}\left[\frac{1}{2}\sqrt{(-2\,C-C[1])\,(-t^2-2\,t\,C[2]-C[2]^2)},\ \frac{4\,C}{2\,C+C[1]}\right]\right\}\right\}$$

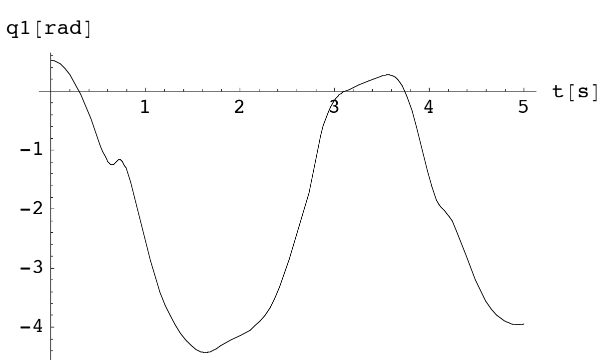
```
Apply[Clear, Names["Global`*"]];
Off[General::spell];
Off[General::spell1];

(*Input data*)

data = {m1 → 1., m2 → 1., L1 → 1., L2 → 1., g → 10};

(*data={m1→m, m2→m,L1→L,L2→L};*)

IC1 = m1 / 12 * (L1^2);
IA = IC1 + m1 (L1 / 2) ^2;
IC2 = m2 / 12 * (L2^2);

(*Position,velocity and acceleration vectors*)

xB = L1 * Cos[q1[t]];
yB = L1 * Sin[q1[t]];
rB = {xB, yB, 0};
rC1 = rB / 2.;
vC1 = D[rC1, t];
aC1 = D[vC1, t];

xC = xB + L2 * Cos[q2[t]];
yC = yB + L2 * Sin[q2[t]];
rC = {xC, yC, 0};
rC2 = (rB + rC) / 2.;
vC2 = D[rC2, t];
aC2 = D[vC2, t];

Print["rC1=", rC1];
Print["rB=", rB];
Print["rC2=", rC2];
Print["rC=", rC];
Print["aC1=", aC1];
Print["aC2=", aC2];

(*Angular velocities and accelerations*)

omega1 = {0, 0, q1'[t]};
omega2 = {0, 0, q2'[t]};
alpha1 = {0, 0, q1''[t]};
alpha2 = {0, 0, q2''[t]};

Print["alpha1=", alpha1];
Print["alpha2=", alpha2];


F01 = {F01x, F01y, 0};
F21 = {F21x, F21y, 0};

Print["Joint reaction at B: F21={F21x,F21y,0}"];

G1 = {0, -m1 g, 0};
G2 = {0, -m2 g, 0};

(*Newton equations*)
```

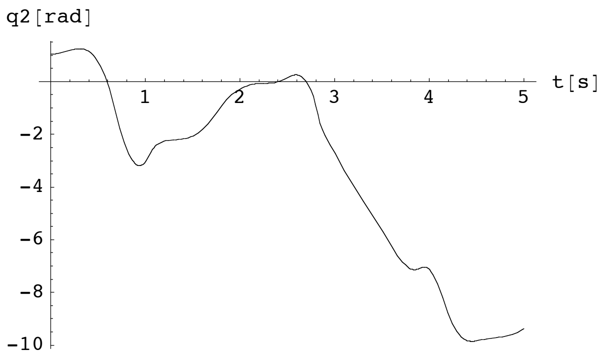
```
"LINK 1"
"Sum M for 1 wrt A:"
"-IA alpha1 + AB x F21 + AC1 x G1 = 0"
EA = -IA * alpha1 + Cross[rB, F21] + Cross[rC1, G1];
eq1 = (EA[[3]] /. data) == 0;
Print["(z): ", eq1, ", (1)"];

"LINK 2"
"Sum F for link 2:"
"-m2 aC2 + G2 + (-F21) = 0"
N2 = -m2 * aC2 + (-F21) + G2;
eq2 = (N2[[1]] /. data) == 0;
eq3 = (N2[[2]] /. data) == 0;
Print["(x): ", eq2, ", (2)"];
Print["(y): ", eq3, ", (3)"];
"Sum M for 2 wrt C2:"
"-IC2 alpha2 + C2B x (-F21) = 0"
E2 = -IC2 * alpha2 + Cross[rB - rC2, -F21];
eq4 = (E2[[3]] /. data) == 0;
Print["(z): ", eq4, ", (4)"];

Print["From Eqs.(2)(3) => {F21x,F21y}"];

sol = Solve[{eq2, eq3}, {F21x, F21y}];

Print["F21x = ", Simplify[F21x /. sol[[1]]]];
Print["F21y = ", Simplify[F21y /. sol[[1]]]];

Print["From Eqs.(1)(4) => equations of motion"];
eI = Simplify[(eq1 /. sol)[[1]]];
eII = Simplify[(eq4 /. sol)[[1]]];
Print[eI, ", (5)"];
Print[eII, ", (6)"];
```

```
rC1={0.5 L1 Cos[q1[t]], 0.5 L1 Sin[q1[t]], 0}

rB={L1 Cos[q1[t]], L1 Sin[q1[t]], 0}

rC2={0.5 (2 L1 Cos[q1[t]] + L2 Cos[q2[t]]), 0.5 (2 L1 Sin[q1[t]] + L2 Sin[q2[t]]), 0}

rC={L1 Cos[q1[t]] + L2 Cos[q2[t]], L1 Sin[q1[t]] + L2 Sin[q2[t]], 0}

aC1={-0.5 L1 Cos[q1[t]] q1′[t]^2 - 0.5 L1 Sin[q1[t]] q1″[t],
  -0.5 L1 Sin[q1[t]] q1′[t]^2 + 0.5 L1 Cos[q1[t]] q1″[t], 0}

aC2={0.5 (-2 L1 Cos[q1[t]] q1′[t]^2 - L2 Cos[q2[t]] q2′[t]^2 - 2 L1 Sin[q1[t]] q1″[t] - L2 Sin[q2[t]] q2″[t]),
  0.5 (-2 L1 Sin[q1[t]] q1′[t]^2 - L2 Sin[q2[t]] q2′[t]^2 + 2 L1 Cos[q1[t]] q1″[t] + L2 Cos[q2[t]] q2″[t]), 0}

alpha1={0, 0, q1″[t]}

alpha2={0, 0, q2″[t]}

Joint reaction at B: F21={F21x,F21y,0}

LINK 1

Sum M for 1 wrt A:
```

```
-IA alpha1 + AB x F21 + AC1 x G1 = 0

(z): -5. Cos[q1[t]] + 1. F21y Cos[q1[t]] - 1. F21x Sin[q1[t]] - 0.333333 q1''[t] == 0, (1)

LINK 2

Sum F for link 2:

-m2 aC2 + G2 + (-F21) = 0

(x): -F21x - 0.5
    (-2. Cos[q1[t]] q1'[t]^2 - 1. Cos[q2[t]] q2'[t]^2 - 2. Sin[q1[t]] q1''[t] - 1. Sin[q2[t]] q2''[t]) == 0, (2)

(y): -10. - F21y - 0.5
    (-2. Sin[q1[t]] q1'[t]^2 - 1. Sin[q2[t]] q2'[t]^2 + 2. Cos[q1[t]] q1''[t] + 1. Cos[q2[t]] q2''[t]) == 0, (3)

Sum M for 2 wrt C2:

-IC2 alpha2 + C2B x (-F21) = 0

(z):
 0. F21y Cos[q1[t]] + 0.5 F21y Cos[q2[t]] + 0. F21x Sin[q1[t]] - 0.5 F21x Sin[q2[t]] - 0.0833333 q2''[t] == 0
 , (4)

From Eqs.(2)(3) => {F21x,F21y}

F21x = 1. Cos[q1[t]] q1'[t]^2 + 0.5 Cos[q2[t]] q2'[t]^2 + 1. Sin[q1[t]] q1''[t] + 0.5 Sin[q2[t]] q2''[t]

F21y = -10. + 1. Sin[q1[t]] q1'[t]^2 + 0.5 Sin[q2[t]] q2'[t]^2 - 1. Cos[q1[t]] q1''[t] - 0.5 Cos[q2[t]] q2''[t]

From Eqs.(1)(4) => equations of motion

(-1. Cos[q2[t]] Sin[q1[t]] + 1. Cos[q1[t]] Sin[q2[t]]) q2'[t]^2 ==
  30. Cos[q1[t]] + 2.66667 q1''[t] + 1. Cos[q1[t] - q2[t]] q2''[t], (5)

(1. Cos[q2[t]] Sin[q1[t]] - 1. Cos[q1[t]] Sin[q2[t]]) q1'[t]^2 ==
  10. Cos[q2[t]] + 1. Cos[q1[t] - q2[t]] q1''[t] + 0.666667 q2''[t], (6)
```

```
solution = NDSolve[{eI, eII, q1[0] == N[Pi] / 6, q2[0] == N[Pi] / 3, q1'[0] == 0, q2'[0] == 0},
   {q1[t], q2[t]}, {t, 0, 5}, MaxSteps → 2000];

Plot[Evaluate[q1[t] /. solution], {t, 0, 5}, AxesLabel → {"t[s]", "q1[rad]"}];
Plot[Evaluate[q2[t] /. solution], {t, 0, 5}, AxesLabel → {"t[s]", "q2[rad]"}];
```




q2[rad]
t[s]
1
2
3
4
5
-2
-4
-6
-8
-10

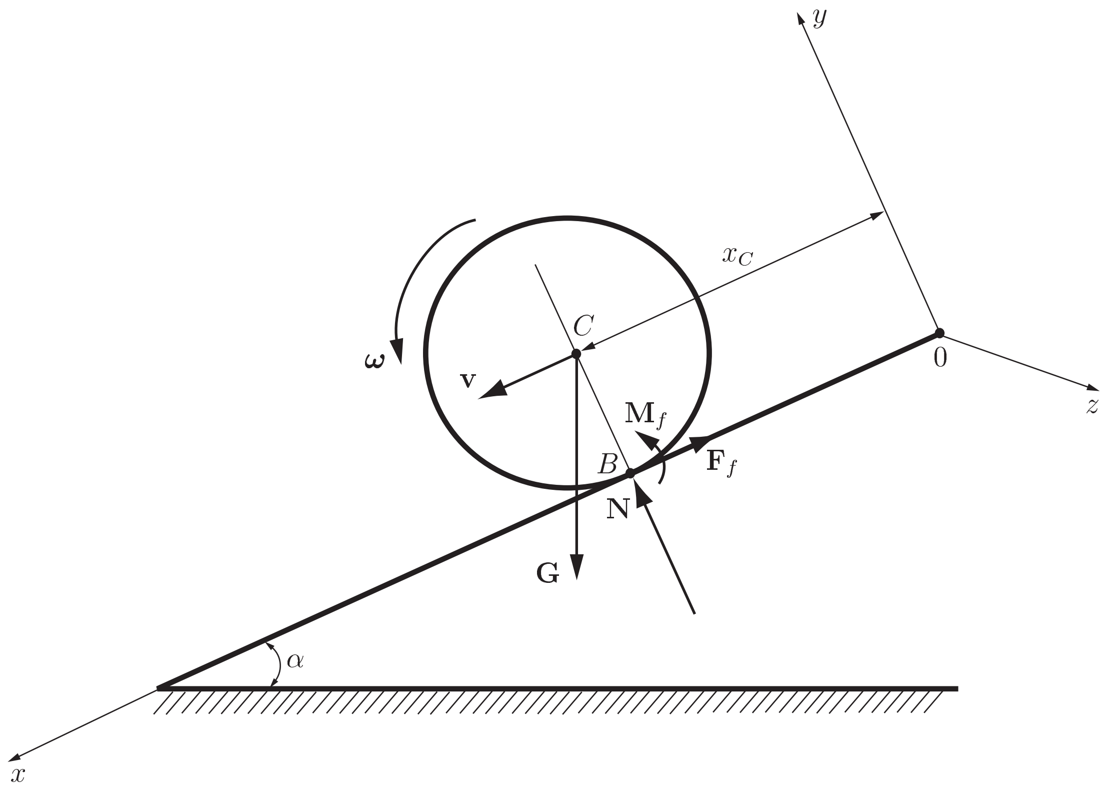


Figure 3

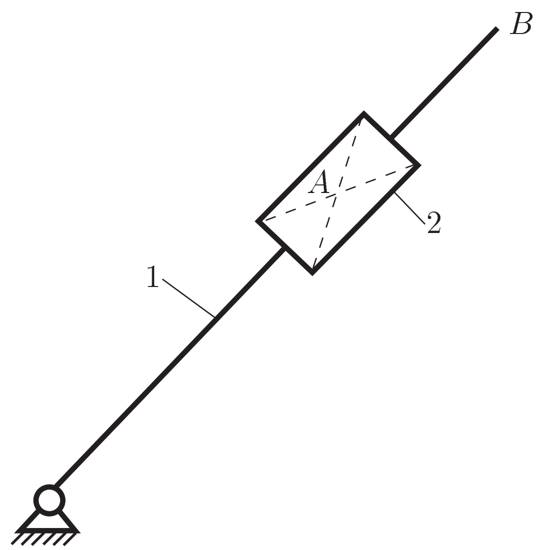


Figure 4

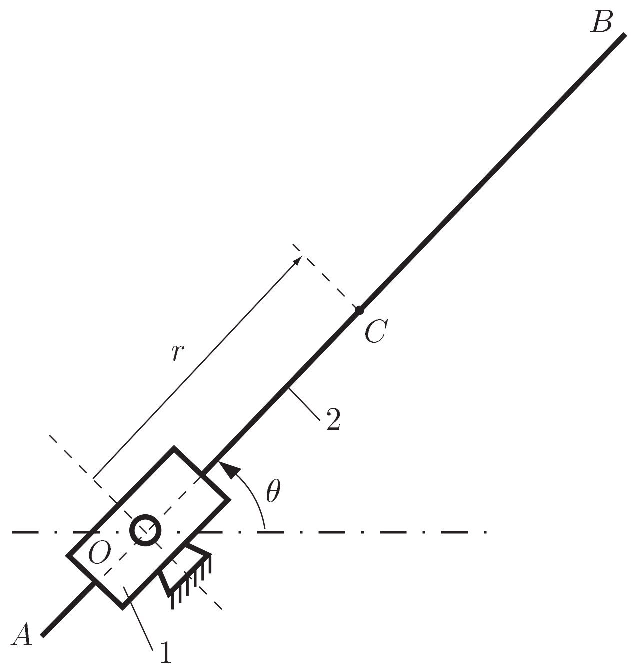


Figure 5

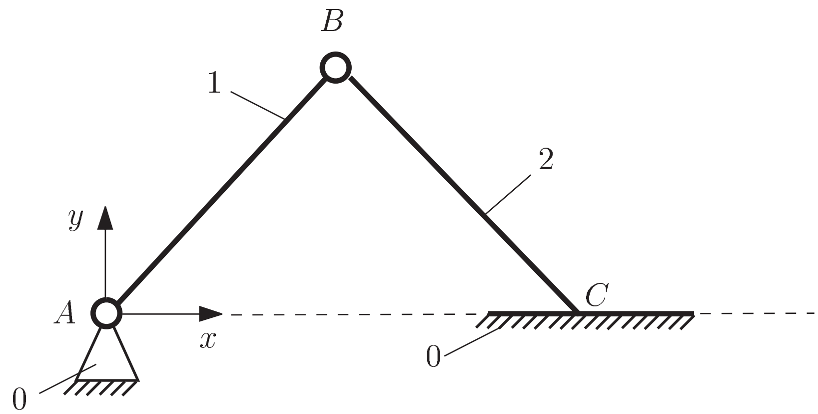


Figure 6

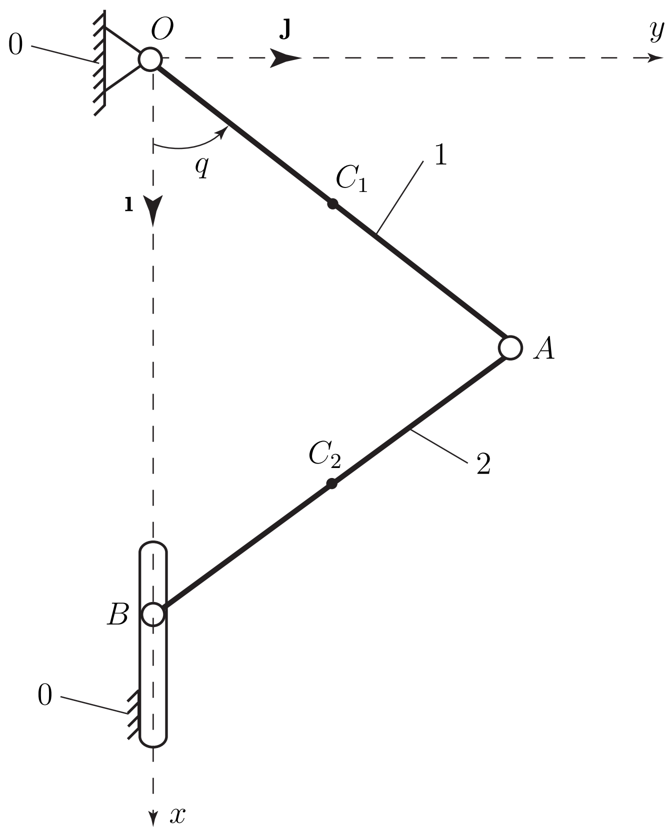


Figure 7

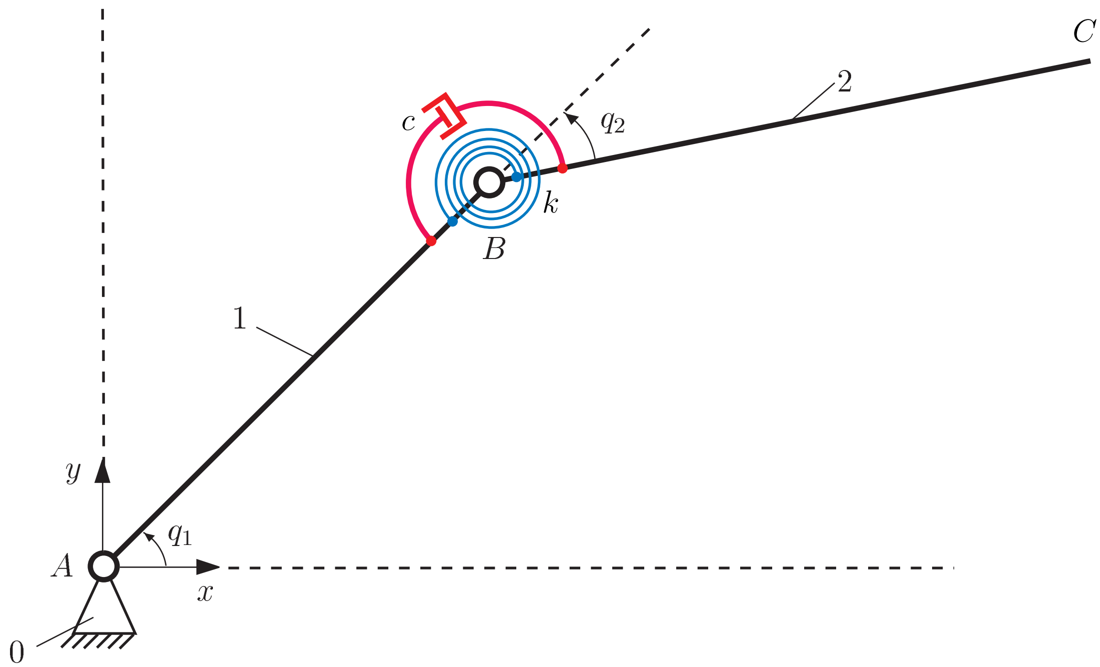


Figure 8